MathEVmatics

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A2 Applied Mathematics for WJEC

Unit 2b:

The (Continuous) Uniform Distribution

Examples and Practice Exercise

Please note: *All solutions have been AI-generated and, as such, may contain errors. Please check carefully and let me know if you find any errors.*

Table of Contents

Where it fits:

Specification Reference: 2.4.2 (Statistical Distributions)

Topics	Guidance
2.4.2 Statistical distributions	
Understand and use the continuous uniform distribution and Normal distributions as models. Find probabilities using the Normal distribution. Link to histograms, mean, standard deviation, points of inflection and the binomial distribution.	Use of calculator / tables to find probabilities. Linear interpolation in tables will **not** be required.
Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the continuous uniform or Normal model may not be appropriate.	The distributions from which the selection can be made are: Discrete: binomial, Poisson, uniform Continuous: Normal, uniform

Learning Objectives

- To understand the idea of a continuous probability distribution, and the idea of a probability density function (PDF);
- To understand the contexts in which a Continuous Uniform Distribution is appropriate, and to be able to find probabilities;
- To understand and use the formulae for the mean and variance of the uniform distribution.

The (Continuous) Uniform Distribution

Definition

The Continuous Uniform Distribution is the simplest continuous probability distribution. It models a situation where a continuous random variable X is equally likely to take any value within a specific interval $[a, b]$.

It is often referred to as the **Rectangular Distribution** because of the shape of its probability density function (PDF).

Notation: We write $X \sim U[a, b]$.

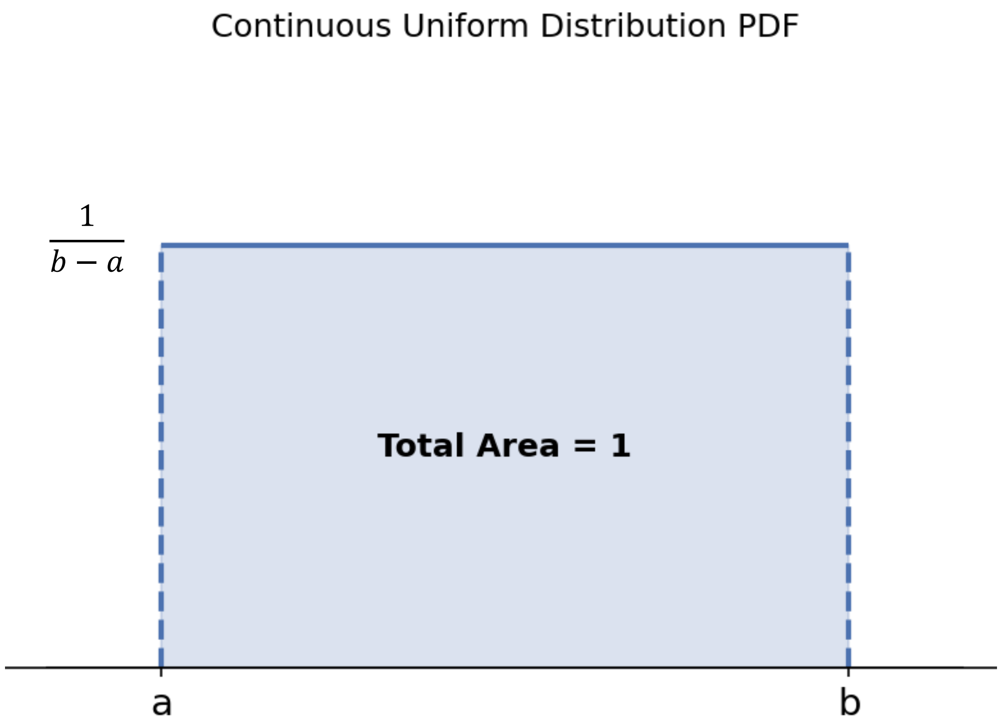
The Probability Density Function (PDF)

Since the total area under any PDF must equal 1, and the width of the interval is $(b-a)$, the height of the rectangle must be

$$\frac{1}{b-a}$$

Therefore,

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



Calculating Probabilities

For a continuous uniform distribution, probability is simply the area of a rectangle.

