MathEVmatics

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A2 Mathematics for WJEC

Unit 10 - Parametric Equations

Examples and Practice Exercises

Unit Learning Objectives

- ***To understand how curves can be defined parametrically;***
- ***To solve coordinate geometry problems involving parametric equations;***
- ***To understand how to differentiate parametric equations.***
- ***Differentiate the inverse trigonometric functions;***

Prerequisite atoms:

Coordinate Geometry, Equations and Inequalities (AS Mathematics)

Differentiation (AS Mathematics)

Functions, Domain and Range (A2 Unit 4)

Differentiation (A2 Units 6 and 9)

When you have completed the unit…

Objective	Met	Know	Mastered
I understand how parametric equations work, and can plot a curve defined parametrically.			
I can convert parametric equations into Cartesian form.			
I can solve coordinate geometry problems involving parametric equations.			
I can differentiate curves defined parametrically.			

Notes/Areas to Develop:

Sometimes, it is useful to describe movement of x and y directions in terms of a third parameter (often t for time, or θ for angles). This is often the case in mechanics when considering the movement of an object or the distribution of a force.

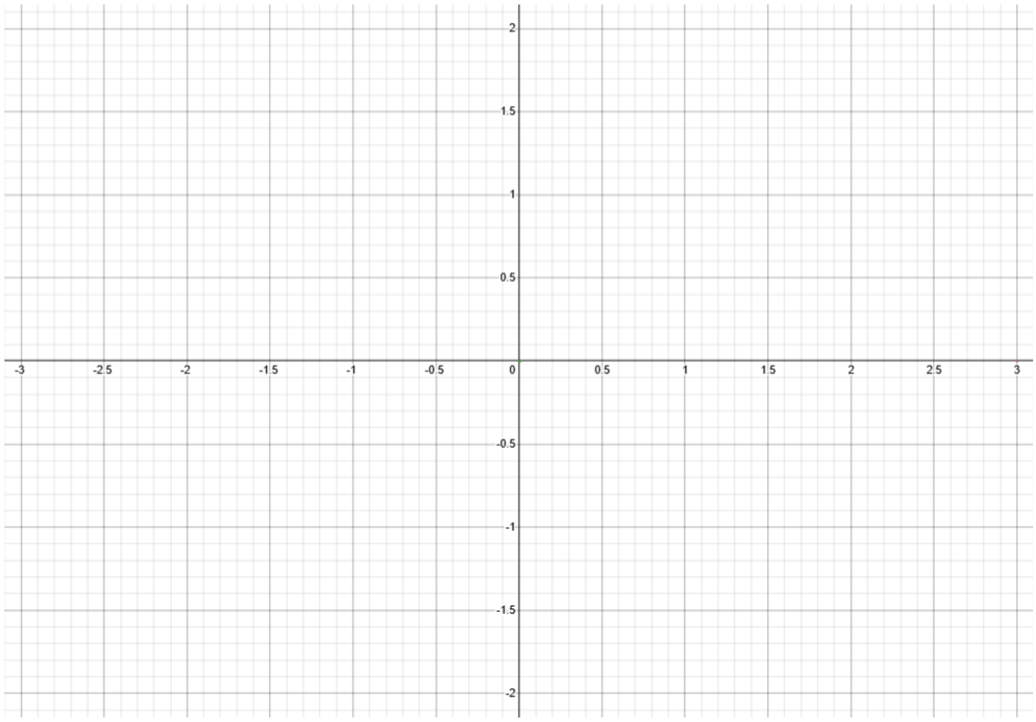
It can also be useful in pure mathematics, where to write the curve in Cartesian form (linking x and y directly) may be too complex or messy.

Example 1: A curve is defined parametrically by the equations $x = 3\cos t\,, y = 2\sin t$, where $0 \leq t \leq 2\pi$.

a) By completing the table of values below and plotting the graph, show that these equations represent an ellipse.

b) Using the identity $sin^2t + cos^2t \equiv 1$, find the Cartesian equation of the ellipse.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$x = 3\cos t$	3			0			-3			0			3
$y = 2\sin t$	0	1		2		1	0	-1		-2		-1	0



Investigate this curve at: https://www.desmos.com/calculator/c2wl8thyhu

Where the parametric equations are non-trigonometric, we can usually find the Cartesian equation relatively simply by a combination of rearrangement and substitution.

Example 2: A curve is defined by parametric equations $x = 3 - t$, $y = 2t^2$, for $-3 \leq t \leq 3$.

