A2 Mathematics for WJEC

(Pure + Applied)

Unit 13: Differential Equations

Examples and Practice Exercises

Unit Learning Objectives

- ***To understand what is meant by a first-order differential equation;***
- ***To solve first-order differential equations by separating variables, understanding the difference between the general and particular solutions;***
- ***To be able to form first-order differential equations in a variety of contexts, and solve mechanics problems involving separation of variables.***

Prior Learning Atoms:

- *Differentiation*
- *Integration*
- *Logarithms and Exponentials*
- *A knowledge of the AS/A2 Mechanics content (for the applied questions)*

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Now you have completed the unit…

Objective	Met	Know	Mastered
I understand what is meant by a (first-order) differential equation.			
I can separate variables to solve a first order differential equation.			
I understand the difference between a general and particular solution, and understand how to find a particular solution using boundary (initial) conditions.			
I can form differential equations.			
I can solve mechanics problems involving differential equations.			

Notes/Areas to Develop:

Introducing Differential Equations

At their heart, a differential equation is a very simple thing – it's an equation involving a derivative. So, for example, we can consider the following:

$$\frac{dy}{dx} = 2x$$

So, we can integrate both sides, and we're done – right?

Well… kinda. The problem is, when we integrate, we get a constant of integration (the dreaded '+c'!). So, in our example, we would get $y = x^2 + c$. The following diagram shows us this graphically, for some selected values of c:

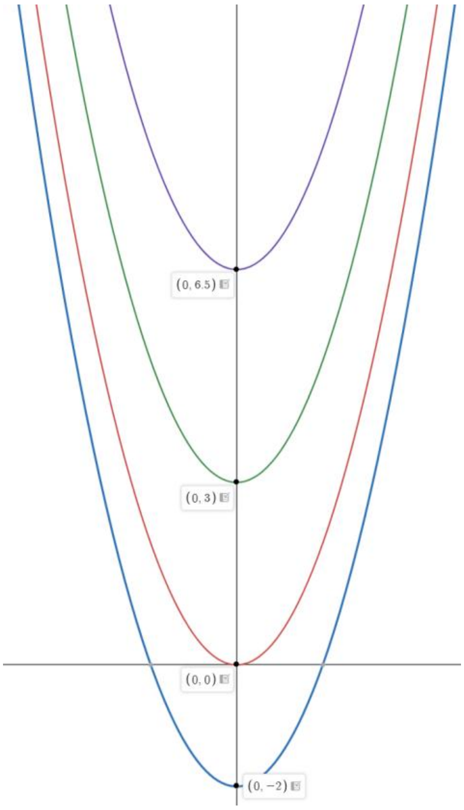

Figure 1: $y = x^2 + c$ for selected values of c. Created with Desmos.

What we actually have here are a 'family' of solutions; an infinite number of solutions depending on the value of c. This is what we call the **general solution** of the differential equation.

If we were given a point on the curve, we could find the precise curve we were wanting (i.e. a specific value of c); this would be our **particular solution.**

Finding the General Solution – Separating Variables

So, how do we actually solve a differential equation?

In Further Mathematics and the real world, this is an important and complicated topic, on which thousands of books exist. However, we have to start somewhere, right?

In A2 Mathematics, we will only consider **first-order** differential equations (that is, equations with only the first derivative, e.g. $\frac{dy}{dx}$ or $\frac{dP}{dt}$) which can be solved by **separating the variables**.

Example 1: By first separating variables, find the general solution to the differential equation

$$\frac{dy}{dx} = xy,$$

giving your answer in the form:

i) $ln|y| = f(x) + c$

ii) $y = Ae^{g(x)}$ *(**Pro tip** – we often need to write a solution in this form!!)*

Example 2: Find the general solution of the first-order differential equation

$$\frac{dP}{dt} = \frac{2t + 3}{2P}$$

giving P in terms of t.

Task 1: Find the general solution of each of the following, giving your answers in a suitable form:

a) $\frac{dy}{dx} = y\ sec^2 x$

b) $\frac{dP}{dt} = kP^3$

Now: Complete Test Your Understanding 1, Page 15, Question 1.