$$\text{Area} = \text{Width} \times \text{Height}$$

Therefore, to find the probability that X lies between x_1 and x_2:

$$P(x_1 < X < x_2) = (x_2 - x_1) \times \frac{1}{b-a} = \frac{x_2 - x_1}{b-a}$$

Example 1

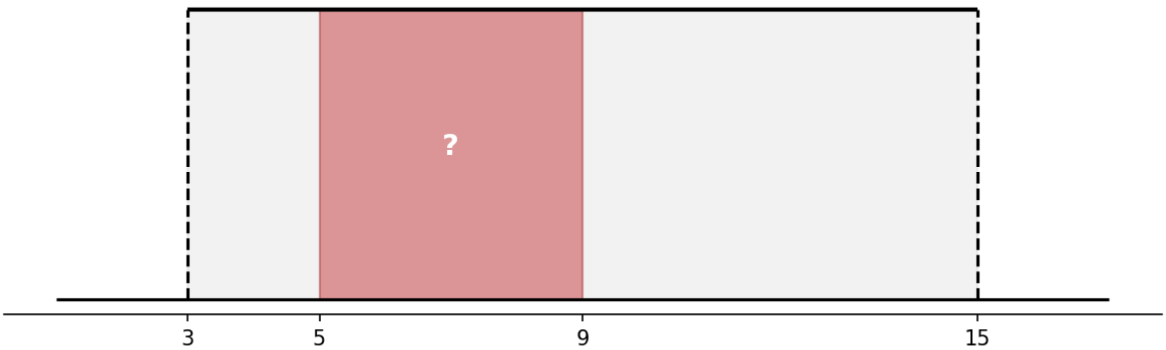
A random variable X has a continuous uniform distribution over the interval $[3, 15]$. Find:

(a) $P(5 < X \leq 9)$

(b) $P(X < 6)$

(c) $P(X > 10)$

(a) $P(5 < X \leq 9)$



The distribution is $X \sim U[3,15]$.

The length of the interval is $15 - 3 = 12$. The height of the PDF is $\frac{1}{12}$.

Width of desired region $= 9 - 5 = 4$.

$$P(5 < X \leq 9) = \text{Width} \times \text{Height} = 4 \times \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

(b) $P(X < 6)$

This is the interval from the minimum value (3) up to 6.

$$P(X < 6) = \frac{6-3}{12} = \frac{3}{12} = \frac{1}{4}$$

(c) $P(X > 10)$

This is the interval from 10 up to the maximum value (15).

$$P(X > 10) = \frac{15 - 10}{12} = \frac{5}{12}$$

Example 2

A random variable X has a continuous uniform distribution over the interval $[a, 20]$.
(a) Given that $P(X \leq 8) = 0.2$, show that $a = 5$.
(b) Find $P(10 \leq X \leq 15)$.
(c) A student claims that $P(X = 14) = 0.6$. Explain the mistake the student has made.
(d) Find $P(X < 12 \mid 8 \leq X \leq 18)$.

(a) Show that $a = 5$

The width of the total interval is $20 - a$. The height is $\frac{1}{20-a}$.
The probability $P(X \leq 8)$ represents the area from a to 8.

$$\frac{8 - a}{20 - a} = 0.2$$

$$8 - a = 0.2(20 - a)$$

$$8 - a = 4 - 0.2a$$

$$4 = 0.8a \Longrightarrow a = \frac{4}{0.8} = 5$$

(b) Find $P(10 \leq X \leq 15)$

Now we know $X \sim U[5,20]$. The total width is 15.

$$P(10 \leq X \leq 15) = \frac{15 - 10}{15} = \frac{5}{15} = \frac{1}{3}$$

(c) Explain the mistake

For any **continuous** distribution, the probability of the variable taking an exact single value is zero. The student has found $P(X < 14)$ which is 0.6.
Correct statement:

$$P(X = 12) = 0$$

(d) Find $P(X < 12 \mid 8 \leq X \leq 18)$

Using the conditional probability formula $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$:

