MathEVmatics

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A2 Mathematics for WJEC

Unit 3: Radian Measure

Examples and Practice Exercises

Unit Learning Objectives

- ***To understand what is meant by a radian;***
- ***To be able to convert between radian and degree measure, and sketch the trigonometric graphs in terms of radians;***
- ***To be able to solve trigonometric equations in radians;***
- ***To understand the formulae for arc length and sector area;***
- ***To know and use the small angle approximations.***

Prerequisite learning:

Trigonometric Graphs and Equations (AS Mathematics)

Sine/Cosine Rule (AS Mathematics)

Basic Skill Check:

1) Sketch the graph of $y = \cos x$ for $0 \leq x \leq 360°$.

2) Solve the equation $\sin x = 0.5$ for $0 \leq x \leq 360°$.

3) Solve the equation $3tan^2x - tanx - 2 = 0$ for $0 \leq x \leq 360°$.

When you have completed the unit…

Objective	Met	Know	Mastered
Understand what a radian is;			
Convert between degrees and radians; draw the trigonometric graphs in terms of radians;			
Find arc lengths and sector areas;			
Solve trigonometric equations involving radians;			
Use the small angle approximations.			

Notes/Areas to Develop:

Radian Measure

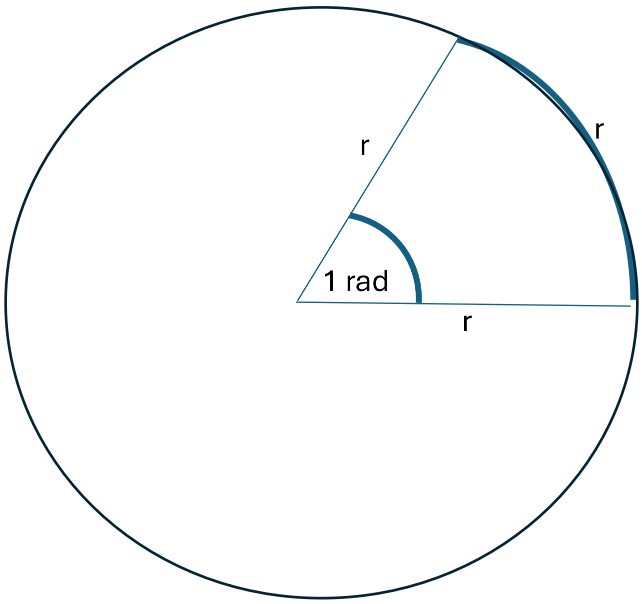
Until this point, we have always thought about angles in terms of degrees.

> *The concept of division of a circle into 360° is ancient – it is generally believed that the Babylonians invented this concept more than 2500 years ago, but it may be even older than this! The Babylonians used a sexagesimal (base-60) system - 60 has many more factors than anything smaller (the technical term for this is that it is highly composite) and thus sub-dividing 60 is convenient. This is also why we use our standard measure of time (60 seconds in a minute, 60 minutes in an hour)!*

However, this is not the only way in which we can measure angles. You will have, no doubt, noticed that your calculator has two other modes: **radians** and *gradians*.

> Gradians (often called grades or grads) were developed in France when the metric system was being developed – it was an attempt to 'metric-ise' angles, where a right angle is 100 grads and thus a full circle is 400 grads. These are used infrequently, most notably by surveyors in some European countries. You will probably never meet them in real life!

Radian measure (developed by Roger Cotes, an English mathematician who worked closely with Newton, though he didn't invent the name) is, in many ways, a more natural angular measure than a degree.



A **radian** is defined as the angle formed by two radii such that the length of the arc subtended is equal to r.

This means, since the circumference of the circle is $2\pi r$, that there are 2π radians in a full circle.

$$2\pi \, radians = 360°$$

Thus:

$$\pi \, radians = 180°$$

$$\text{Degrees} \xrightarrow{\times \frac{\pi}{180}} \text{Radians}$$

$$\text{Radians} \xrightarrow{\times \frac{180}{\pi}} \text{Degrees}$$

(The way I tend to remember this is that, to get 'into' radians I need to introduce pi, whereas to get back to degrees I need to get rid of pi.)

Task 1: Convert the following into radians:

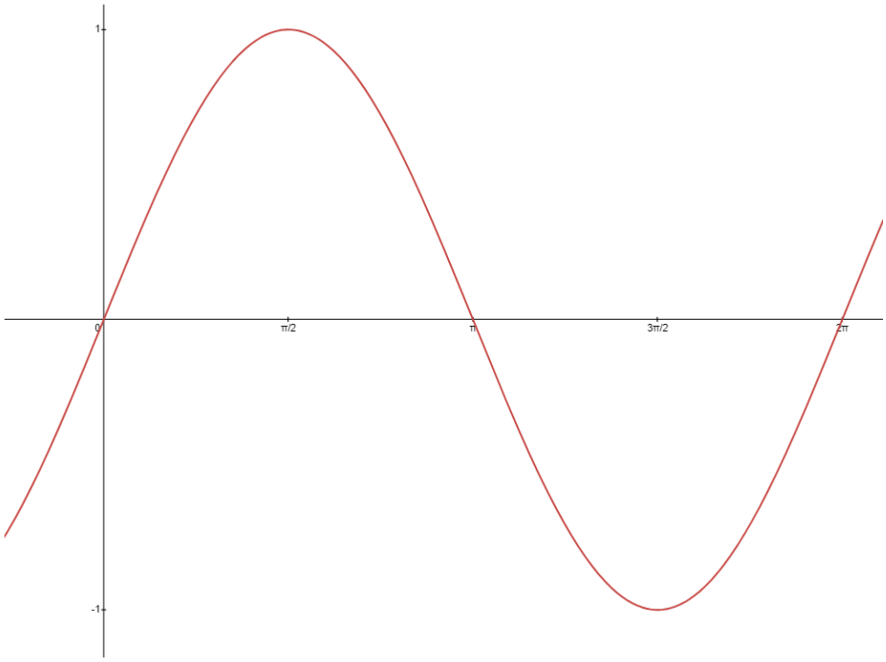
a) 90° b) 270° c) 45° d) 30° e) 10° f) 120°