Finding the Particular Solution – Using Initial (Boundary) Conditions

At this stage, we know how to find the general solution; however, in the exam we will almost always require a **particular solution**, i.e. one without an unknown constant 'c'. For this, we will need to be given **initial conditions** (also referred to as **boundary conditions**) – these are an initial set of values linking the two variables in the problem, thus allowing us to calculate the value of 'c'.

Example 1: The variable x satisfies the differential equation

$$3\frac{dx}{dt} = t^2(1+3x), \text{ where } x \geq 0.$$

Given that when $t = 0$, $x = 1$, solve the differential equation and hence find the value of t when $x = 5$.

Example 2: A situation is modelled by the differential equation

$$\frac{dy}{dx} = ky.$$

i) Show that $y = Ae^{kx}$

ii) When $x = 1, y = 2$ and the rate of change of y with respect to x is 6. Find the values of A and k.

Task 1: A scientist monitors the growth of algae on a lake's surface. At time t years the amount of lake surface covered by algae is $S\ m^2$. The rate of increase of S is modelled by the differential equation

$$\frac{dS}{dt} = kS, \text{ where } k > 0.$$

At the start of monitoring, the area of algae covering the lake is $0.2\ m^2$, and after one year the area covered is $1.2\ m^2$. Find an expression for S in terms of t.

Now: Complete Test Your Understanding 1, Page 15, remaining questions.

Forming and Solving Differential Equations

Often, the examiner will require us to form the initial differential equation (or, at least, to show that a given differential equation follows) – this is particularly so in Mechanics (Unit 4).

This often requires us to use our GCSE knowledge of proportion:

- If $\frac{dy}{dx}$ is directly proportional to e.g. x, then $\frac{dy}{dx} = kx$
- If $\frac{dy}{dx}$ is inversely proportional to e.g. y, then $\frac{dy}{dx} = \frac{k}{y}$

Example 1: The rate of increase of a population P of rabbits after t years is proportional to the number of rabbits at that time.

a) Form a differential equation using this information.

b) Given that the initial population is 80 rabbits, and that the initial rate of increase is 200, solve the differential equation, giving your answer in the form $P = Ae^{bt}$.

c) Find the time taken for the population to reach 5000,

d) Comment on the long-term suitability of this model.

Task 1: The rate of radioactive decay of a substance is to be modelled. The rate of decay is found to be proportional to the number of particles N remaining at time t, where t is in days. Letting the constant of proportionality be λ,

a) Write down a differential equation which satisfies the above information.

b) Given that the initial quantity of the substance contained 4,000 particles, and that the number of particles had halved after 10 days, find the value of λ to 3 significant figures.

c) Find how long is needed for the substance to contain one tenth of the initial number of particles.

Now: Complete Test Your Understanding 2, Page 17.

Applications to Mechanics

Differential equations can be set in the context of mechanics problems in Unit 4. The method for solving them is the same (and, most often, the first part of the question is a 'show that' so that you can solve the differential equation in later parts even if you are unable to form it) – however, a few particularly nasty questions require you to form a suitable differential equation.

Example 1: A parcel with mass $4\ kg$ is projected along a rough horizontal surface with an initial velocity of $8\ ms^{-1}$.

The box experiences a variable resistive force of $0.25v^2N$, where v is the velocity (in metres per second) of the box at time t seconds.

a) Show that v satisfies the differential equation

$$16\frac{dv}{dt} + v^2 = 0$$

Hint: Mr Newton will be called into action a LOT here. What is his Second Law?

b) Hence, show further that $v = \frac{16}{t+2}$.

c) Explain why this model is not particularly realistic.

Example 2: Newton's Law of Cooling states that the rate at which a liquid with temperature $\theta\ °C$ cools in a room with ambient temperature $A\ °C$ is proportional to the difference in temperature between the liquid and the room.

a) Using λ as the constant of proportionality, where $\lambda > 0$, write down a differential equation satisfied by θ.

b) A mug of hot tea is left in a room with a temperature of $20\ °C$. Initially, the tea is at a temperature of $80\ °C$. After 5 minutes, the tea is at a temperature of $60\ °C$.

i) Show that $\lambda = \frac{1}{5}\ln\left(\frac{3}{2}\right)$

ii) Show that a complete model for the temperature $\theta\ °C$ at time t minutes of the tea is given by

$$\theta = Me^{-\lambda t} + n$$

Where M and n are constants to be determined.

c) Find, to 1 decimal place, the temperature of the tea after 12 minutes.

