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A2 Mathematics for WJEC

Unit 7 – Sequences and Series

Examples and Practice Exercises

Unit Learning Objectives

- ***To understand arithmetic sequences and series, knowing the formulae for the general term and sum of terms.***
- ***To understand geometric sequences and series, knowing the formulae for the general term and sum of terms.***
- ***Understand when a geometric series converges so that the sum to infinity can be found.***
- ***Apply these skills in context to various modelling situations.***
- ***Understand other types of sequences defined as recurrence relations, including periodic sequences and increasing/decreasing sequences***

Prerequisite atoms:

Sigma Notation (AS Statistics)

Basic Algebra (GCSE)

When you have completed the unit…

Objective	Met	Know	Mastered
I can understand and solve problems with arithmetic sequences/series.			
I can understand and solve problems with geometric sequences/series.			
I can understand when a sequence is arithmetic/geometric, and can apply this knowledge to model real-life situations.			
I can understand and use sigma notation.			
I can solve problems involving recurrence relations.			

Notes/Areas to Develop:

Arithmetic (Linear) Sequences

We met arithmetic sequences at GCSE (under the name 'linear' sequences). At A2 we will extend our understanding of the general term and introduce formal methods.

In an arithmetic sequence, the difference between any two terms is fixed. We call this a **common difference.**

The general term u_n is given by the formula:

$$u_n = a + (n-1)d$$

where a is the first term of the sequence, and d is the common difference.

Example 1: Find the general term (nth term formula) for each of the following sequences:

a) 3, 7, 11, ⋯ b) 3.5, 5, 6.5, ⋯ c) 13, 7, 1, ⋯

Task 1:

a) Find the general term of the sequence 18, 25, 32, 39, ⋯

b) Use this to find:

i) The 20th term of the sequence,

ii) The first term in the sequence which exceeds 1000

Task 2: A sequence is given by the general term

$$u_n = pn + q.$$

Given that $u_3 = 7$ and $u_7 = 23$, find the values of p and q.

Task 3: Calculate the number of terms in each of the following sequences:

a) 3, 7, 11, ..., 83, 87

b) 3, 9, 15, ..., 129, 135.

Hint: Find the general term first, then consider the last term.

Now: *Complete Test Your Understanding 1, Pg 21.*

Arithmetic Series

So far, we have considered sequences.

A **series** is a sequence where we are summing the terms. So, for example:

3, 5, 7, 9 is a sequence, whereas

3 + 5 + 7 + 9 is a series.

Example 1: Find the sum of the first 100 positive integers.

History Time!

It is rumoured that a very young chap by the name of Carl Frederich Gauss was a naughty boy at school.

As a punishment, his teacher set him the tedious task of computing the sum of the first hundred natural number.

Within seconds, Gauss approached the front of the class and placed his slate on the desk, with the correct answer.

He couldn't have cheated… his teacher probed him on 'how' he had solved it so quickly, and immediately realised after his explanation that Gauss was, rather than naughty, so exceptionally talented so as to have been completely bored by learning the work of elementary school.

Gauss went on to become one of the most famous, prolific and important mathematicians of all time.

Example 2:

Prove that the sum of the first n terms of an arithmetic series is given by the formula

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Note that the examiner can, and sometimes does, ask for this proof in the exam! Learn it!

Example 3: Find the sum of the first 50 terms of the arithmetic series

82 + 75 + 68 + 61 + …

Task 1: Find the least number of terms required for the arithmetic series 3 + 7 + 11 + 15 + … to exceed 1000.

NOW: *Complete Test Your Understanding 2, Page 22.*

Geometric Sequences

Next, we will look at a different type of sequence. Consider for example the sequence:

3, 6, 12, 24, ...

This time, we are not adding the same thing each time, but instead multiplying by a ***common ratio.***

In a geometric sequence, the general term u_n is given by the formula:

$$u_n = ar^{n-1}$$

where a is the first term of the sequence, and r is the common ratio.

