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AS Mathematics for WJEC

Indices and Surds

Examples and Practice Exercises

Unit Learning Objectives

- ***To know and to use the laws of indices;***
- ***To know and to use the rules of surds;***
- ***To be able to rationalise denominators in expressions containing surds;***
- ***To be able to convert between index form and surd form;***
- ***To solve problems involving surds and indices.***

Assumed Prior Knowledge: Basic Algebra and Fractions

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Self-Assessment – For when you have completed the unit…

Objective	Met	Know	Mastered
To know and to use the laws of indices.			
To know and to use the rules of surds.			
To be able to rationalise denominators in expressions containing surds.			
To be able to convert between index form and surd form.			
To solve problems involving surds and indices.			

Notes/Areas to Develop:

Laws of Indices

You have met all of the laws of indices at GCSE. They are as follows:

1. $x^a \times x^b = x^{a+b}$
2. $x^a \div x^b = x^{a-b}$
3. $(x^a)^b = x^{ab}$
4. $x^0 = 1 \qquad (x \neq 0)$
5. $x^{-a} = \frac{1}{x^a}$
6. $x^{\frac{1}{a}} = \sqrt[a]{x}$
7. $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$

Example 1: Simplify the following expressions:

a) $15y^8 \div 5y^2$

$= 3y^6$

b) $(4a^3b^5)^2 \times 3a^2b$

$= 16a^6b^{10} \times 3a^2b$

$= 48a^8b^{11}$

Example 2: Simplify the following expressions:

a) $\frac{c^8 + c^5}{c^2}$

$= \frac{c^8}{c^2} + \frac{c^5}{c^2}$

$= c^6 + c^3$

b) $\frac{35x^6 + 20x^4}{5x^2}$

$= \frac{35x^6}{5x^2} + \frac{20x^4}{5x^2}$

$= 7x^4 + 4x^2$

Example 3: Evaluate the following without a calculator:

a) $4^{\frac{5}{2}}$

$= (\sqrt{4})^5$

$= 2^5$

$= 32$

b) $27^{-1.5}$

$= 27^{-3/2}$

$= \left(\frac{1}{27}\right)^{3/2}$

$= \left(\frac{1}{3\sqrt{3}}\right)^3$

$= \frac{1}{81\sqrt{3}}$

c) $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

$= \left(\frac{27}{64}\right)^{2/3}$

$= \left(\frac{3}{4}\right)^2$

$= \frac{9}{16}$

Example 4: Simplify the following expressions:

a) $\frac{x^5}{x^{-3}}$

$= x^8$

b) $8y^3 \div \sqrt[3]{64y^6}$

$= 8y^3 - 4y^2$

$= 2y$

c) $\frac{8c^2 + c^5}{2c^4}$

$= \frac{8c^2}{2c^4} + \frac{c^5}{2c^4}$

$= \frac{4}{c^2} + \frac{c}{2}$

Example 5: Given that $a = \frac{1}{9}p^3$, find and simplify an expression for $a^{1.5}$

$a = \frac{1}{9}p^3 = \frac{p^3}{9}$

$a^{3/2} = \left(\sqrt{\frac{p^3}{9}}\right)^3 = \left(\frac{p^{1.5}}{3}\right)^3 = \frac{p^{4.5}}{27}$

Test Your Understanding 1

Question 1

Simplify the following expressions fully.

a) $3x^5 \times 4x^2$

$12x^7$

b) $\dfrac{27a^6}{3a^{-2}}$

$9a^8$

c) $2p^3 \times 4p^2 \times 5p^{\frac{1}{2}}$

$40p^{11/2}$

d) $x^{\frac{3}{2}} \times x^{-2.5}$

$\dfrac{1}{x}$

e) $(x^2)^{\frac{3}{2}}$

x^3

f) $\sqrt{x} \div \sqrt[3]{x}$

$x^{1/6}$

g) $\dfrac{\left(\sqrt[3]{x}\right)^2}{\sqrt[4]{x}}$

$x^{5/12}$

Question 2

Simplify the following.

a) $\dfrac{8a^4 + 12a^7}{2a}$

$4a^3 + 6a^6$

b) $\dfrac{7a^3 + 2a^4}{a^3}$

$7 + 2a$

c) $\dfrac{6x^2 + 9x^5}{3\sqrt{x}}$

$2x^{15} + 3x^{45}$

d) $\dfrac{c + 3c^3}{c^3}$

$\dfrac{1}{c^2} + 3$

Question 3

Evaluate.

a) $9^{\frac{5}{2}}$

243

b) $64^{0.5}$

8

c) $27^{-\frac{1}{3}}$

$\dfrac{1}{3}$

d) $(-3)^{-4}$

$\dfrac{1}{81}$

e) $\left(\dfrac{2}{7}\right)^0$

1

f) $\left(\dfrac{16}{25}\right)^{\frac{3}{2}}$

$\dfrac{64}{125}$

g) $\left(\dfrac{343}{512}\right)^{-\frac{1}{3}}$

$\dfrac{8}{7}$

h) $\left(1\dfrac{7}{9}\right)^{-1.5}$

$= \left(\dfrac{16}{9}\right)^{-3/2}$

$= \dfrac{27}{64}$

Question 4

Simplify.