Task 2: Write the following in degrees (to 1 d.p. where necessary):

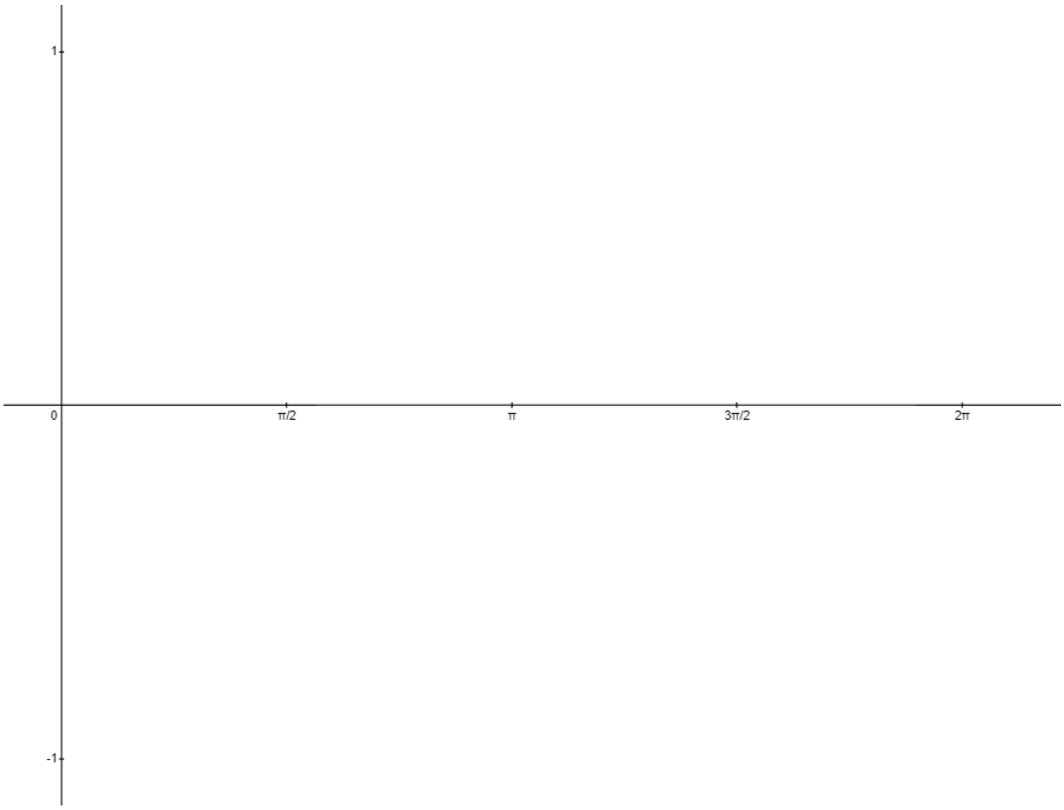
a) $\frac{\pi}{3}$ b) $\frac{5\pi}{6}$ c) $\frac{7\pi}{18}$ d) 1 e) 0.3

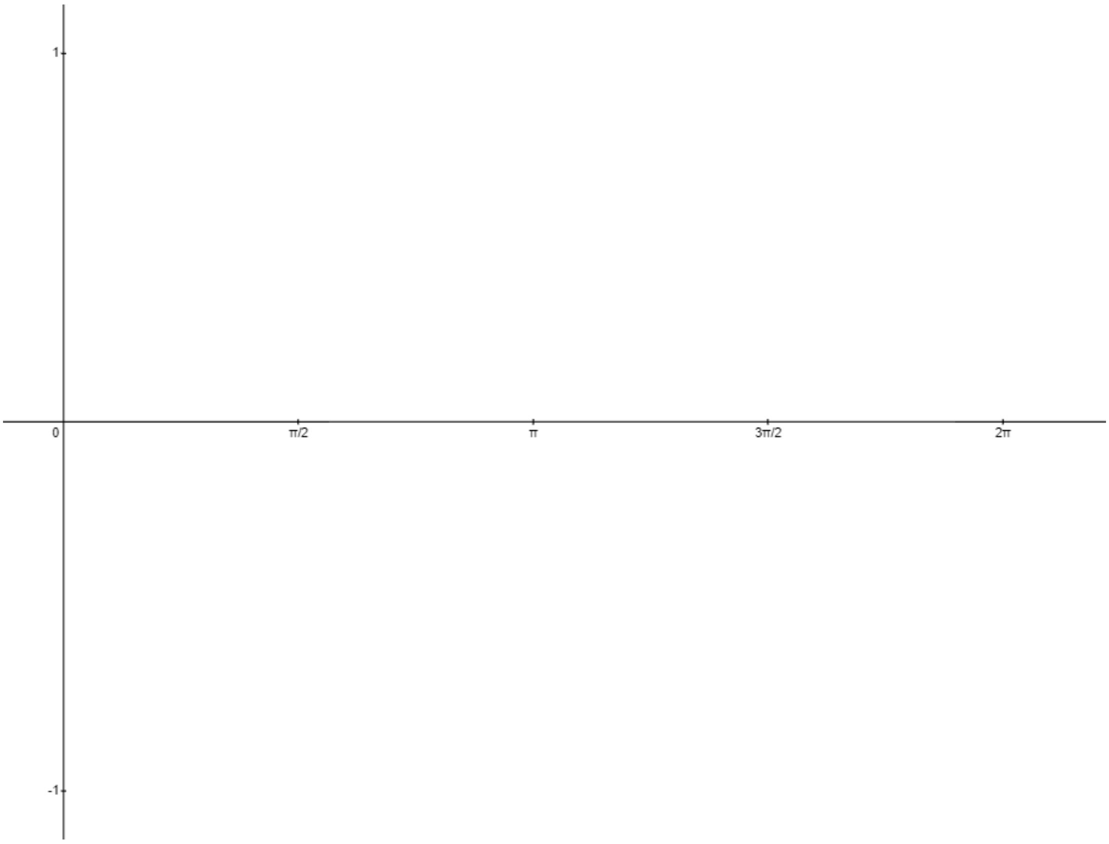
This means we can now redraw our trigonometric graphs from GCSE in terms of radian measure.