Example 1: Find the general term for each of the following sequences:

a) 2, 6, 18, 54, ...

b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$

c) 4, -8, 16, -32, ⋯

Some key points:

- Example 1b) is an example of a **convergent** sequence. If the common ratio is $|r| < 1$ we have a converging sequence – this means that it has a limit that it 'tends towards' (in this case zero. These types of sequences are incredibly important in analysis.
- Example 1a) and 1c) are **divergent** sequences – they shoot off towards infinity.
- Example 1c) is also an example of an **alternating** sequences – a negative common ratio means that the terms alternate between positive and negative.

Task 1: By first finding the general term, find the 20th term of each of the following sequences:

a) 2, 5, 12.5, ...

b) 80, -40, 20, -10, ...

Task 2: The second and fifth terms of a geometric sequence are -4 and $8\sqrt{2}$.

Find the tenth term in the sequence.

Task 3:

A geometric sequence begins $x, 2, x+3, \dots$

Find the possible values of x and, given that the sequence is alternating, find the 10th term in the sequence.

NOW: *Complete Test Your Understanding 3, Page 24.*

Geometric Series

Example 1: Prove that the sum of the first n terms of a geometric series is given by:

$$S_n = \frac{a(1-r^n)}{1-r}$$

This formula is given to you in the examination; however, again, you can be asked to prove it!

Task 1: Find the sum of the first ten terms of the series 2 + 6 + 18 + 54 + ...

Example 2: Find the sum of the series 256 – 128 + 64 - ... + 1

There's a really useful trick here for when our ratio is negative – we are not able to take logs of a negative number!

Task 2:

Find the least number of terms required for the series 3 + 6 + 12 + 24 + … to exceed one million.

Space for additional notes:

NOW: *Complete Test Your Understanding 4, Page 26, Questions 1-6.*

Converging Geometric Series and the Sum to Infinity

If $|r| < 1$, then our terms are getting smaller and smaller, and no matter how many terms we add, there will be a finite upper limit to the value. This is called a convergent series, and the upper limit is found by

$$S_\infty = \frac{a}{1-r}$$

This is easy to prove from our summation formula $S_n = \frac{a(1-r^n)}{1-r}$.

As $|r| < 1$, $r^n \to 0$ *as* n *increases to infinity!!*

Example 1: Find the sum to infinity of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

History Time!

This series is related to one of the most famous 'ancient' philosophy/maths paradoxes, known as Zeno's paradox.

Zeno considered a race between a tortoise and Achilles, with the tortoise given a 1 metre head start. Achilles could cover twice as much ground as the tortoise, so when Achilles reached the point the tortoise started, the tortoise was now only ½ a metre ahead. Similarly, when Achilles reached this new point, the tortoise would still be ¼ of a metre ahead. Continuing this argument, Zeno claimed, showed that Achilles would never catch the tortoise…

Task 1: Find the sum to infinity of the series $4 - 1 + \frac{1}{4} - \frac{1}{16} + \cdots$.

Task 2: The second term of a geometric series is 1.6 and the fifth term is 0.0128.

Show that the series is convergent, and hence find the sum to infinity of the series.

Space for additional notes:

NOW: *Complete Test Your Understanding 4, Page 27, Questions 7-12.*

Putting it all together:

Modelling with Sequences and Series

Task 1:

Vaishnavi starts a new company selling spicy noodles. She predicts that in year 1 she will make a profit of £30,000, in year 2 she will make a profit of £45,000, in year 3 she will make a profit of £60,000 and so on, such that her profits grow in an arithmetic progression until they reach £270,000. She then expects her profits to remain constant at this value.

a) Calculate Vaishnavi's anticipated profits after 20 years.

b) Her financial advisers, Evans and Browne, instead advise her that it is more likely that her profits will grow by 4% per annum. Calculate her revised anticipated profits by this new model.

Task 2:

Anya decides to save some of her earnings each week. She saves one pound in the first week, two pounds in the second week, three pounds in the third week and so on, such that her savings form an arithmetic series.

a) How much will she have saved after one year?

Her friend Meera instead advises her to save £1 in week one, £1.10 in week two, £1.21 in week three and so on, such that her savings form a geometric series. Meera says she will earn approximately twice as much this way.

b) Show that Meera is **incorrect**.

NOW: *Complete Test Your Understanding 5, Page 28.*

This section is just a brief recap of sigma notation and how it can apply to the topics already learnt!