We want

$$\frac{P(8 \leq X < 12)}{P(8 \leq X \leq 18)}$$

Numerator (Area between 8 and 12):

$$\frac{12-8}{15} = \frac{4}{15}$$

Denominator (Area between 8 and 18): Since the max value is 20, the interval $8 \leq X \leq 18$ is valid.

$$\frac{18-8}{15} = \frac{10}{15}$$

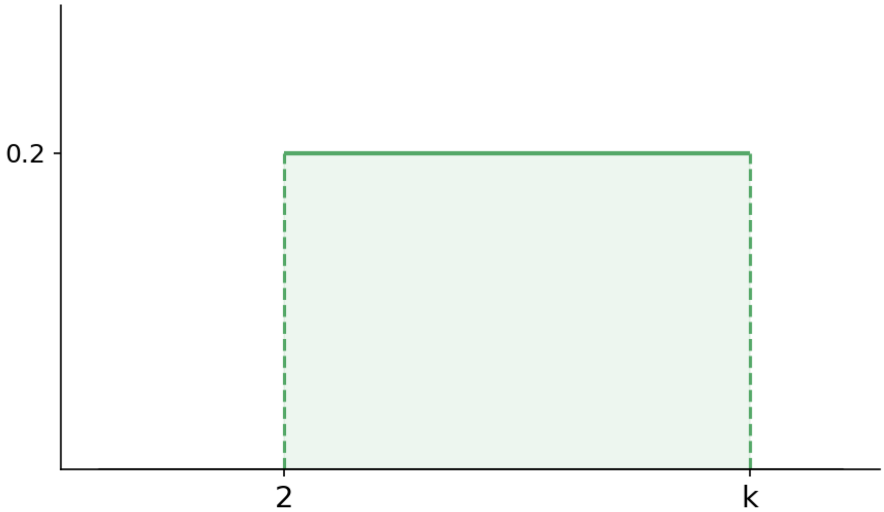
$$\text{Probability} = \frac{4/15}{10/15} = \frac{4}{10} = 0.4$$

Task 1

Given that $W \sim U[2,6]$, find $P(2.5 < W < 5)$.

Task 2

The random variable X has the probability density function shown in the diagram below.



(a) Calculate the value of k.
(b) Hence find $P(X < 3.5)$.

Task 3

Three independent random variables A, B and C each have a continuous uniform distribution over the interval $[0, 10]$.

(a) Find $P(A > 7)$.

(b) Find the probability that A, B and C are **all** greater than 7.

(c) Find the probability that **exactly one** of the variables is greater than 7.

Example 3 (Contextual Problem)

A delivery truck is scheduled to arrive at a warehouse. The time T, in minutes past 10:00 AM, that the truck arrives is modelled by a continuous uniform distribution over the interval $[10,50]$.

(a) State the latest time the truck is expected to arrive.
(b) State an assumption made when using this model.
(c) What is the probability that the truck arrives before 10:25 AM?
(d) Given that the truck has not arrived by 10:20 AM, find the probability that it arrives after 10:40 AM.
(e) A second truck's arrival time Y is modelled as $Y \sim U[15,45]$. Which truck has the higher probability of arriving within a 10-minute window?

(a) Latest Time
The interval is $[10,50]$. The max value is 50 minutes past 10:00 AM.
Time = 10:50 AM.

(b) Assumption
We assume the truck is equally likely to arrive at any moment during the 40-minute interval. (e.g., traffic conditions do not make a later arrival more probable).

(c) $P(T < 25)$

$$T \sim U[10,50]$$

Interval width $= 40$.

$$P(T < 25) = \frac{25 - 10}{40} = \frac{15}{40} = \frac{3}{8}$$

(d) $P(T > 40 \mid T > 20)$

$$= \frac{P(T > 40 \text{ AND } T > 20)}{P(T > 20)} = \frac{P(T > 40)}{P(T > 20)}$$

Numerator $P(40 < T < 50)$:

$$\frac{50 - 40}{40} = \frac{10}{40}$$

Denominator $P(20 < T < 50)$:

$$\frac{50 - 20}{40} = \frac{30}{40}$$

$$\text{Probability} = \frac{10}{30} = \frac{1}{3}$$

(e) Comparison

Truck 1 (T): Width = 40. Prob of 10-min window = $\frac{10}{40} = 0.25$.