Task 1: A ball of mass $2\ kg$ is thrown vertically upwards, with initial velocity $30\ ms^{-1}$.

At time t seconds, the ball has velocity $v\ ms^{-1}$.

During upward motion, the object experiences a resistance to motion of R netwons, where R is directly proportional to V. When $v = 0.2\ ms^{-1},\ R = 0.06$ newtons.

a) Show that $\frac{dv}{dt} = -(9.8 + 0.15v)$

b) Hence find an expression for v in terms of t.

c) Use your model to determine the value of t when the object reaches its highest point.

Now: You are ready to face the Grade Enhancer™.

Test Your Understanding 1

Question 1

For each of the following differential equations, by first separating variables, find the general solution, giving your answer in a suitable form.

a) $\frac{dy}{dx} = (x+1)^2$ b) $\frac{dy}{dx} = \frac{cos2x}{y}$ c) $\frac{dP}{dt} = \frac{e^{3t}}{P^2}$

d) $\frac{dA}{dt} = \frac{A}{t}$ e) $\frac{dS}{dv} = Sv^3$ f) $\frac{d\theta}{dt} = \sec t \tan t\,(1+2\theta)$

g) $\frac{dy}{dx} = 2xy - 3x$ h) $\frac{dx}{dt} = te^x$ i) $\frac{dP}{dN} = P^2 \sec^2 N$

Question 2

For each of the following differential equations, and the given boundary conditions, find a particular solution in a suitable form.

a) $\frac{dy}{dx} = 3cos2x$, when $x = \frac{\pi}{2}, y = 1$ b) $\frac{dy}{dx} = \frac{2x}{y}$, when $x = 3, y = 5$

c) $\frac{dy}{dx} = \frac{1+y}{x}$, when $x = -1, y = e^2$ d) $\frac{dP}{dt} = Pt^2$, when $t = 0, P = e^3$

e) $\frac{dP}{dA} = P \sin A$, when $A = \frac{\pi}{2}, P = 3$ f) $(1+2x)\frac{dy}{dx} = (1+y)$, when $x = 2, y = 1$

g) $\frac{dN}{dt} = \frac{\sqrt{t}}{N}$, when $t = 9, N = 4$ h) $\frac{dy}{dx} = \frac{x}{e^x siny}$, when $x = -1, y = \pi$

Question 3

A particular species is such that the rate of increase of the population P at time t months after the start of monitoring is given by

$$\frac{dP}{dt} = 0.3P$$

Given that the initial population was 12, find a model for P in the form $P = Ae^{3t}$, stating the value of A, and hence find the time when the population first exceeds 10000.

Question 4

The rate of increase of area $A\ cm^2$ of moss on a building t weeks after monitoring begins satisfies the differential equation

$$\frac{dA}{dt} = \frac{k\sqrt[3]{t}}{A}$$

When monitoring begins, the area of moss is $8\ cm^2$. After 8 weeks, the area has increased to $20\ cm^2$.

i) Use the above information to show that $A = 21\sqrt[3]{t^4 + 64}$

ii) Find, according to the model and to 3 significant figures, the area of moss covering the building after one year.

iii) Comment on the suitability of this model for large values of t.

Question 5

A cup of coffee with temperature θ is left in a room with an ambient temperature of $18\ °C$. The rate at which the coffee cools is given by the differential equation

$$\frac{d\theta}{dt} = -k(\theta - 18).$$

The initial temperature of the coffee is $75\ °C$. After 8 minutes, the temperature of the coffee is $50\ °C$.

Solve the differential equation, finding the value of k, and thus find the temperature of the coffee after 15 minutes.

Test Your Understanding 2

Question 1 *(WJEC 2011)*

The size N of the population of a small island may be modelled as a continuous variable. At time t, the rate of increase of N is directly proportional to the value of N.

(a) Write down the differential equation that is satisfied by N. [1]

(b) Show that $N = Ae^{kt}$, where A and k are constants. [3]

(c) Given that $N = 100$ when $t = 2$ and that $N = 160$ when $t = 12$,

(i) show that $k = 0{\cdot}047$, correct to three decimal places,

(ii) find the size of the population when $t = 20$. [7]

Question 2 *(WJEC 2010)*

The value, £V, of a car may be modelled as a continuous variable. At time t years, the rate of decrease of V is directly proportional to V^2.