You have met sigma notation in e.g. statistics as a way of showing summations. This makes it an ideal notation for a cruel examiner wishing to set series summation questions!

For example:

$$\sum_{k=1}^{n}(2k+1) = 3+5+7+9+\cdots+(2n+1)$$

Think about what type of sequence we've just generated – and the types of questions the examiner could now set us!

Example 1: Calculate

$$\sum_{k=1}^{15}(4k-3)$$

Examiner Tip: Always work out the first few terms of the sequence when given a question like this, so that we can see how it 'behaves' and (if possible) can work out what formula we need.

Task 1: Calculate

$$\sum_{k=1}^{10}(2k+5)$$

Examiner Tip: This next example is a really important 'trick' that is loved across examining boards as a way to test understanding. Make sure you understand it. Understand?

Example 2:

a) Calculate

$$\sum_{k=1}^{12} 4 \times 3^{k-1}$$

b) Hence, calculate

$$\sum_{k=7}^{12} 4 \times 3^{k-1}$$

NOW: *Complete Test Your Understanding 6, Page 30.*

Recurrence Relations

So far, we have looked at sequences where we can write a 'position-to-term' rule; that is, we can work out any term by knowing where in the list it is (e.g. for the 5^{th} term we let $n = 5$ etc.).

Another way of defining a sequence is in relation to the previous term(s) – this is how the Fibonacci sequence works:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Each term is the sum of the two previous terms!

A recurrence relation looks something like:

$$u_{n+1} = f(u_n)$$

where u_n is the 'current' term and u_{n+1} is the next one we want to find.

Example 1:

Find the first four terms of the following sequences:

a) $u_{n+1} = 3u_n - 2$, $u_1 = 5$

b) $u_{n+1} = 3u_n - 2$, $u_1 = 2$

Notice that the recurrence relation was the same in both parts, but changing the starting number drastically alters the sequence produced!

There are some key words we should understand to describe the behaviour of sequences.

A sequence such as 1, 3, 6, 10, ⋯ is **increasing.**

A sequence such as 20, 10, 5, 2.5, ⋯ is **decreasing** (and also **convergent,** as it is a geometric sequence with common ratio ½).

A sequence can also form a 'loop', for example 1, 0, 1, 0, 1, 0, ⋯ - this is called a **periodic** sequence. The **order**/**period** of the sequence is the number of terms in each separate repeat (here the order would be 2).

Task 1: Describe the following sequences:

a) $u_{n+1} = 2u_n,\ u_1 = 3$

b) $u_{n+1} = 3 - u_n,\ u_1 = 2$

c) $u_n = \sin(90n°)$

d) $u_{n+1} = u_n - 5,\ u_1 = 3$

The examiner can be quite creative with the recurrence relations questions – but there are still standard tricks to be aware of, especially with periodic sequences.

Example 2:

A sequence is defined by the recurrence relation

$$u_1 = p, \qquad u_{n+1} = u_n + (-1)^n$$

a) Show that $u_3 = u_1$ and that $u_4 = u_2$, and hence state the period of the sequence.

b) Given that $p = 5$, find:

i) u_{200}

ii)

$$\sum_{k=1}^{200} u_k$$

Task 2:

A sequence u_1, u_2, u_3, ... is defined by the recurrence relation

$$u_1 = 20$$
$$u_{n+1} = u_n + 5\cos\left(\frac{n\pi}{2}\right) - 3(-1)^n$$

a) Show that $u_2 = 12$, and find the values of u_3 and u_4.

b) Show further that the sequence is periodic with order 4.

c) Hence, work out

$$\sum_{k=1}^{41} u_k$$

NOW: *Complete Test Your Understanding 7, Page 32.*

THEN: You are ready for the Grade Enhancer™, Page 34.