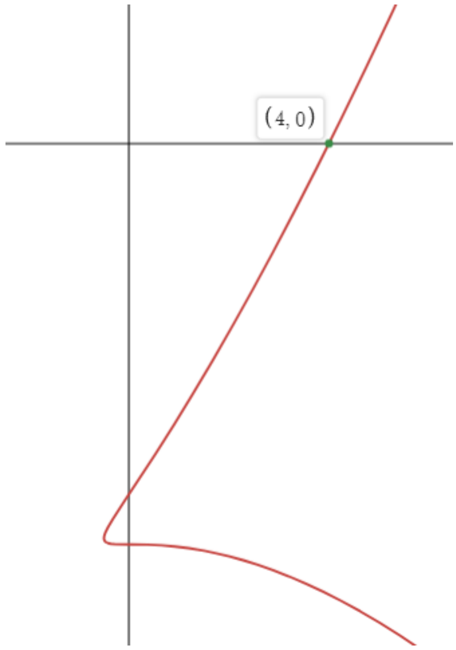
a) Find the coordinates of the point where $t = 1$.

b) Find a Cartesian equation of the curve, stating the domain and range.

Key Point: To find the domain and range of a parametric curve, we can just consider the possible values of the x (domain) and y (range) parameters -it arguably makes life easier!

Example 3:

The curve shown in the image is defined parametrically by equations $x = pt^2 - t$, $y = t^3 - 8$, where p is a constant.



a) Given that the curve passes through $(4, 0)$, find the value of p.

b) Find the coordinates of the points where the curve intersects the y-axis.

Example 4

The curve C is defined by parametric equations $x = 3t$, $y = t^2$. The line with equation $x + y + 2 = 0$ meets C at the points A and B. Find the coordinates of A and B.

Test Your Understanding 1

Question 1:

A curve is defined parametrically by the equations $x = t^2$, $y = 2t$, for $-3 \leq t \leq 3$.

a) By creating a table of values for t, x and y, plot the curve on graph paper.

b) Find a Cartesian equation for the curve in the form $y = f(x)$, stating the domain and range.

Question 2:

a) Find the coordinates of the point on the curve $x = 3t^2$, $y = 1 - 4t^3$ where $t = -2$.

b) Find the coordinates of the **points** on the curve where $x = 3$.

Question 3

Determine Cartesian equations for the following curves defined parametrically.

a) $x = \cos t\,, y = 2\sin t$

b) $x = 3\cos t\,, y = sin^2 t$

c) $x = 2t - 3, y = 5 - 3t$

d) $x = t,\ \ y = 1 + \frac{1}{t^2}$

Question 4

The curve defined by parametric equations $x = 5 + t$, $y = 3 - t$ meets the x-axis at A and the y-axis at B. Find the coordinates of A and B.

Question 5

Find the coordinates of the points where the curve defined by the parametric equations $x = t^2 - 1, y = \frac{1}{t} - 1$ meets the y-axis.

Question 6

A curve is defined parametrically as $x = \frac{t-1}{t+1}$, $y = 2t^2$, $\ t \neq -1$.

Find the coordinates of any points of intersection with the x- or y- axes.

Question 7

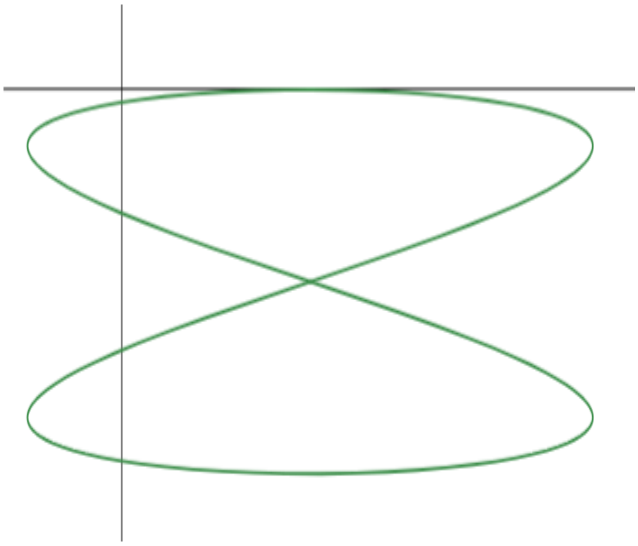
A line L_1 is defined parametrically by the equations $x = 3t + 2, y = 1 - t$. The line L_2 has Cartesian equation $y = 2 - x$. Find the point of intersection of L_1 and L_2.

Question 8

Find the coordinates of the points of intersection of the line $y = 6 - 3x$ and the curve with parametric equations $x = t^2$, $y = 3t$.

Question 9

The curve shown in the diagram on the right is defined parametrically by the equations $x = 2 + 3\sin 2\theta\,, y = \cos\theta - 1$ over the interval $0 \le \theta < 2\pi$.



a) Show that the curve meets the x-axis at the point $(2, 0)$.

b) Find the coordinates of the points where the curve meets the y-axis.

Question 10

Given that the line with equation $y = 2x - k$ does not intersect the curve defined by parametric equations $x = 1 - t, y = 3t^2 + 1$, find the range of possible values for k.

Question 11

Two curves are defined parametrically as follows:

$$C_1 : x = 2t, y = t^2$$

$$C_2 : x = t, y = 3t$$

Find the coordinates of the points where C_1 and C_2 intersect.

Hint: For questions where two parametric equations meet, it is usually best to convert to Cartesian equations first!