For example, the following is the graphs of $y = \sin x$ for $0 \leq x \leq 2\pi$.



Task 3: Sketch the graphs of $y = cos\ x$ and $y = tan\ x$ for $0 \leq x \leq 2\pi$.





It is really important that we can think and sketch in terms of radians, as we will now be expected to solve equations in radians.

Examiner Tip: *The examiner will not usually state 'radians' or 'degrees', but they will give you a clue with the defined domain in which you are asked to solve.*

Example 1:

Solve $cos\theta = \frac{1}{2}$ for $0 \leq \theta \leq 2\pi$.

Example 2:

Solve $\tan 2\theta = 1$ for $0 \leq \theta \leq 2\pi$.

Space for additional notes/graphs:

Example 3:

Solve $7 + cos\theta = 10sin^2\theta$ for $0 \leq \theta \leq 2\pi$.

Space for additional notes/graphs:

Now: Complete TEST YOUR UNDERSTANDING 1, Page 19.

Arcs and Sectors

At GCSE we considered arcs and sectors, using the idea that they are fractions of a circle.

We can use radian measure to derive and simplify more efficient formulae to work with arcs and sectors.

Remember that circumference is given by $C = 2\pi r$, and that a circle has a total angle of 360. Thus, the length l of an arc would be given by

$$l = \frac{\theta}{360} \times 2\pi r,$$

where θ is the central angle.

However, since we now know that 360° is equal to 2π radians,

$$l = \frac{\theta}{2\pi} \times 2\pi r$$

$$\therefore l = r\theta$$

This is, as Borat would say, very nice.

Space for additional notes:

In the same way, we can derive a formula for the area of a sector.

Starting with

$$A = \frac{\theta}{360} \times \pi r^2$$

for the area of a sector with central angle θ and radius r, we can again replace 360° with 2π. Thus,

$$A = \frac{\theta}{2\pi} \times \pi r^2,$$

which leads to our formula,

$$A = \frac{1}{2} r^2 \theta.$$

KEY POINT

These formulae are not given, so you need to learn them! (Or, better yet, understand them so that you can derive them!)

$$l = r\theta$$

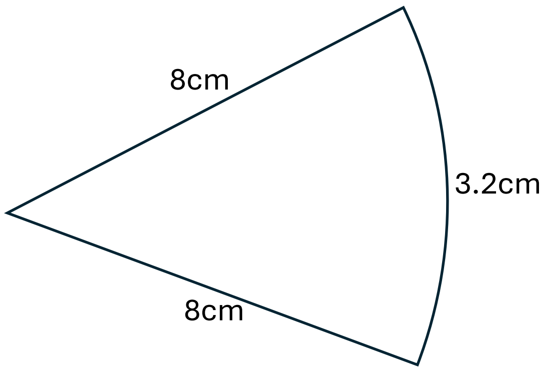
$$A = \frac{1}{2} r^2 \theta$$

Example 1:

Calculate the arc length and area of a sector of radius 2.8 cm with central angle 0.6 radians.

Example 2:

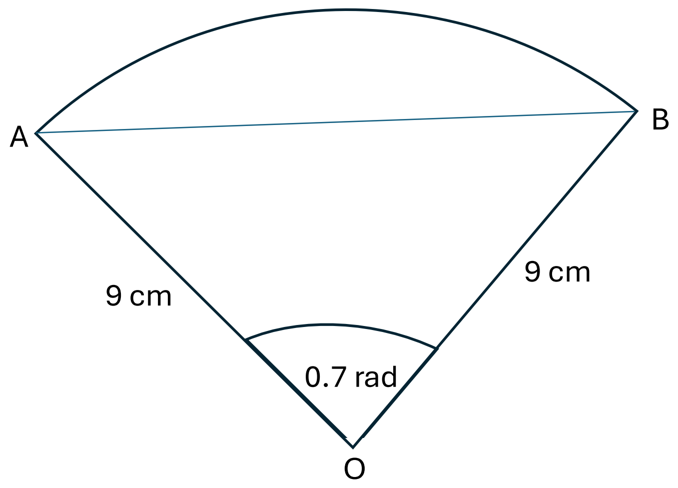
Find the area of the following sector.



Example 3:

Find an expression for the perimeter P of a sector with angle θ and radius r. Hence write r in terms of P and θ.

Task 1:



Calculate the area bounded by the chord AB and the arc AB.

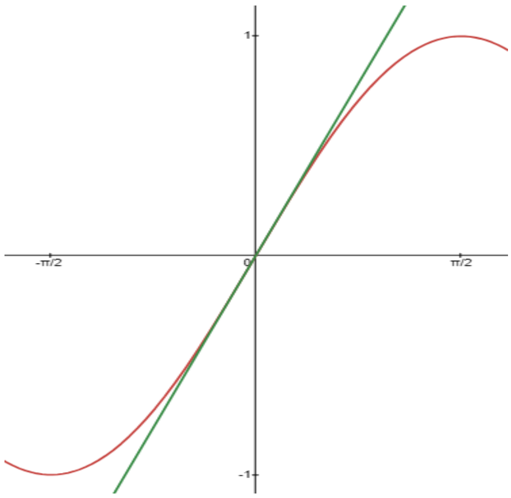
Now: Complete TEST YOUR UNDERSTANDING 2, Page 21.