(a) Write down a differential equation satisfied by V. [1]

(b) Given that $V = 12000$ when $t = 0$, show that

$$V = \frac{12000}{at+1},$$

where a is a constant. [4]

(c) The value of the car at the end of two years is £9000. Find the value of the car at the end of four years. [4]

Question 3 *(WJEC 2009)*

The value of an electronic component may be modelled as a continuous variable. The value of the component at time t years is £P. The rate of decrease of P is directly proportional to P^3.

(a) Write down a differential equation that is satisfied by P. [1]

(b) The value of the component when $t = 0$ is £20. Show that

$$\frac{1}{P^2} = \frac{1}{400} + At,$$

where A is a positive constant. [5]

(c) Given that the value of the component when $t = 1$ is £10, find the time when the value is £5. [4]

Question 4 *(WJEC 2014)*

The value £V of a long term investment may be modelled as a continuous variable.
At time t years, the rate of increase of V is directly proportional to the value of V.

(a) Write down a differential equation satisfied by V. [1]

(b) Show that $V = Ae^{kt}$, where A and k are constants. [3]

(c) The value of the investment after 2 years is £292 and its value after 28 years is £637.

(i) Show that $k = 0{\cdot}03$, correct to two decimal places.

(ii) Find the value of A correct to the nearest integer.

(iii) Find the initial value of the investment. Give your answer correct to the nearest pound. [6]

Question 5 *(WJEC 2008)*

A neglected large lawn contains a certain type of weed. The area of the lawn covered by the weed at time t years is $W\,\text{m}^2$. The rate of increase of W is directly proportional to W.

(a) Write down a differential equation that is satisfied by W. [1]

(b) The area of the lawn covered by the weed initially is $0{\cdot}10\,\text{m}^2$ and one year later the area covered is $2{\cdot}01\,\text{m}^2$. Find an expression for W in terms of t. [6]

Question 6 *(WJEC 2017)*

The size N of the population of a small island may be modelled as a continuous variable. At time t years, the rate of increase of N is assumed to be directly proportional to the value of $\sqrt{N}$.

(a) Write down a differential equation satisfied by N. [1]

(b) When $t = 5$, the size of the population was 256. When $t = 7$, the size of the population was 400. Find an expression for N in terms of t. [6]

Grade Enhancer™ - Apply your knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Section A: Pure Maths

Question 1 *(WJEC 2019)*

Wildflowers grow on the grass verge by the side of a motorway. The area populated by wildflowers at time t years is A m^2. The rate of increase of A is directly proportional to A.

a) Write down a differential equation that is satisfied by A. [1]

b) At time $t = 0$, the area populated by wildflowers is $0{\cdot}2$ m^2. One year later, the area has increased to $1{\cdot}48$ m^2. Find an expression for A in terms of t in the form pq^t, where p and q are rational numbers to be determined. [7]

Question 2 *(WJEC 2018)*

The variable y satisfies the differential equation

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 2y, \qquad \text{where } x \geqslant 0.$$

Given that $y = 1$ when $x = 0$, find an expression for y in terms of x. [5]

Question 3 *(WJEC 2023)*

The rate of change of a variable y with respect to x is directly proportional to y.

a) Write down a differential equation satisfied by y. [1]

b) When $x = 1$ and $y = 0{\cdot}5$, the rate of change of y with respect to x is 2.
Find y when $x = 3$. [6]

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Question 4 *(WJEC 2024)*

(a) Given that $y = \frac{1+\ln x}{x}$, show that $\frac{dy}{dx} = \frac{-\ln x}{x^2}$. [2]

(b) Hence, solve the differential equation

$$\frac{dx}{dt} = \frac{x^2 t}{\ln x},$$

given that $t = 3$ when $x = 1$.

Give your answer in the form $t^2 = g(x)$, where g is a function of x. [5]

Question 5 *(WJEC 2013)*

Part of the surface of a small lake is covered by green algae. The area of the lake covered by the algae at time t years is $A\,\text{m}^2$. The rate of increase of A is directly proportional to $\sqrt{A}$.