a) $(81p^6)^{\frac{1}{2}}$ $9p^3$

b) $(125y^{12})^{\frac{4}{3}}$ $625y^{16}$

c) $\left(\frac{81}{16}p^6\right)^{-\frac{3}{2}}$ $\frac{64}{729p^9}$

Question 5

Given that $y = \frac{8}{27}x^3$, express the following in the form ax^n, where a and n are rational constants to be determined.

a) y^2 $\frac{64x^6}{729}$

b) $y^{\frac{1}{3}}$ $\frac{2}{3}x$

c) $3y^{-\frac{2}{3}}$

$3\left(\frac{8x^3}{27}\right)^{-\frac{2}{3}}$

$= 3\left(\frac{27}{8x^3}\right)^{\frac{2}{3}} = \frac{27}{4x^2}$

Question 6

Show that

$$\frac{(8x^3)^{\frac{2}{3}} \times 0.5x^{-2}}{32^{\frac{1}{5}}} = 1$$

$(8x^3)^{\frac{2}{3}} = 4x^2$

$\frac{4\cancel{x^2} \times 0.5\cancel{x^{-2}}}{32^{\frac{1}{5}}} = \frac{2}{2} = 1$

CHALLENGE QUESTION

Write the following in the form ax^n, where a and n are rational constants to be determined.

$$\frac{(2\sqrt{x})^3 \times 3\sqrt[3]{x}}{12x^{-1.25}}$$

Simplifying with Surds

Similarly to indices, there are rules of surds which allow us to simplify them.

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$
- $(\sqrt{a})^2 = \sqrt{a^2} = a$

Example 1: Simplify the following.

a) $\sqrt{18}$

$= \sqrt{9} \times \sqrt{2}$

$= 3\sqrt{2}$

b) $\frac{\sqrt{24}}{2}$

$= \frac{\sqrt{4} \times \sqrt{6}}{2}$

$= \frac{\cancel{2}\sqrt{6}}{\cancel{2}}$

$= \sqrt{6}$

c) $3\sqrt{20} + 4\sqrt{45} - \sqrt{125}$

$= 3\sqrt{4}\sqrt{5} + 4\sqrt{9}\sqrt{5} - \sqrt{25}\sqrt{5}$

$= 6\sqrt{5} + 12\sqrt{5} - 5\sqrt{5}$

$= 13\sqrt{5}$

Example 2: Expand and simplify.

a) $\sqrt{3}(\sqrt{2} - 4)$

$= \sqrt{6} - 4\sqrt{3}$

b) $(3 + \sqrt{2})(5 - 2\sqrt{2})$

$= 15 + 5\sqrt{2} - 6\sqrt{2} - 4$

$= 11 - \sqrt{2}$

Rationalising the Denominator

We will be expected to rationalise the denominator in fractional expressions containing surds.

Example: Rationalise and simplify.

a) $\frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{10\sqrt{5}}{5}$

$= 2\sqrt{5}$

b) $\frac{1}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}}$

$= \frac{5-\sqrt{3}}{(5+\sqrt{3})(5-\sqrt{3})}$

$= \frac{5-\sqrt{3}}{22}$

c) $\frac{1}{(1+\sqrt{5})^2} \times \frac{(1-\sqrt{5})^2}{(1-\sqrt{5})^2}$

$= \frac{(1-\sqrt{5})^2}{(1+\sqrt{5})^2(1-\sqrt{5})^2}$

$= \frac{(1-\sqrt{5})^2}{16}$

Definition:

A *surd* is a root which is irrational (e.g. $\sqrt{2}$).

Test your Understanding 2

Question 1

Simplify.

a) $\sqrt{44}$ $=2\sqrt{11}$

b) $\sqrt{50}$ $=5\sqrt{2}$

c) $\sqrt{90}$ $=3\sqrt{10}$

d) $\frac{\sqrt{12}}{2}$ $=\sqrt{3}$

e) $\sqrt{20}+\sqrt{5}$ $=3\sqrt{5}$

Question 2

Simplify fully.

a) $\sqrt{28}+\sqrt{63}$ $=5\sqrt{7}$

b) $5\sqrt{80}-3\sqrt{20}$ $=14\sqrt{5}$

c) $\sqrt{12}+3\sqrt{48}-\sqrt{75}$

$2\sqrt{3}+12\sqrt{3}-5\sqrt{3}$

$=9\sqrt{3}$

Question 3

Write $\sqrt{75}+2\sqrt{12}$ in the form $n\sqrt{3}$ where n is an integer.