Small Angle Approximations

Another important use of radians is in finding **approximate** values for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

These approximations turn out to be incredibly important for finding the *derivatives* of the trigonometric functions – we'll meet this later!

Consider the following graph of $y = \sin x$ and $y = x$.

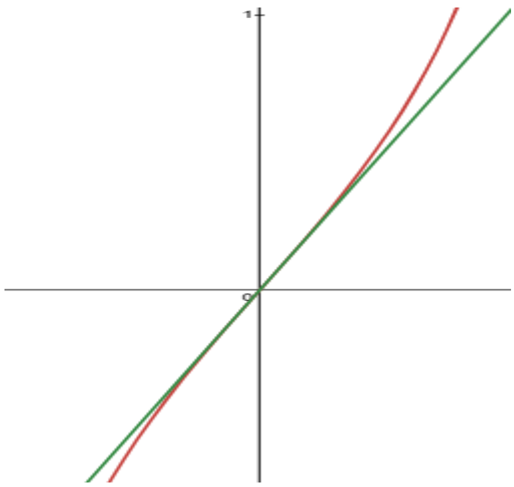


Can you see that, the closer x is to zero, the closer the line is to the curve – indeed, when x is small enough, the two graphs essentially look the same!!

This means, that for suitably small values of θ,

$$\sin\theta \approx \theta.$$

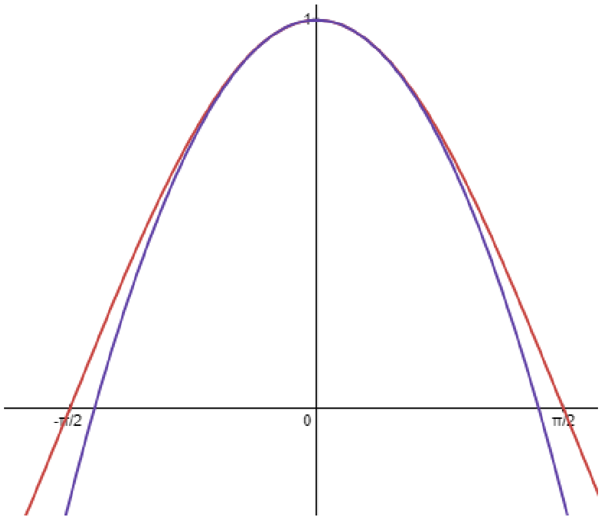
Similarly, for $y = \tan x$ and $y = x$.



Thus, we can see that

$$\tan\theta \approx \theta.$$

Finally, consider the following:



This time, we are comparing $y = \cos x$ with $y = 1 - \frac{x^2}{2}$ (as we can see, the cosine graph looks quadratic-ish in the section drawn), and once again these are very close as x gets closer to zero. Hence,

$$\cos\theta \approx 1 - \frac{\theta^2}{2}.$$

These three small angle approximations are given in the formula booklet, but it is still useful to commit them to memory if you can!!

KEY POINT

The small angle approximations (SAAs) for sine, cosine and tangent are:

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

Example 1:

When θ is small, find approximate expressions for the following:

a) $\theta \tan 3\theta$ b) $1 - \cos\theta$ c) $\frac{4sin2\theta}{8}$

Space for any additional notes:

Example 2:

When θ is small, find the approximate value of $\frac{1-\cos 2\theta}{\theta \sin 2\theta}$

Space for any additional notes:

Task 1:

Given that θ is sufficiently small, find approximate expressions for $1 - \cos 4\theta$ and $tan^2 2\theta$. Hence, find

$$\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{tan^2 2\theta}$$

Space for any additional notes:

Task 2:

Show that, when θ is sufficiently small, the equation $1 - \cos 4\theta = \frac{17}{2}(\sin 2\theta - \frac{4}{17})$ can be written as

$$8\theta^2 - 17\theta + 2 = 0.$$

Hence find the solutions of this equation and comment on their validity.

Now: Complete TEST YOUR UNDERSTANDING 3, Page 25.

Test Your Understanding 1

Question 1

Write the following angles in radians:

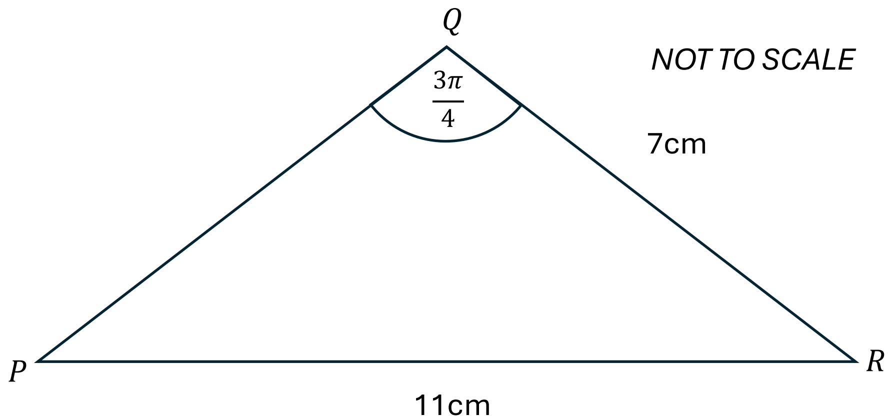
a) 20° b) 150° c) 18° d) 2° e) 300°

Question 2

Find the following values:

a) $\sin\frac{\pi}{4}$ b) $\cos\pi$ c) $\tan\frac{7\pi}{6}$ d) $\cos\frac{2\pi}{3}$ e) $\sin\frac{3\pi}{2}$

