MathEVmatics

Supercharge your learning

A2 Mathematics for WJEC

Unit 2 - Binomial Expansion

Examples and Practice Exercises

Unit Learning Objectives

- ***To use the A2 form of the binomial expansion to expand expressions of the form $(1+x)^n$, where n is a fractional or negative power.***
- ***To understand that this expansion gives an infinite series, and is only valid for specific values of x in each case.***
- ***To be able to expand expressions of the form $(a+bx)^n$, and find the range of validity for these expressions.***
- ***To be able to solve problems involving multiple expansions or approximations using the techniques learnt.***

Prerequisite atoms:

Binomial Expansion (AS Form)

Inequality notation (GCSE mathematics)

Use of modulus notation will be introduced here; this will be studied in more detail in Unit 4.

Atom Check:

Expand $(5x-3)^4$ using the AS form of the Binomial Expansion.

© MathEYmatics 2024

When you have completed the unit…

Objective	Met	Know	Mastered
I can use the A2 Binomial Expansion to expand expressions of the form $(1+x)^n$			
I can extend this method to expand expressions of the form $(a+bx)^n$			
I understand that these series expansions are only valid for particular ranges of values of $\lvert x \rvert$ *and can find these ranges of validity.*			
I can solve problems, including approximation problems, using these techniques.			

Notes/Areas to Develop:

© MathEYmatics 2024

Extending the Binomial Expansion

In AS mathematics we were introduced to the Binomial Expansion formula as a quick way to expand repeated brackets.

However, the idea of the Binomial Expansion goes much deeper, and can be applied to any expression of the form $(1+x)^n$, even where n is not a positive integer. This formula was originally developed by a certain chap called Mr I. Newton, who you might have heard of…

The expansion of $(1+x)^n$, where n is a negative or fractional coefficient, is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \text{ (Valid if } |x| < 1\text{)}$$

(This is given in the formula booklet!!)

Some things we should definitely note here:

- The formula is infinite! We generally only get asked to find expressions as far as the term in x^3 (at most!), however…
- If n happened to be a positive integer, this formula would actually be valid for any value of x and would give the same output as the Y12 form of expansion we learnt!
- However, we only use this form in cases where n is a non-natural number (i.e. a fraction or negative), and in these cases the series is only valid for 'small' values of x.
- As always, the more terms we take, the more accurate the expansion would become – we will see just how accurate it is later in this unit!

Example 1: Find the first four terms of the expansion of $(1+x)^{\frac{1}{2}}$.

Example 2: Find the expansion, up to the term in x^2, of

$$(1+x)^{-1},$$

and state the range of values of x for which the expansion is valid.

Example 3: Find the first four terms in the binomial expansion of

$$(1-2x)^{\frac{1}{2}},$$

and state the range of values for which the expansion is valid.

Task 1: For each of the following, find the first three terms in the binomial expansion, and state the range of validity of eah expansion.

a) $(1+x)^{-2}$

b) $(1-x)^{\frac{1}{3}}$

c) $(1+4x)^{\frac{5}{4}}$

© MathEYmatics 2024

Task 2:

a) Find the first four terms of the binomial expansion of

$$\left(1-\frac{x}{2}\right)^{-2}$$

b) Hence, find the expansion up to the term in x^2 of

$$(1+x)\left(1-\frac{x}{2}\right)^{-2}$$

Advanced Links *(to Further Mathematics)****:***

The binomial expansion is actually a specific case of a more general technique called a Taylor series – an extremely powerful tool in Analysis at degree level. You will meet Maclaurin Series in A2 Further Pure – this is a specific type of Taylor series (and you can derive the binomial expansion using it)!

Now: *Complete Test Your Understanding 1, Pg 15.*

Expanding $(\boldsymbol{a}+\boldsymbol{b}\boldsymbol{x})^{\boldsymbol{n}}$

For us to use the A2 form of the binomial, the bracket *must* be in the form $(1+mx)^n$.

This means, if the examiner decides to be awkward, we will have to do some 'jiggery-pokery' to get our expression into the correct form prior to expanding.