Truck 2 (Y): Width = $45 - 15 = 30$. Prob of 10-min window = $\frac{10}{30} = 0.333\ldots$

Truck 2 has the higher probability (the distribution is narrower, so the probability density is higher).

Test Your Understanding 1

Q1. The continuous random variable X is uniformly distributed over the interval $[2, 10]$.
(a) Write down the probability density function $f(x)$.
(b) Find $P(X > 7)$.
(c) Find $P(3 < X < 8)$.

Q2. A random variable $Y \sim U[5,17]$.
(a) Find $P(Y < 8)$.
(b) Find the value of k such that $P(Y < k) = 0.25$.

Q3. A piece of string is 30cm long. It is cut at a random point L cm from one end. The location of the cut is modelled by the distribution $L \sim U[0,30]$.
(a) Find the probability that the cut is made within the first 5cm.
(b) Find the probability that the longer piece of string is at least twice the length of the shorter piece.

Q4. A random variable X has a continuous uniform distribution over the interval $[a, b]$. The width of the distribution is 12 units, and $P(X < 7) = 0.25$. Find the values of a and b.

Q5. The waiting time, W minutes, for a bus is uniformly distributed between 0 and 15 minutes.
(a) Find the probability that a passenger waits more than 10 minutes.
(b) Given that a passenger has already waited 5 minutes, find the probability that they will wait for at least another 5 minutes.

Q6. A random variable $X \sim U[-3,6]$.
Find the probability that $X^2 < 4$.
(Hint: Solve the inequality $x^2 < 4$ first to find the range of x values).

Q7. Two independent random variables X and Y are defined as $X \sim U[0,5]$ and $Y \sim U[0,5]$.
(a) Find the probability that $X > 4$.
(b) Find the probability that both X and Y are greater than 4.

Q8. The quadratic equation $x^2 + 4kx + k = 0$ has coefficients dependent on a parameter k.
The value of k is chosen randomly from the interval $U[0,1]$.
Find the probability that the equation has **real roots**.
(Hint: Use the discriminant $b^2 - 4ac \geq 0$).

Q9. In a computer game, the length of time T seconds after the game begins before a booster appears is modelled by a continuous uniform distribution over the interval [40,120].

(a) State the longest time a player will have to wait for a booster to appear according to this model.
(b) State an assumption that needs to be made in using this distribution.
(c) What is the probability that a booster appears less than 60 seconds after the game begins?
(d) Mike has been playing the game for one minute and a booster has not yet appeared. What is the probability that Mike will have to wait more than 25 seconds for the booster to appear?
(e) James plays another computer game, in which boosters appear sometime between 1 minute and 4 minutes after the game begins. Letting X seconds denote the length of time after the game begins before a booster appears, write down the distribution of X.

Q10. A drinks machine dispenses hot drinks into cups. It is programmed to cut off the flow of liquid randomly at any value between 225 and 240ml.

Letting V be the random variable "volume of liquid dispensed into a cup",

(a) Write down both the distribution and the probability density function for V.
(b) Find the probability that the machine dispenses:
    (i) At most 236ml.
    (ii) Exactly 236ml.
(c) Calculate the 90th percentile of V.
(d) Find the value of x such that $P(X \geq x) = 3P(X \leq x)$
(e) Give an interpretation, in context, of your answer to part (d).

Mean and Variance of the Uniform Distribution

The formulae for mean and variance of a Continuous Uniform Distribution are given in the WJEC formula booklet.

If $X \sim U[a, b]$, then

Mean: $\frac{1}{2}(a+b)$

Variance: $\frac{1}{12}(b-a)^2$

Since the standard deviation is simply the square root of the variance, this means that

Standard Deviation: $\sqrt{\frac{1}{12}(b-a)^2}$.

Whilst the result for the mean should be obvious (it is simply the halfway point of the rectangle), the result for the variance is generally proven using results learnt in Further Mathematics Unit 2. Proofs can easily be found online if you are interested in reading further.

Since the uniform distribution is symmetrical, it should be noted that the median and mean are equal.