Question 3



a) Find the size of angle $P\hat{R}Q$ in radians.

b) Find the length of PQ

c) Find the area of triangle PQR

Question 4

Solve each of the following equations in the interval $0 \leq \theta \leq 2\pi$.

a) $\sin\theta = 0.7$ b) $3\cos\theta = 1$ c) $10\sin\theta = -3$

d) $3\tan\theta - 2 = 5$ e) $\cos 3\theta = 0.5$ d) $\tan\left(\theta - \frac{\pi}{5}\right) = 1$

Question 5

Solve $5cos^2\theta - 2cos\theta - 3 = 0$ in the interval $-\pi \leq \theta \leq \pi$.

Question 6

Solve $3tan^2\theta = \tan\theta$ for $0 \leq \theta \leq 2\pi$.

Question 7

Solve $5\sin\theta = 3\tan\theta$ for $0 \leq \theta \leq 2\pi$.

Question 8

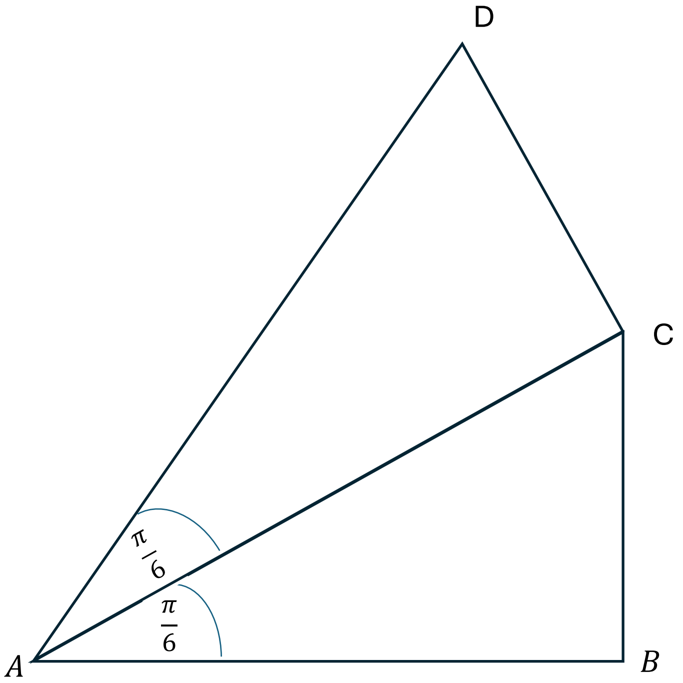
(a) Show that the equation $cos^2\beta - 4\cos\beta - 2 = 0$ can be written in the form

$$\cos\beta = m \pm \sqrt{n} \text{ where } m, n \in \mathbb{Z}$$

(b) Hence solve $cos^2 2\beta - 4\cos 2\beta - 2 = 0$ in the interval $0 \leq \beta \leq 2\pi$.

CAREFUL!! EXAMINERS ARE SNEAKY PEOPLE...

Question 9 (CHALLENGE QUESTION)



ABC and ACD are right angled triangles.

a) Given that AB = 2cm, find the lengths of CD and AD.

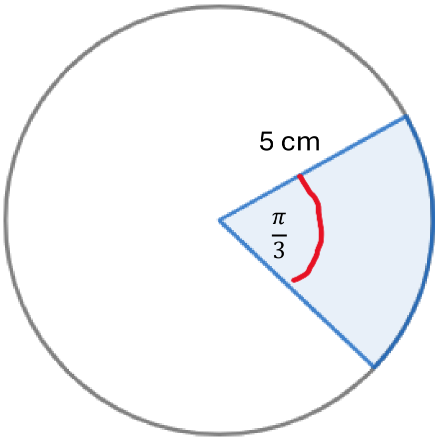
*b) Instead, given that AB = a cm, show that $AD = 2CD$

Test Your Understanding 2

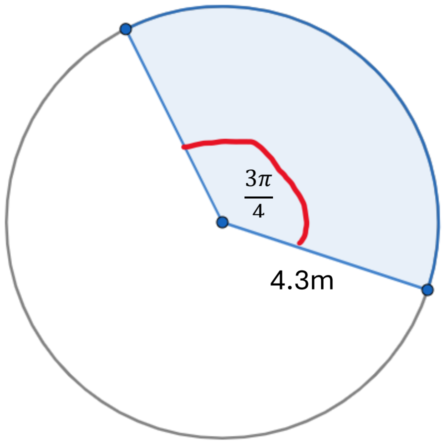
Question 1

For each diagram, find (i) the arc length and (ii) the area for the shaded region.