Spoiler alert: The examiner ALWAYS decides to be awkward. That's what they do.

Example: Find the first three terms in the binomial expansion of

a) $(2+x)^{-1}$

b) $(8+12x)^{1/3}$

In each case, state the range of validity for the expansion.

Task:

Find, as far as the term in x^3, the binomial expansion of $\sqrt{4+2x}$, and state the range of values of x for which the expansion is valid.

NOW: *Complete Test Your Understanding 2, Page 16.*

© MathEYmatics 2024

Solving Problems with the Binomial Expansion

Examiners are predictable people. We have a stock of 'types' of question we like to ask. In this topic, there are essentially six 'classes' of problem-solving question the examiner can use. These are listed below (from least to most likely):

- Partial Fractions into Binomial
- Binomial followed by solution to equation;
- Multiple expansions within expression;
- Binomial followed by 'changing the x';
- Answer to expansion (partially) given, unknown in original expression.
- Binomial followed by approximation of a root. (At least twice as likely as any other question!)

Let's look at an example of each – the onus here is on you 'having a go', with the exception of the final type.

Task 1:

a) Express $\frac{2x-4}{(x+1)(x-1)}$ as partial fractions

b) Hence, find (as far as the term in x^2), the binomial expansion of $\frac{2x-4}{(x+1)(x-1)}$.

Task 2:

a) Find the first three terms in the binomial expansion of $(9 + 2x)^{1/2}$

b) Hence, find the approximate solution to the equation

$$(9 + 2x)^{\frac{1}{2}} = \frac{539}{54}x^2 + \frac{58}{3}x + 1$$

(Do NOT get scared by the scary numbers – it is designed to work nicely!)

Task 3: Find the first three terms in the binomial expansion of $\frac{\sqrt{1+x}}{2+x}$, stating the range of validity for this expansion.

Task 4:

a) Find the first three terms in the expansion of $(1+3x)^{\frac{1}{2}}$.

b) Hence, find the expansion of $(1+3y+9y^2)^{\frac{1}{2}}$ up to and including the term in y^2.

Task 5:

In the expansion of $(1 + ax)^{-2}$ the coefficient of x^2 is 27. Find the value of a and hence the coefficient of the x^3 term.

Example:

You are given that $(1-2x)^{\frac{1}{2}} = 1 - x - \frac{1}{2}x^2 - \cdots$

By writing $x = 0.01$ in this expansion, find an approximate value for $\sqrt{2}$.

Task 6:

You are given that $(4-x)^{\frac{1}{2}} = 2 - \frac{1}{4}x - \frac{1}{64}x^2 - \cdots$

By writing $x = \frac{1}{9}$ in this expansion, find an approximate value for $\sqrt{35}$. Calculate the percentage error in your approximation, and state how you could achieve a more accurate approximation.

NOW – YOU ARE READY FOR THE GRADE ENHANCER™, page 17.

If you do not feel confident with any of the above questions, please ask your teacher for further practice prior to attempting the Grade Enhancer™.

© MathEYmatics 2024

Test Your Understanding 1

Question 1

Find the first three terms in each of the following binomial expansions:

a) $(1+x)^{\frac{1}{2}}$ b) $(1+x)^{-3}$ c) $(1+x)^{3/4}$

Question 2

Expand each of the following in ascending powers of x, as far as the term in x^2, and state the range of values of x for which each expansion is valid.

a) $(1+2x)^{\frac{1}{2}}$ b) $(1-3x)^{-2}$ c) $\left(1+\frac{x}{2}\right)^{1/3}$

Question 3

Find the first four terms in the binomial expansion of $\sqrt{1-x}$

Question 4

Find the first three terms in the binomial expansion of $\frac{1}{\sqrt[3]{1-2x}}$

Question 5

$$f(x)=\frac{3}{1+4x}-\frac{2}{1-5x}$$

a) Find the binomial expansion of $f(x)$ as far as the term in x^2.

b) Use your expansion to approximate $f(0.01)$, and hence find the percentage error in the approximation.

c) Explain why this expansion cannot be used to approximate $f(0.4)$.