Test Your Understanding 1

Question 1

For each of the following sequences, the general term u_n.

a) 3, 5, 7, 9, … b) 2, 6, 10, 14, … c) -1, 4, 9, 14, …

d) 32, 25, 18, 11, … e) 8, 5.5, 3, 0.5, … f) $2p, 3p + 2q, 4p + 4q, 5p + 6q, …$

Question 2

For each of the following sequences, find the first three terms, and the values of a and d.

a) $u_n = 3n + 4$ b) $u_n = 5n - 3$ c) $u_n = 4 - 3n$

Question 3

Calculate how many terms are in the following sequences:

a) 8, 11, 14, …, 29 b) 44, 42, 40, …, 4 c) 3, 8, 13, …, 228

Question 4

The second term in an arithmetic sequence is 12 and the fifth term is 21. Find the values of a and d, and hence find the general term u_n.

Question 5

The tenth term in an arithmetic sequence is 48 and the thirtieth term is 18. Find the 100th term of the sequence.

Question 6

A sequence has nth term $u_n = pn + q$. Given that u_8 is twice the value of u_{10}, and that the first term is 33, find the values of p and q.

Question 7

An arithmetic sequence starts $8x, 15, 2x, …$. Find the value of the 10th term.

Challenge

A sequence has first term $p^2, p > 0$, and common difference p. The fourth term is 28. Find p.

Test Your Understanding 2

Question 1

Find the sums of the first ten terms in the following series:

a) $4 + 7 + 10 + 13 + \ldots$ b) $25 + 29 + 33 + 37 + \ldots$ c) $22 + 17 + 12 + 7 + \ldots$

Question 2

Find the sums of the following series:

a) $3 + 5 + 7 + \ldots + 31$ b) $12 + 22 + 32 + \ldots + 222$ c) $41 + 35 + 29 + \ldots - 13$

Question 3

Find how many terms of the sequence are in required for the following sums:

a) $3 + 7 + 11 + \cdots = 136$ b) $2 + 3.5 + 5 + \cdots = 325$ c) $47 + 43 + 39 + \cdots = -1240$

Question 4

Find the sum of the first 100 even numbers.

Question 5

Find the sum of the first 20 multiples of 12.

Question 6

The first term of an arithmetic sequence is 5, and the sum of the first 12 terms is 456. Find the 20^{th} term of the sequence.

Question 7

An arithmetic sequence is such that $u_4 = 17$, $u_8 = 29$ and $S_n = 294$.

a) Show that $3n^2 + 13n - 588 = 0$.

b) Hence find the value of n.

Question 8

Find how many terms are needed for the sum 3 + 9 + 15 + 21 + … to exceed 1000.

Challenge Question 1

Show that the sum of the first n natural numbers is given by $\frac{1}{2}n(n+1)$

Challenge Question 2 – Beyond the Boundaries™

a) An arithmetic sequence has first term $\ln 2$ and third term $\ln 8$. Show that $a = d$.

b) Another arithmetic sequence has first term $\ln 3$ and common difference $p \ln 3$.

i) Show that the general term is given by $u_n = \ln(3^{p(n-1)+1})$

ii) Given that the tenth term is $\ln(3^{37})$, find the value of p.

iii) Hence, find the sum of the first ten terms of the sequence, giving your answer in the form $\ln(3^q)$ where q is a positive integer to be determined.

Test Your Understanding 3

Question 1

For each of the following geometric sequences, find the general term u_n:

a) 5, 10, 20, … b) 2, -4, 8, … c) 2, 1, 0.5, …

Question 2

Which of the following are geometric sequences? If they are, write down the values of a and r.

a) 3, 6, 12, 24, … b) 4, -2, -1, -0.5, … c) $2x, x^2, 0.5x^3, 0.25x^4$, …

Question 3

The first three terms of a geometric sequence are 4, x and 8. Given that all terms of the sequence are positive, find:

a) The value of x b) The common ratio c) The value of u_{10}.

Question 4

A geometric sequence has n[th] term formula $u_n = 3 \times 2^n$. Find the first and seventh terms of the sequence.

Question 5

A geometric sequence is such that $u_3 = 9$ and $u_7 = 81$. Given that all terms in the sequence are positive, find the values of the first term and the common ratio.

Question 6

The first three terms of a geometric sequence are 3, $x - 2$ and 12. Given that the sequence alternates between positive and negative terms, find the value of x, the common ratio and the 10[th] term of the sequence.

Question 7

A geometric sequence has first term 4 and common ratio 3. Find the first term that exceeds one million.