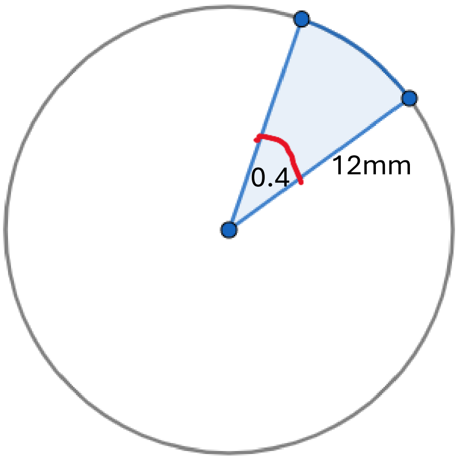
a)



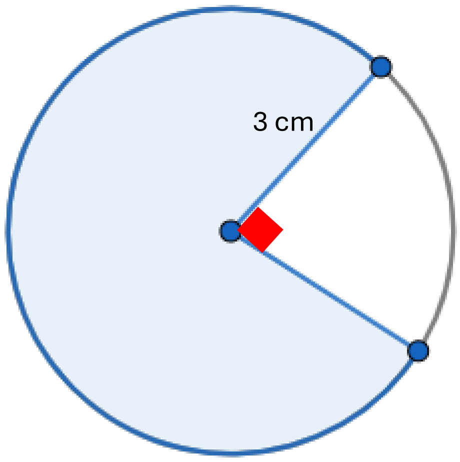
b)



c)

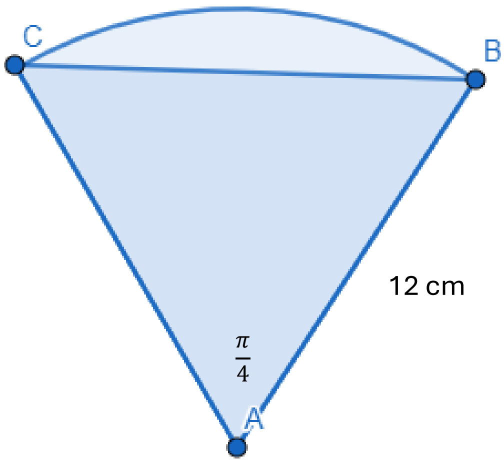


d)



Question 2

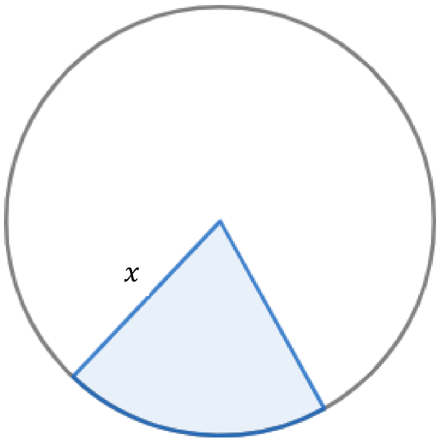
Find the area of the segment bounded by the chord BC and arc BC.



Question 3

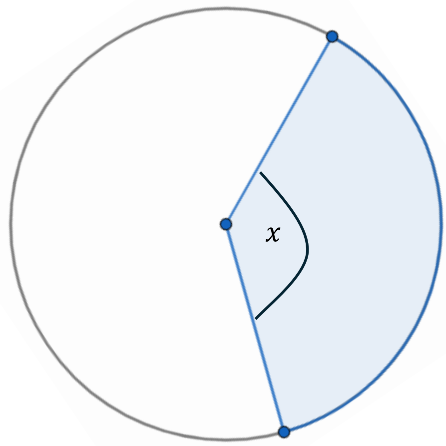
For each of the following, find the value of x.

a)



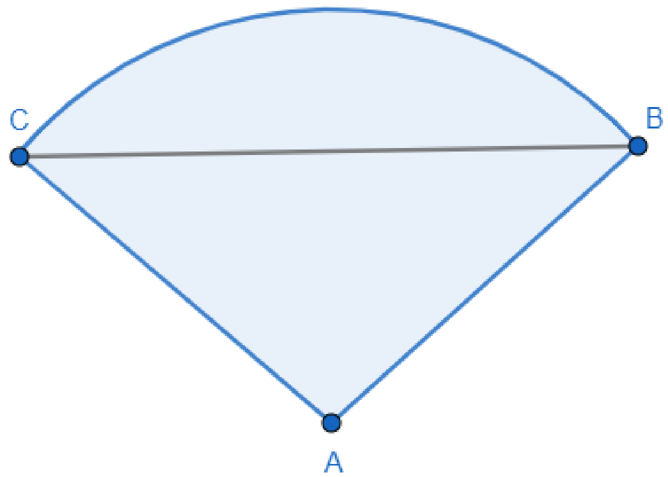
Angle at centre: 0.8 radians

b)



Area of sector: 15cm^2

Question 4



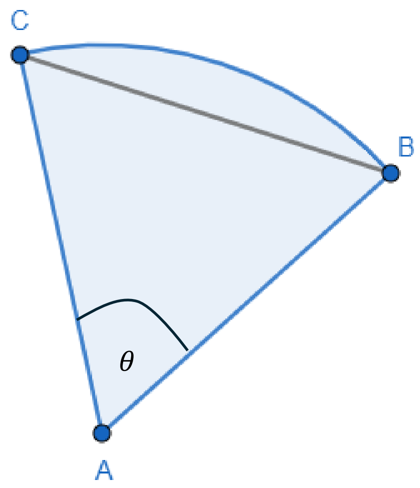
In the diagram above, $AB = AC = 8cm$ and the exact perimeter of the sector ABC is $(16 + 6\pi)\ cm$.

Find:

a) the angle $B\hat{A}C$

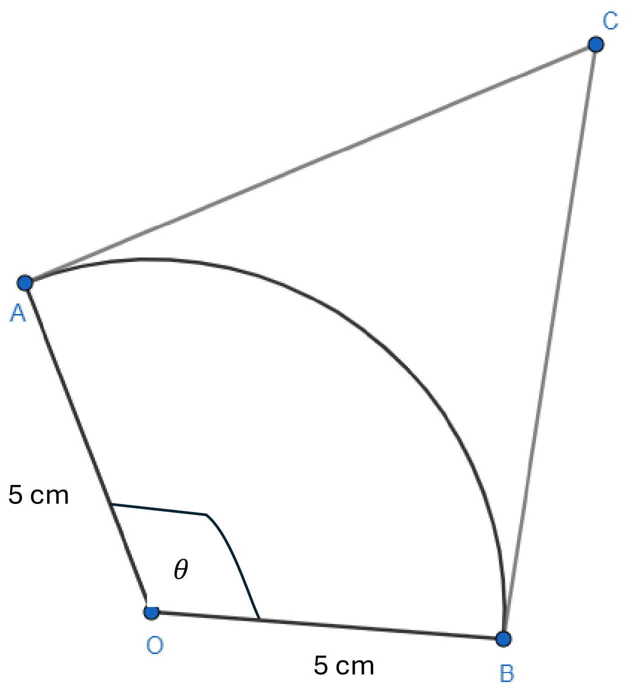
b) the length BC.