Example 1:

A square has sides of length L cm, where L is a continuous random variable such that $L \sim U[d, 9]$.

(a) Given that the 20th percentile is $4.2cm$, find the value of d.
(b) Find the mean and standard deviation of L.
(c) Find the probability that the area of the square is less than $50cm^2$.

(a) Finding d

$$\frac{4.2-d}{9-d} = 0.2$$

$$\therefore d = 3$$

(b) Finding the mean and standard deviation

Mean = $\frac{3+9}{2} = 6$cm

Variance = $\frac{1}{12}(9-3)^2 = 3cm$, $\therefore$ standard deviation = $\sqrt{3}cm$.

(c) Find probability that area is less than $50cm^2$

$$P(L^2 < 50) = P(L < \sqrt{50}) = P(3 < L < \sqrt{50}) = \frac{\sqrt{50}-3}{9-3} = 0.6785.$$

Example 2:

The continuous random variable X is such that $X \sim U[a, b]$. Given that the mean is 7.5 and the variance is $\frac{3}{4}$, find the values of a and b.

$$\frac{a+b}{2} = 7.5 \qquad \frac{1}{12}(b-a)^2 = \frac{3}{4}$$

From the second equation,

$$(b-a)^2 = 9$$

Since $b > a, (b-a) = 3, \therefore b = a + 3$

Substituting this into the first equation,

$$2a + 3 = 15,$$

$$\therefore a = 6, b = 9$$

Task 1:

A string of length 20cm is cut at a randomly chosen point. Let L be the length, in cm, of the longer piece of string.

(a) Write down the distribution of L.
(b) State the mean and variance of L.
(c) Find the probability that the length of the longer piece will be less than 16cm.
(d) The longer piece of string is shaped to form a square. Find the probability that the area of the square is greater than 9 cm^2.

Task 2:

Mr Evans arrives at the train station at a random point in time after school. The trains at the station are scheduled to leave at 15 minute intervals.

(a) Assuming that Mr Evans gets on the next train,
   - **i.** Suggest an appropriate distribution to model his waiting time;
   - **ii.** State the mean and variance of this distribution;

(b) The probability that Mr Evans misses the next available train due to making a phone call is 0.25. If he misses the first available train, he is certain to get on the next train.
   - **i.** Find the probability that Mr Evans waits between 12 and 20 minutes.
   - **ii.** Given that Mr Evans waits between 12 and 20 minutes, find the probability that he catches the first train.

Test Your Understanding 2

1. The continuous random variable X is uniformly distributed on $[2,6]$. Find:
(a) $P(2.5 < X < 4.5)$

(b) The value of k such that $P(3X < k - X) = 0.3$.

2. The continuous random variable X is uniformly distributed on the interval $[2,8]$. Find:
(a) $P(3.5 < X < 6)$

(b) $P(X > 5 \mid X > 3)$

(c) The 80^{th} percentile of X.

3. The continuous random variable X is uniformly distributed over the interval $[10,20]$.
(a) State the mean and variance of X.
(b) Find the interquartile range of X.
(c) Find $P(12 < X < 16 \mid X < 18)$.

4. The continuous random variable $Y \sim U[a, b]$. Given that $P(Y < 8) = 0.2$ and that $P(Y > 12) = 0.4$, find the value of a and the value of b.

5. The continuous random variable $Y \sim U[a, b]$. Given that the mean of Y is 4 and the variance is 3, find the value of a and the value of b.

6. The radius of a copper pipe, R cm, is modelled as a continuous uniform random variable

$$R \sim U[2,6].$$

(a) Find the mean and standard deviation of R.
(b) Let A denote the cross-sectional area of the pipe. Find $P(A \leq 16\pi)$.

7. The time in minutes that a customer waits on hold at a call centre follows a continuous uniform distribution defined over the interval $[1,9]$. Find:
(a) The probability that a customer waits more than 6 minutes on one call.
(b) The probability that a customer waits less than 3 minutes on each of three successive calls.
(c) The probability that a customer waits less than 5 minutes, given that they have already waited 2 minutes.