Question 8

A geometric sequence starts 5, 15, 45, …

Determine whether 35,375 is a term in the sequence.

Challenge Question

The first, second and fifth terms of an arithmetic sequence are the first three terms of a geometric sequence. Find the value of the common ratio r of the geometric sequence.

Test Your Understanding 4

Question 1

a) Find the sum of the first 10 terms of the geometric series $2 + 4 + 8 + 16 + ...$

b) Find the sum of the first 8 terms of the geometric series $4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + ...$

c) Find the sum of the first 15 terms of the geometric series $100 + 10 + 0.1 + 0.01 + ...$

Question 2

Find the sum of the geometric series $512 + 256 + 128 + ... + 1$.

Question 3

A geometric series has first term 4 and common ratio $\frac{3}{4}$. Find the value of S_{20}, giving your answer to 4 significant figures.

Question 4

Find the least number of terms required for the sequence 1 + 3 + 9 + 27 + ... to exceed a million.

Question 5

A geometric sequence has first term 20 and fifth term 80. Given that all of the terms are positive, find the sum of the first five terms, giving your answer in the form $p + q\sqrt{2}$.

Question 6

A geometric series contains only positive terms, and has first term a and common ratio r. The sum of the first two terms is 1.4, whilst the sum of the first four terms is 1.624. Find the values of a and r.

Challenge 1

The first three terms of a geometric series are $x - 4, 5$ and $x + 20$. Given that the series is convergent, find the value of x and the sum of the first 20 terms.

Question 7

For each of the following series, determine (with a reason) whether the series is geometric and convergent and, if so, find the sum to infinity.

a) $2 + 1 + 0.5 + 0.25 + \cdots$

b) $2 - 1 + 0.5 - 0.25 + \cdots$

c) $1 + 2 + 4 + 8 + \cdots$

d) $20 + 10 + 5 + 2.5 + \cdots$

e) $1 + 1.5 + 2.25 + 3.375 + \cdots$

f) $3 + 2 + 1 + 0 + \cdots$

Question 8

A geometric series has first term 10 and common ratio 0.4. Find the sum to infinity.

Question 9

A geometric series has first term 5 and sum to infinity 9. Find the value of r and the 5th term.

Question 10

A geometric series is such that $u_3 = 1.25$ and $S_\infty = \frac{80}{3}$. Find the values of a and r, given that $r > 0$ and a is an integer.

Question 11

A geometric sequence has first term 30 and sum to infinity 240. Find the smallest number of terms required for the sum to exceed 210.

Challenge Question 2

Use the sum to infinity to prove that $0.\dot{1}\dot{7} = \frac{17}{99}$

Challenge Question 3

A geometric series has a common ratio -0.2 and $S_\infty = 1000$. Find S_5.

Test Your Understanding 5

Question 1

A company predicts annual profits of £25000 for 2024, and predicts that the profits will increase by 10% per annum.

a) Explain why the annual profits predicted by this model form a geometric sequence.

b) Find the total predicted profit predicted by the model for 2024 to 2034 inclusive, giving your answer to 3 significant figures.

c) Explain why this model may not be reliable.

Question 2

Louis saves some money each week over a period of one year. He saves 12p in the first week, 18p in the second week, 24p in the third week and so on, such that his savings form an arithmetic series.

a) Calculate how much Louis saves in one year.

Louis' friend Max also saves some money each week. He saves 1p in week 1, 2p in week 2, 4p in week 3 and so on.

b) Find how many weeks until Max's savings exceed Louis' amount for the year.

Question 3

A company has an incentive scheme to reduce sick days. An employee earning £a per week will earn an extra £d each week they have no sick days.

An employee with no sick days would earn £1,010 in the final week of the year.

An employee with no sick days would earn £39,260 in total in the year.

a) Use the information above to form two equations in a and d.

b) Hence, find the values of a and d.

Question 4

A house-building programme launched by the government of Evansland promises to build new houses from 2025, with 10000 being built in the first year, and an increase of 10% in the number built each year.

a) Calculate the number of houses built in 2034.

b) Calculate the total number of houses built by 2034 by the Evansland model, to the nearest house.