Question 5



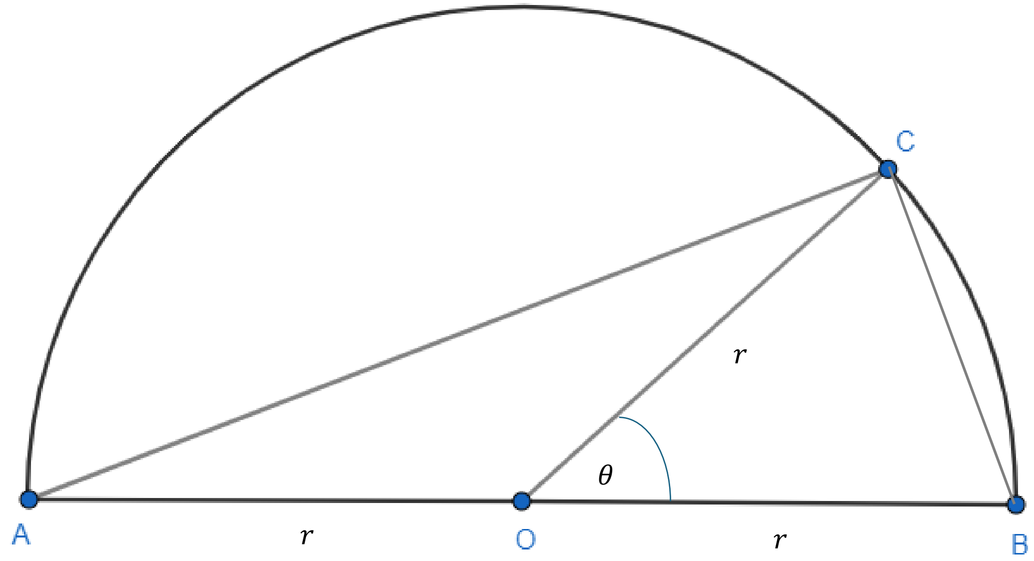
The above diagram shows a sector of a circle. Given that the radius $AB = AC = 10$ cm and the length of the chord BC is 18.65cm, find the angle θ and hence the length of the arc BC.

Question 6



AOB is a sector of a circle. AC and BC are tangents to the circle at A and B respectively. Given that the angle θ is 2 radians, find the area of the region ABC.

Question 7 (CHALLENGE QUESTION)



NOT DRAWN TO SCALE

In the above circle, the area of triangle AOC is eight times the area of the segment bounded by the arc BC and the chord BC.

Show that $8\theta - 9\sin\theta = 0$.

HINT 1: Start off writing expressions for anything you can!

HINT 2: Remember that $sin(180 - \theta) = sin\,\theta$

Test Your Understanding 3

Question 1

Use the small angle approximations to find expressions for each of the following:

a) $\frac{1}{2}\sin\theta$ b) $\theta\tan 2\theta$ c) $\frac{1}{\cos 3\theta}$

Question 2

Given that θ is in radians and suitably small, show that

$$1 + \cos\theta - 3cos^2\theta \approx 2.5\theta^2 - 1$$

Question 3

Given that θ is in radians and suitably small that terms in θ^3 can be ignored, estimate the solution(s) of the equation

$$\frac{\cos(\sqrt{6}\theta)}{1 - \sin\theta} = 0.5$$

Question 4

A curve C has equation $y = f(x)$ such that $f'(x) = \frac{1}{2}(2x + cosx)$.

The curve has a stationary point at $x = \alpha$ where α is small and in radians.

Use the small angle approximations to find an estimate of the value of α to 4 decimal places.

Grade Enhancer – Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1

(a) Given that θ is in radians and small enough that terms in θ^3 or higher powers can be ignored, show that

$$4\cos\theta + cos^2 2\theta \approx 5 - 6\theta^2$$

(b) Write 5° in radians in the form $\frac{\pi}{n}$ where n is an integer.

(c) Hence, find an approximation for the value of $4\cos\theta + cos^2 2\theta$ when $\theta = 5°$.

[5 marks]

Question 2 *(WJEC 2018)*

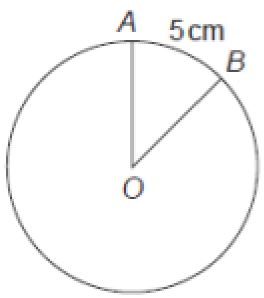
Use small angle approximations to find the small negative root of the equation

$$\sin x + \cos x = 0.5$$

[3 marks]

Question 3 (WJEC 2018)

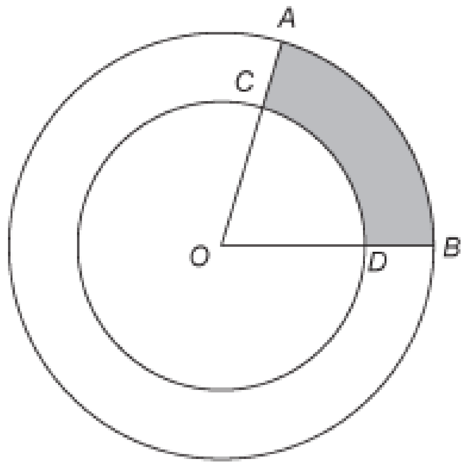
The diagram below shows a circle centre O, radius 4 cm. Points A and B lie on the circumference such that arc AB is 5 cm.