8. A cereal dispenser fills bowls with cornflakes. It is electronically controlled to cut off the flow randomly between 40g and 60g. The random variable X is the mass of cereal dispensed.
(a) Specify the distribution of X.
(b) Find the probability that the machine dispenses:
(i) Less than 45g.
(ii) Exactly 45g.
(c) Calculate the interquartile range of X.
(d) Determine the value of x such that $P(X \geq x) = 0.8$.
(e) Sarah fills five bowls from the dispenser. Find the probability that exactly two bowls contain less than 45g.

9. In the quadratic equation $y^2 + ky + k + 3 = 0$, the value of k is chosen at random from the interval $[-4,4]$.
Calculate the probability that the equation will have real roots.

10. A tour bus departs from a terminal every 20 minutes. A tourist arrives at the terminal at a random point in time.

(a) Suggest an appropriate distribution to model the tourist's waiting time and give the parameters.
(b) State the mean and variance of this distribution.
(c) State an assumption you have made in suggesting this distribution.
(d) Now assume that the probability that the tourist misses the next available bus due to buying a snack is 0.1. If they miss the first bus, they are certain to catch the next one.
(i) Find the probability that the tourist waits between 15 and 25 minutes.
(ii) Given that the tourist waits between 15 and 25 minutes, find the probability that they missed the first bus.

11. A string PQ of length 12cm is cut at a random place C into two pieces PC and CQ.
(a) Write down the probability distribution of the random variable X, where X cm denotes the length PC.
(b) Sketch the probability distribution of X.
(c) Find the probability that the shorter piece has length less than 4cm.
(d) If a rectangle is drawn with side lengths equal to PC and CQ, find the probability that the area of the rectangle is greater than $32cm^2$.

12. An isosceles triangle ABC has $AB = AC = 5$cm. The angle $B\hat{A}C$, denoted by θ, is a random variable uniformly distributed on the interval $\left(0, \frac{5\pi}{6}\right)$. The length BC is denoted by X cm.

(a) Show that $X = 10\sin\left(\frac{\theta}{2}\right)$.

(b) Find $P(X \leq 5)$.

(c) Find the probability that the area of the triangle is greater than $\frac{25}{4}cm^2$.

13. (Challenge Q) A string of length L is cut at a point chosen at random on the string. Let A be the area of a rectangle, whose sides are formed by these pieces.

(a) If X denotes the length of the longer piece of string, write down the distribution of X.

(b) Give an expression for the length of the shorter piece of string in terms of L and X.

(c) Show that $A > \frac{2L^2}{9}$ means that $X^2 - LX + \frac{2L^2}{9} < 0$.

(d) Hence, find the probability that the area of the rectangle exceeds $\frac{2L^2}{9}$.

14. (Challenge Q) The continuous random variable X represents the error, in mm, made when a machine cuts iron sheeting to a target length. The distribution of X is uniform over the interval [-2, 2].

(a) Find $P(X < -0.75)$

(b) Find $P(|X| \leq 1.8)$

A supervisor takes a random sample of 10 lengths of sheeting cut by the machine.

(c) Find the probability that more than half of the sample are within 1.8mm of the target length.

On average, the machine produces 600 cuts per hour. If a cut is $1.8mm$ or more outside the target length, it is disposed of.

(d) Show that, in a randomly selected hour, the chance that at least 90 cuts have to be rejected is approximately 0.02%.

Teacher Solutions

Uniform Distribution - Task 1

$W \sim U[2,6]$. Width $= 4$.

$$P(2.5 < W < 5) = \frac{5-2.5}{4} = \frac{2.5}{4} = 0.625$$

Task 2

(a) Area must be 1. Width $= k - 2$. Height $= 0.2$.

$$0.2(k-2) = 1 \Longrightarrow k - 2 = 5 \Longrightarrow k = 7$$

(b)

$$P(X < 3.5) = 0.2 \times (3.5 - 2) = 0.2 \times 1.5 = 0.3$$

Task 3

(a) Width $= 10$. Interval > 7 is $[7,10]$ (length 3).

$$P(A > 7) = \frac{3}{10} = 0.3$$

(b) Independence means we multiply probabilities.

$$P(A > 7) \times P(B > 7) \times P(C > 7) = 0.3^3 = 0.027$$

(c) This is a Binomial situation where $n = 3, p = 0.3$. We want 1 success.

$$3 \times (0.3)^1 \times (0.7)^2 = 3 \times 0.3 \times 0.49 = 0.441$$

Test Your Understanding 1 - Solutions

Q1.