$5\sqrt{3}+4\sqrt{3}$ $=9\sqrt{3}$ $(n=9)$

Question 4

Expand and simplify the following.

a) $\sqrt{5}(3-\sqrt{5})$ $3\sqrt{5}-5$

b) $\sqrt{2}(\sqrt{8}+\sqrt{3})$ $4+\sqrt{6}$

c) $(5+\sqrt{3})(5-\sqrt{3})$ 22

d) $(2+3\sqrt{3})(6-5\sqrt{3})$

$8\sqrt{3}-33$

Question 5

Rationalise and simplify where possible.

a) $\frac{1}{\sqrt{6}}$ $\frac{\sqrt{6}}{6}$

b) $\frac{\sqrt{2}}{\sqrt{14}}$ $\frac{\sqrt{7}}{7}$

c) $\frac{16}{\sqrt{20}}$ $=\frac{8}{\sqrt{5}}=\frac{8\sqrt{5}}{5}$

d) $\frac{1}{2+\sqrt{3}}$ $2-\sqrt{3}$

e) $\frac{3}{2-\sqrt{3}}$ $3(2+\sqrt{3})$

f) $\frac{\sqrt{7}}{6+\sqrt{7}}$ $\frac{\sqrt{7}(6-\sqrt{7})}{29}$

g) $\frac{3-\sqrt{2}}{3+\sqrt{2}}$ $\frac{(3-\sqrt{2})^2}{7}$

h) $\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}-\sqrt{3}}$ $\frac{(\sqrt{3}-\sqrt{2})(3\sqrt{2}+\sqrt{3})}{15}$

i) $\frac{2}{(2+\sqrt{3})^2}$ $2(2-\sqrt{3})^2$

Question 6

Show that $\frac{3-2\sqrt{5}}{\sqrt{5}-1}$ can be written in the form $a+b\sqrt{5}$, where $a, b \in \mathbb{Q}$.

$\frac{(3-2\sqrt{5})(\sqrt{5}+1)}{4} = \frac{3\sqrt{5}+3-10-2\sqrt{5}}{4} = \frac{-7}{4} + \frac{\sqrt{5}}{4}$ $a = -\frac{7}{4}$

$b = \frac{1}{4}$

Question 7

A rectangle has a length of $(5+\sqrt{3})\ cm$, and an area of $66\ cm^2$. Find the exact width of the rectangle.

$w = \frac{66}{5+\sqrt{3}} = \frac{66(5-\sqrt{3})}{22} = 3(5-\sqrt{3})\ cm$

Question 8

Write the following in the form 2^n.

a) 64 2^6

b) $\frac{1}{32}$ 2^{-5}

c) $\sqrt{8}$ $2^{3/2}$

d) $\sqrt[3]{\frac{1}{16}}$ $2^{-4/3}$

CHALLENGE QUESTION

Write $2k\sqrt{12}+8\sqrt{27}-\sqrt{3k^4}$ in the form $a\sqrt{3}$, stating a in terms of k.

$= 2k\sqrt{4}\sqrt{3} + 8\sqrt{9}\sqrt{3} - \sqrt{k^4}\sqrt{3}$

$= 4k\sqrt{3} + 24\sqrt{3} - k^2\sqrt{3}$ $= (24+4k-k^2)\sqrt{3}$

<u>***Grade Enhancer – Apply your Knowledge!***</u>

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1

Showing all of your working, simplify the following expressions fully.

(a) $\dfrac{6\sqrt{x}}{(\sqrt{x}+2)^2-(\sqrt{x}-2)^2}$

[3 marks]

(b) $\dfrac{3\sqrt{5}+2\sqrt{11}}{\sqrt{11}-\sqrt{3}}$

[3 marks]

Question 2

Expand $(x+\sqrt{y})^3$

[3 marks]

Question 3

(a) Given that $y = 2^x$, express 4^x in terms of y.

[1 mark]

(b) Hence, solve the equation

$$4^x - 9(2^x) + 8 = 0$$

[4 marks]

Question 4 *(WJEC 2022)*

Showing all your working, simplify the following expression.

$$5\sqrt{48} + \frac{2+5\sqrt{3}}{5+3\sqrt{3}} - (2\sqrt{3})^2$$

[6 marks]

Question 5 *(WJEC 2019)*

Solve the following simultaneous equations.

$$3^{3x} \times 9^{y} = 27$$

$$2^{-3x} \times 8^{-y} = \frac{1}{64}$$

[6 marks]

Question 6

A triangle ABC is such that $A\hat{B}C = 90°$, $AB = \left(6 - 3\sqrt{3}\right)\,cm$ and $AC = \left(91 - 26\sqrt{3}\right)\,cm$.

Find:

(a) the length of BC,

[5 marks]

(b) the area of the triangle.

[3 marks]

Challenge Question *(STEP 1 2019)*

Find integers m and n such that $\sqrt{3 + 2\sqrt{2}} = m + n\sqrt{2}$.

Extension Question

(a) Expand and simplify $(\sqrt{x} + \sqrt{x+1})(\sqrt{x} - \sqrt{x+1})$

(b) Hence, find the value of $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{24}+\sqrt{25}}$

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