Differentiation of Parametric Curves

Since we have now looked at parametric curves, it makes sense to consider how we would find their gradient and to solve other problems linked to our prior learning, such as tangents/normals and stationary points.

Again, the **chain rule** comes to our rescue here! If I have two functions x and y given in terms of t, then I can easily find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Then, by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Example 1: A curve C is defined parametrically by $x = t^3 - t$, $y = 2t - t^2$. Find the equation of the tangent to C at the point P where $t = 3$.

Task 1: A curve is defined by parametric equations $x = 2\cos\theta$, $y = 3\sin\theta$. Find the equation of the normal to the curve at the point where $\theta = \frac{\pi}{6}$.

Task 2: Find the coordinates of the stationary points on the curve given by the parametric equations $x = 2 - t, y = 2t^2 - t^3$. *You do not need to determine the nature of these points.*

Test Your Understanding 2

Question 1

For each of the following, find an expression for $\frac{dy}{dx}$, in terms of the parameter t.

a) $x = 2t$, $y = t^2 - t$

b) $x = 4t^3$, $y = 5t - t^2$

c) $x = \frac{2}{t}, y = t^3 \ (t \neq 0)$

d) $x = e^t$, $y = \ln t \ (t > 0)$

e) $x = \sec t \,, y = \tan t$

f) $x = 5t^2, y = \cos t$

Question 2

Find the gradient of the curve at the given value of t.

a) $x = 3t^2, y = 2 + 3t$ when $t = 2$

b) $x = 1 - \frac{1}{t}, y = 1 + \frac{1}{t}$ when $t = 3$.

Question 3

Find the equation of the tangent to the curve $x = 3\cos t \,, y = 2\sin t$ at the point where $t = \frac{3\pi}{4}$, giving your answer in the form $y = ax + b\sqrt{2}$.

Question 4

A curve is defined parametrically by the equations $x = t^3, y = 3t^2 - t$.

a) Verify that the point $P(1, 2)$ lies on the curve.

b) Find the equation of the normal to the curve at P.

Question 5

A curve is defined by parametric equations $x = e^t$, $y = e^t + e^{-t}$. Find the equation of the tangent to the curve at the point where $t = 0$.

Question 6

A curve is defined parametrically by the equations $x = -\cos 2t$ and $y = 8\sin t$. Show that $\frac{dy}{dx} = 2\text{cosec}\, t$.

Question 7

A curve is defined by parametric equations $x = t^2 + t, y = t^2 - 10t + 5$. Find the coordinates of the point where $\frac{dy}{dx} = 2$, the equation of the tangent at this point, and show that the tangent does not intersect the curve at any other point.

Grade Enhancer™ - Apply your Knowledge!

These 'Grade Enhancer' questions are either WJEC Past Paper questions, or designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(WJEC 2023)*

The curve C_1 has parametric equations $x = 3p + 1$, $y = 9p^2$.

The curve C_2 has parametric equations $x = 4q$, $y = 2q$.

Find the Cartesian coordinates of the points of intersection of C_1 and C_2. [7]

Question 2 *(WJEC 2022)*

The parametric equations of the curve C are

$$x = 3 - 4t + t^2, \qquad y = (4 - t)^2.$$

a) Find the coordinates of the points where C meets the y-axis. [3]

b) Show that the x-axis is a tangent to the curve C. [5]

Question 3 *(WJEC 2019)*

A curve C has parametric equations $x = \sin\theta$, $y = \cos 2\theta$.

a) The equation of the tangent to the curve C at the point P where $\theta = \frac{\pi}{4}$ is $y = mx + c$. Find the exact values of m and c. [6]

b) Find the coordinates of the points of intersection of the curve C and the straight line $x + y = 1$. [5]

Question 4 *(WJEC 2012, partial and reworded)*

A curve C has parametric equations

$$x = t^2, y = 2t.$$

Show that the normal to C at the point where $t = p$ has equation

$$y + px = p^3 + 2p.$$

[5]

Question 5 *(WJEC 2013, partial and reworded)*

The curve C has parametric equations

$$x = at, y = \frac{b}{t}$$

where a and b are positive constants.

The point P lies on C such that $t = p$.

a) Show that the equation of the tangent to C at the point P is

$$ap^2y + bx - 2abp = 0$$

b) The tangent to C at the point P meets the x-axis at the point A and the y-axis at the point B. Find the area of the triangle AOB, where O denotes the origin, giving your answer in its simplest form.

Question 6 *(WJEC 2011)*

The curve C has the parametric equations

$$x = 3\cos t, y = 4\sin t.$$

The point P lies on C and has parameter p.

(a) Show that the equation of the tangent to C at the point P is

$$(3\sin p)y + (4\cos p)x - 12 = 0.$$ [5]

(b) The tangent to C at the point P meets the x-axis at the point A and the y-axis at the point B. Given that $p = \frac{\pi}{6}$,

(i) find the coordinates of A and B,

(ii) show that the exact length of AB is $2\sqrt{19}$. [4]

Total Mark Available is 50.