(a)

$$f(x) = \begin{cases} \frac{1}{8}, & \text{for } 2 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$\frac{10-7}{8} = \frac{3}{8}$$

(c)

$$\frac{8-3}{8} = \frac{5}{8}$$

Q2.

(a) Width $= 12$.

$$P(Y < 8) = \frac{8-5}{12} = \frac{3}{12} = 0.25$$

(b)

$$\frac{k-5}{12} = 0.25 \Longrightarrow k - 5 = 3 \Longrightarrow k = 8$$

Q3.

(a) $\frac{5}{30} = \frac{1}{6}$

(b) Let cut be at x. Pieces are x and $30 - x$.
Case 1: x is the longer piece.

$$x \geq 2(30 - x) \Longrightarrow 3x \geq 60 \Longrightarrow x \geq 20$$

Case 2: $30 - x$ is longer.

$$30 - x \geq 2x \Longrightarrow 30 \geq 3x \Longrightarrow x \leq 10$$

Valid regions: $[0,10]$ and $[20,30]$. Total length $= 20$.
Required probability $= \frac{20}{30} = \frac{2}{3}$

Q4.

Width $= b - a = 12$.

$$P(X < 7) = \frac{7-a}{12} = 0.25$$

$$7 - a = 3 \Longrightarrow a = 4$$

$$b = 4 + 12 = 16$$

So $a = 4, b = 16$.

Q5.

(a) $W \sim U[0,15]$.

$$P(W > 10) = \frac{5}{15} = \frac{1}{3}$$

(b) We want

$$P(W \geq 10 \mid W \geq 5)$$

Wait is total time. "Another 5 mins" means total wait ≥ 10.

$$P(W \geq 10 \mid W \geq 5) = \frac{P(W \geq 10)}{P(W \geq 5)} = \frac{5/15}{10/15} = \frac{5}{10} = 0.5$$

Q6.

$$X^2 < 4 \Longrightarrow -2 < X < 2$$

Distribution is $U[-3,6]$, width $= 9$. Interval $(-2,2)$ has length 4. Therefore

$$P(X^2 < 4) = \frac{4}{9}$$

Q7.

(a) $\frac{1}{5}$

(b) $\frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$ (0.04).

Q8.

Discriminant $\Delta = (4k)^2 - 4(1)(k) = 16k^2 - 4k$.
For real roots, $\Delta \geq 0$, $\therefore 16k^2 - 4k \geq 0 \Longrightarrow 4k(4k-1) \geq 0$. Critical values: $k = 0, k = 0.25$.
Curve is "U" shaped, so we need "outside" the roots: $k \leq 0$ or $k \geq 0.25$.
Since $k \sim U[0,1]$, k cannot be <0, $\therefore$ we need $0.25 \leq k \leq 1$.
Length of valid interval $= 0.75$.
Total interval $= 1$.
Probability $= 0.75$.

Q9.

Using $T \sim U[40, 120]$:

(a) 2 minutes (or 120 seconds)

(b) E.g. A booster is equally likely to appear within any subintervals of equal length between 40 and 120 seconds **or** A booster can appear at any moment within the interval.

(c) $P(T < 60) = \frac{60-40}{120-40} = \frac{1}{4}$

(d) $P(T > 85 | T \geq 60) = \frac{P(T>85)}{P(T\geq 60)} = \frac{35}{60} = \frac{7}{12}$

(e) $X \sim U[60, 240]$

Q10.