The smaller, altogether-less-pleasant country of Jonesville also launches a house-building programme in 2025. They promise to build 10000 houses in the first year, and an increase of 1000 houses each year such that the number built form an arithmetic sequence.

c) Calculate the number of houses built in Jonesville and thus the difference in the number of houses built by the two models.

Question 5

Maddie makes and sells TayTay posters, charging increasing amounts per sale. Her profit for the first ten posters follows a geometric sequence. She makes £0.80 profit on the fourth poster sold, and £51.20 on the tenth poster sold.

a) Calculate her profit in total for the first ten posters.

Once she has sold ten posters, she sells her remaining stock of 90 posters at a fixed profit of £2.50.

b) Calculate her average profit per poster across the 100 posters.

Challenge Question

In 2024 Abi gave a gift of £60 to a charity. Each year after, Abi increased the gift given by £15. After n years she had given £3375.

Calculate the year in which her total donations reached £3375, and the amount given in this year.

Test Your Understanding 6

Question 1

For each of the following series,

i) identify whether it is arithmetic, geometric or neither;

ii) find the value of the sum.

a) $$\sum_{k=1}^{5}(2k+3)$$

b) $$\sum_{k=1}^{5}(k^2)$$

c) $$\sum_{k=1}^{5}3\times 2^k$$

d) $$\sum_{k=2}^{5}\sin(90k°)$$

e) $$\sum_{k=0}^{3}3\times k^3$$

f) $$\sum_{k=3}^{10}12-3k$$

Question 2

A series is given by 1 + 5 + 9 + 13 + … + 77.

a) Find how many terms are in the series.

b) Find the sum of the series.

c) Write this series using sigma notation.

Question 3

Evaluate the following summations:

a) $$\sum_{k=1}^{10}(3k+2)$$

b) $$\sum_{k=1}^{15}37-2k$$

c) $$\sum_{k=1}^{12}2\times 0.5^k$$

d) $$\sum_{k=3}^{10}(2\mathrm{k}+13)$$

e) $$\sum_{k=11}^{50}(5k-1)$$

f) $$\sum_{k=3}^{10}800\left(\frac{1}{4}\right)^k$$

Question 4

Evaluate

$$\sum_{k=1}^{\infty} 5\left(\frac{1}{2}\right)^{k}$$

Question 5

You are given that

$$\sum_{k=1}^{n} (3k+2) = 1648$$

a) Show that $3n^2 + 7n - 3296 = 0$

b) Hence find the value of n.

Question 6

Given that

$$\sum_{k=0}^{n} 3 \times 2^{k} = 6141$$

find the value of n.

Challenge Question

A convergent geometric series is given by $2 + 4x + 8x^2 + \cdots$.

Given that

$$\sum_{k=1}^{\infty} (2x)^{k-1} = \frac{8}{3}$$

find the value of x.

Test Your Understanding 7

Question 1

Generate u_2, u_3 and u_4 in each of the following recurrence relations:

a) $u_{n+1} = u_n - 2,\ u_1 = 5$

b) $u_{n+1} = 3u_n + 1,\ u_1 = -2$

c) $u_{n+1} = \frac{1}{2}u_n + 5,\ u_1 = 10$

d) $u_{n+1} = (u_n)^2 - 1,\ u_1 = 1$

e) $u_{n+1} = \frac{2}{u_n},\ u_1 = 4$

f) $u_{n+1} = \sqrt{u_n},\ u_1 = 256$

Question 2

A recurrence relation is defined by $u_{n+1} = ku_n - 2,\ n \geq 1,$ where $u_1 = 3$.

a) Find expressions for u_2 and u_3.

b) Given that $u_3 = 63$, find the possible values of k.

c) Given that k is not an integer, find the value of u_2.

Question 3

A recurrence relation is defined by $u_{n+1} = au_n + b,\ n \geq 1,$ where $u_1 = 4$.

Given that $u_2 = 17$ and $u_3 = 82$, find the values of a and b.