Challenge Question

a) Show that $\frac{2-4x}{(1+x)(1-x)} \equiv \frac{A}{1+x}+\frac{B}{1-x}$

b) Hence, show that $\frac{2-4x}{(1+x)(1-x)} \approx 2-4x+2x^2$

© MathEYmatics 2024

Test Your Understanding 2

Question 1

Find the first three terms in each of the following binomial expansions:

a) $(3+x)^{-1}$ b) $(4-x)^{\frac{1}{2}}$ c) $(2+3x)^{-2}$

Question 2

Expand each of the following in ascending powers of x, as far as the term in x^2, and state the range of values of x for which each expansion is valid.

a) $(2+x)^{-3}$ b) $(8-16x)^{1/3}$ c) $(4+x)^{-1/2}$

Question 3

Find the first four terms in the binomial expansion of $\sqrt{9+4x}$

Question 4

Find the first three terms in the binomial expansion of $\frac{2}{(2+x)^{-1}}$

Question 5

The expansion of $\frac{1}{\sqrt{a+bx}}$ is $\frac{1}{5}+\frac{2x}{125}+\frac{6x^2}{3125}+\cdots$.

Find the values of a and b and hence the coefficient of the x^3 term in the expansion.

Super Challenge Question *(STEP 2, 2012) – Beyond the Boundaries™*

Write down the general term in the expansion in powers of x of $(1-x^6)^{-2}$.

(i) Find the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}.$$

Obtain also, and simplify, formulae for the coefficient of x^n in the different cases that arise.

(ii) Show that the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}(1-x)^{-1}$$

is 55, and find the coefficients of x^{25} and x^{66}.

Grade Enhancer™ - Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(WJEC 2016)*

(a) (i) Expand $\dfrac{1}{\sqrt{1+2x}}$ in ascending powers of x up to and including the term in x^2.

(ii) State the range of values of x for which your expansion is valid. [3]

(b) Use your expansion in part *(a)* to find an approximate value for one root of the equation

$$\frac{6}{\sqrt{1+2x}} = 4 + 15x - x^2.$$ [2]

Question 2 *(WJEC 2019)*

Expand $\dfrac{4-x}{\sqrt{1+2x}}$ in ascending powers of x up to and including the term in x^3. State the range of values of x for which the expansion is valid. [6]

Question 3 *(WJEC 2017)*

(a) Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2. State the range of values of x for which your expansion is valid. [3]

(b) Use your answer to part *(a)* to expand $(1+4y+8y^2)^{-\frac{1}{2}}$ in ascending powers of y up to and including the term in y^2. [3]

Question 4 *(WJEC 2014)*

Expand

$$6\sqrt{1-2x} - \frac{1}{1+4x}$$

in ascending powers of x up to and including the term in x^2.
State the range of values of x for which your expansion is valid. [7]

© MathEYmatics 2024

Question 5 *(WJEC 2018)*

Write down the first three terms in the binomial expansion of $(1-4x)^{-\frac{1}{2}}$ in ascending powers of x. State the range of values of x for which the expansion is valid. By writing $x = \frac{1}{13}$ in your expansion, find an approximate value for $\sqrt{13}$ in the form $\frac{a}{b}$, where a, b are integers. [5]

Question 6 *(WJEC 2015)*

Expand $\left(1+\frac{x}{8}\right)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2.

State the range of values of x for which your expansion is valid.
Hence, by writing $x = 1$ in your expansion, find an approximate value for $\sqrt{2}$ in the form $\frac{a}{b}$, where a and b are integers whose values are to be found. [5]

Question 7 *(WJEC 2022)*

Find the first three terms in the binomial expansion of $\frac{2-x}{\sqrt{1+3x}}$ in ascending powers of x. State the range of values of x for which the expansion is valid.

By writing $x = \frac{1}{22}$ in your expansion, find an approximate value for $\sqrt{22}$ in the form $\frac{a}{b}$, where a, b are integers whose values are to be found. [8]

Total Mark Available is 42.