(a) Write down a differential equation satisfied by A. [1]

(b) The area of the lake covered by the algae at time $t = 3$ is $64\,\text{m}^2$ and the area covered at time $t = 5{\cdot}5$ is $196\,\text{m}^2$. Find an expression for A in terms of t. [6]

Marks Available for Section A: 34

Section B: Mechanics

Question 1 *(WJEC 2018)*

An object of mass 0·5 kg is thrown vertically upwards with initial speed $24\,\text{ms}^{-1}$. The velocity of the object at time t seconds is $v\,\text{ms}^{-1}$. During the upward motion, the object experiences a resistance to motion R N, where R is proportional to v. When the velocity of the object is $0{\cdot}2\,\text{ms}^{-1}$ the resistance to motion is 0·08 N.

a) Show that the upward motion of the object satisfies the differential equation

$$\frac{dv}{dt} = -9{\cdot}8 - 0{\cdot}8\,v.$$ [3]

b) Find an expression for v at time t. [6]

c) Determine the value of t when the object is at the highest point of the motion. [2]

Question 2 *(WJEC 2019)*

A box of mass 2 kg is projected along a horizontal surface with an initial velocity of $5\,\text{ms}^{-1}$. The box experiences a variable resistive force of $0{\cdot}4v^2$ N, where $v\,\text{ms}^{-1}$ is the velocity of the box at time t seconds.

a) Show that v satisfies the equation

$$5\frac{dv}{dt} + v^2 = 0.$$ [2]

b) Find an expression for v in terms of t. [4]

c) Briefly explain why this model is not particularly realistic. [1]

Question 3 (*WJEC 2022*)

Megan wants to make some ice cubes. She fills a plastic ice cube tray with water of temperature 10°C and places it directly into the freezer. The temperature inside the freezer remains constant at −18°C. The temperature of the water at time t hours after being placed in the freezer is denoted by θ°C. For Megan's scenario, Newton's law of cooling states that the rate of decrease of θ is directly proportional to the difference between the temperature of the water and the temperature inside the freezer.

a) Using k as the constant of proportionality, where k is positive, write down a differential equation satisfied by θ. [1]

b) Show that, for $\theta > -18$,

$$kt = \ln\left(\frac{28}{\theta + 18}\right).$$

[4]

c) Given that the temperature of the water in the tray is 6°C after 1 hour, determine the total amount of time that Megan needs to wait for the temperature of the water to reach −5°C. Give your answer to the nearest hour. [3]

Question 4 *(WJEC 2023)*

A train is moving along a straight horizontal track. At time t seconds, its velocity is $v\,\text{ms}^{-1}$, its acceleration is $a\,\text{ms}^{-2}$, and a is inversely proportional to v. At time $t = 1$, $v = 5$ and $a = 1{\cdot}8$.

a) **i)** Write down a differential equation satisfied by v.

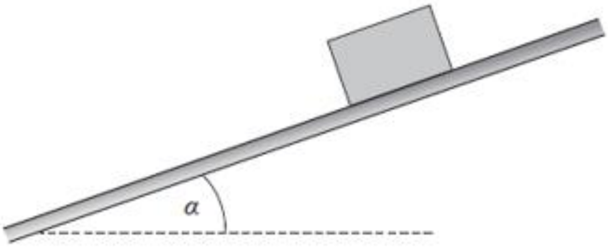
ii) Show that $v^2 = 18t + 7$.

[6]

b) Find the time at which the magnitude of the velocity is equal to the magnitude of the acceleration. [2]

Question 5 (*WJEC 2024)*

The diagram below shows a parcel, of mass m kg, sliding down a rough slope inclined at an angle α to the horizontal, where $\sin\alpha = \frac{7}{25}$.



The coefficient of friction between the parcel and the slope is $\frac{1}{12}$. In addition to friction, the parcel experiences a variable resistive force of mv N, where v ms^{-1} is the velocity of the parcel at time t seconds.

(a) Show that the motion of the parcel satisfies the differential equation

$$5\frac{\mathrm{d}v}{\mathrm{d}t} = g - 5v. \qquad [5]$$

(b) Given that the parcel is initially at rest on the slope, find an expression for v in terms of t and g. [5]

(c) To avoid damage, the speed of the parcel must be restricted to a maximum of 2 ms^{-1} down the slope. Determine whether or not the speed of the parcel exceeds 2 ms^{-1}. [1]

Marks Available for Section B: 45

TOTAL MARKS AVAILABLE: 79 MARKS.