Question 4

A sequence is defined by:

$$u_1 = 5$$

$$u_{n+1} = 7 - u_n$$

for $n \geq 1.$

a) Show that the sequence is periodic, stating the order.

b) Find

$$\sum_{k=1}^{101} u_n$$

Question 5

A recurrence relation is defined as:

$$u_1 = k$$

$$u_{n+1} = 3u_n - 5$$

Find

$$\sum_{k=1}^{4} u_n$$

and hence show that this sum is divisible by 5.

Question 6

A recurrence relation is defined by

$$u_1 = 1$$

$$u_{n+1} = \sqrt{(u_n)^2 + 5}$$

Find the values of u_2 and u_3 as surds, and show that u_4 gives an integer value.

Challenge Question

A sequence $u_1, u_2, u_3, \ldots$ is defined by the recurrence relation

$$u_1 \quad = 10$$

$$u_{n+1} = u_1 + 5\cos\left(\frac{n\pi}{4}\right) - 3(-1)^n$$

a) Show that the sequence is periodic with order 8.

c) Hence, work out

$$\sum_{k=1}^{16} u_k$$

Grade Enhancer™ - Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 (*WJEC 2014*)

(a) A geometric series has first term a and common ratio r. Prove that the sum of the first n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r}.$$ [3]

(b) The fourth term of a geometric series is -108 and the seventh term is 4.

(i) Find the common ratio of the series.

(ii) Find the sum to infinity of the series. [6]

Question 2 (*WJEC 2014*)

(a) An arithmetic series has first term a and common difference d.
Prove that the sum of the first n terms of the series is given by

$$S_n = \tfrac{n}{2}[2a + (n-1)d].$$ [3]

(b) The first term of an arithmetic series is 3 and the common difference is 2. The sum of the first n terms of the series is 360.
Write down an equation satisfied by n. Hence find the value of n. [3]

(c) The tenth term of another arithmetic series is seven times the third term. The sum of the eighth and ninth terms of the series is 80. Find the first term and common difference of this arithmetic series. [4]

Question 3 (*WJEC 2022*)

The sum to infinity of a geometric series with first term a and common ratio r is 120.
The sum to infinity of another geometric series with first term a and common ratio $4r^2$ is $112\frac{1}{2}$. Find the possible values of r and the corresponding values of a. [6]

Question 4 (*WJEC 2022*)

Geraint opens a savings account. He deposits £10 in the first month. In each subsequent month, the amount he deposits is 20 pence greater than the amount he deposited in the previous month.

a) Find the amount that Geraint deposits into the savings account in the 12th month. [2]

b) Determine the number of months it takes for the total amount in the savings account to reach £954. [3]

Question 5 *(WJEC 2023)*

The 12th term of an arithmetic series is 41 and the sum of the first 16 terms is 488. Find the first term and the common difference of the series. [5]

Question 6 *(WJEC 2023)*

A tree is 80 cm in height when it is planted. In the first year, the tree grows in height by 32 cm. In each subsequent year, the tree grows in height by 90% of the growth of the previous year.

a) Find the height of the tree 10 years after it was planted. [4]

b) Determine the maximum height of the tree. [2]

Question 7 *(WJEC 2018)*

Find seven numbers which are in arithmetic progression such that the last term is 71 and the sum of all of the numbers is 329. [5]

Question 8 *(WJEC 2018)*

a) Explain why the sum to infinity of a geometric series with common ratio r only converges when $|r| < 1$. [1]

b) A geometric progression V has first term 2 and common ratio r. Another progression W is formed by squaring each term in V. Show that W is also a geometric progression. Given that the sum to infinity of W is 3 times the sum to infinity of V, find the value of r. [6]

c) At the beginning of each year, a man invests £5000 in a savings account earning compound interest at the rate of 3% per annum. The interest is added at the end of each year. Find the total amount of his savings at the end of the 20th year correct to the nearest pound. [3]

Question 9 *(WJEC 2014)*

The nth term of a number sequence is denoted by t_n. The $(n + 1)$th term of the sequence satisfies

$$t_{n+1} = 1 - \frac{1}{t_n},$$

for all positive integers n. Given that $t_1 = 4$,

(a) evaluate t_2, t_3, and t_4, [2]

(b) describe the behaviour of the sequence and hence, without carrying out any further calculation, write down the value of t_{50}. [2]

Total Mark Available is 60.