a) Calculate the angle subtended at O by the arc AB. [2]

b) Determine the area of the sector OAB. [2]

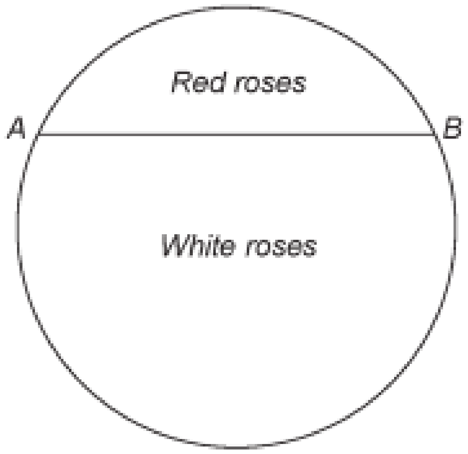
Question 4 *(WJEC 2014)*



The diagram shows two concentric circles with a common centre O. The radius of the larger circle is 7 cm and the radius of the smaller circle is 4 cm. The points A and B lie on the larger circle and OA and OB cut the smaller circle at the points C and D respectively. The area of the shaded region $ACDB$ is $23{\cdot}1\,\text{cm}^2$. Find the perimeter of $ACDB$. [6]

Question 5 *(WJEC 2015)*

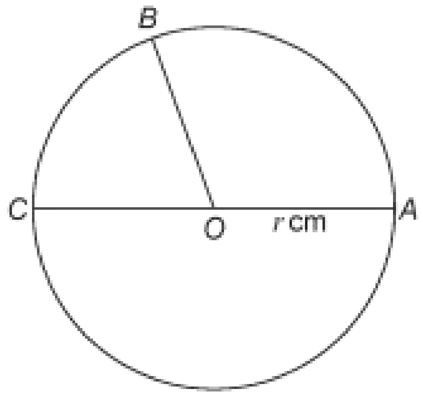
Gwyn wants to turn part of his garden into a circular flower bed. In order to do this, he digs out a shallow circular hole of radius r m and then divides it into two segments by means of a thin plank AB, as shown in the diagram. He plants red roses in the minor segment and white roses in the major segment.



Let the centre of the flower bed be denoted by O. Show that when $A\hat{O}B$ equals $2{\cdot}6$ radians, the area of the flower bed containing white roses is approximately twice the area containing red roses.

[5]

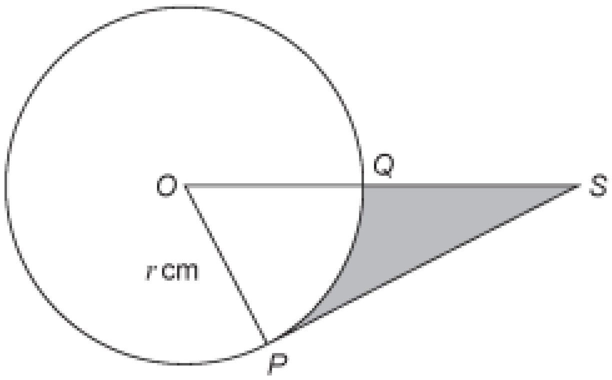
Question 6 (*WJEC 2016)*



The diagram shows a sketch of a circle with centre O and radius r cm. Three points A, B and C lie on the circle. The line AC is a diameter of the circle and $A\hat{O}B = 2{\cdot}15$ radians.

Given that the area of sector BOC is $26\,\text{cm}^2$ less than the area of sector AOB, find the value of r. Give your answer correct to one decimal place. [5]

Question 7 (*WJEC 2014)*



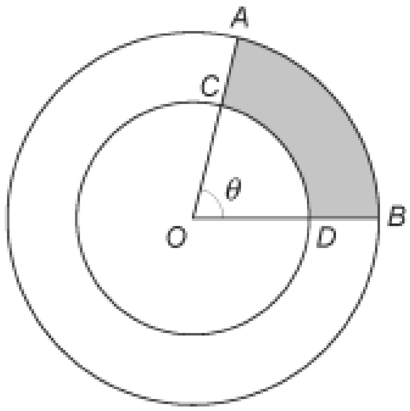
The diagram shows a circle with centre O and radius r cm. The points P and Q are on the circle and $P\hat{O}Q = 0{\cdot}9$ radians. The tangent to the circle at P intersects the line OQ produced at the point S.

(a) Find an expression in terms of r for

(i) the area of sector POQ,
(ii) the length of PS,
(iii) the area of triangle POS. [3]

(b) Given that the area of the shaded region is $95{\cdot}22\,\text{cm}^2$, find the value of r. [3]

Question 8 (*WJEC 2017)*



The diagram shows two concentric circles with common centre O. The radius of the larger circle is R cm and the radius of the smaller circle is r cm. The points A and B lie on the larger circle and are such that $A\hat{O}B = \theta$ radians. The smaller circle cuts OA and OB at the points C and D respectively. The sum of the lengths of the arcs AB and CD is L cm. The area of the shaded region $ACDB$ is K cm^2.

(a) (i) Write down an expression for L in terms of R, r and θ.

(ii) Write down an expression for K in terms of R, r and θ. [2]

(b) Given that $AC = x$ cm, use your results to part *(a)* to find an expression for K in terms of x and L. [3]

Question 9

Solve the equation $2\sin 3\theta - \sqrt{3} = 0$ in the interval $0 \leq \theta \leq 2\pi$.

[4 marks]

A TOTAL OF 43 MARKS ARE AVAILABLE.