(a) $V \sim U[225, 240]$

$$f(x) = \begin{cases} \frac{1}{15}, & \text{for } 225 \leq x \leq 240 \\ 0, & \text{otherwise} \end{cases}$$

(b) $P(V \leq 236) = \frac{236-225}{240-225} = \frac{11}{15}$

(c) The 90th percentile means that 90% of the data lies beneath this value, i.e. we want

$$P(V \leq x) = 0.9$$

$$\frac{x - 225}{240 - 225} = 0.9$$

$$x = 238.5$$

(d)

$$P(V \geq x) = 3P(V \leq x)$$

Since $P(V \geq x) = 1 - P(V \leq x)$,

$$1 - P(V \leq x) = 3P(V \leq x)$$

$$\therefore 4P(V \leq x) = 1$$

$$P(V \leq x) = \frac{1}{4}$$

$$\frac{x - 225}{240 - 225} = \frac{1}{4}$$

$$x = 228.75\text{ml}$$

(e) E.g. One-quarter of the cups dispensed by the machine have 228.75ml or less.

Mean and Variance Task 1: The String Problem

Part (a): Distribution of L

$$L \sim U[10,20]$$

The Probability Density Function (PDF) is:

$$f(l) = \frac{1}{20-10} = \frac{1}{10} \text{ for } 10 \le l \le 20$$

Part (b): Mean and Variance

For a continuous uniform distribution $U(a,b)$, the mean and variance are given by:

$$Mean = \frac{a+b}{2} \text{ and } Variance = \frac{(b-a)^2}{12}$$

Given $a = 10$ and $b = 20$:

Mean: $E(L) = \frac{10+20}{2} = \frac{30}{2} = 15$ cm

Variance: $\mathrm{Var}(L) = \frac{(20-10)^2}{12} = \frac{10^2}{12} = \frac{100}{12} = \frac{25}{3} \approx 8.33$ cm^2

Part (c): Probability $L < 16$

We require $P(L < 16)$.

$$P(L<16) = \frac{16-10}{20-10} = \frac{3}{5}$$

Part (d): Probability Area $> 9cm^2$

The longer piece of string (length L) is used to form a square. Let s be the side length of the square.

$$\text{Perimeter} = 4s = L \Longrightarrow s = \frac{L}{4}$$

The area A of the square is s^2: $A = \left(\frac{L}{4}\right)^2 = \frac{L^2}{16}$

We require the probability that the Area is greater than 9:

$$P(A > 9) = P\left(\frac{L^2}{16} > 9\right)$$

$$P(L^2 > 144)$$

Taking the square root (length must be positive):

$$P(L > 12)$$

$$= P(12 < L < 20)$$

$$= \frac{20-12}{20-10} = \frac{4}{5}$$

Task 2: Mr Evans and the Train

(a): Distribution of Waiting Time

$$W \sim U[0,15]$$

The PDF is:

$$f(w) = \frac{1}{15} \text{ for } 0 \leq w \leq 15$$

(b): Mean and Variance

Using standard formulae for $U[0,15]$:

Mean:

$$\frac{0+15}{2} = 7.5$$

Variance:

$$\frac{(15-0)^2}{12} = \frac{225}{12} = 18.75$$

(c): Probability Total Wait is between 12 and 20 minutes

Let T be the **total** waiting time. Let C be the event he catches the first train ($P(C) = 0.75$).
Let M be the event he misses the first train ($P(M) = 0.25$).

We are looking for $P(12 < T < 20)$. We must consider two mutually exclusive scenarios:

Scenario 1: He catches the first train.

- In this case, total wait $T = W$.
- We need $P(12 < W < 20)$.
- Since W is defined on $[0, 15]$, the valid range is

$$12 < W < 15$$

Probability for the wait time:

$$P(12 < W < 15) = \frac{15-12}{15} = \frac{3}{15} = 0.2$$

Joint probability for Scenario 1:

$$P(\text{Scenario 1}) = P(12 < W < 15) \times P(C) = 0.2 \times 0.75 = 0.15$$

Scenario 2: He misses the first train.

- If his wait time $W > 15$ then he must catch the second train. He will be waiting between 0 and five further minutes here.

Probability for the wait time:

$$P(0 < W < 5) = \frac{5-0}{15} = \frac{5}{15} = \frac{1}{3}$$

Joint probability for Scenario 2:

$$P(\text{Scenario 2}) = P(0 < W < 5) \times P(M) = \frac{1}{3} \times 0.25 = \frac{1}{12}$$

Total Probability:

$$P(12 < T < 20) = 0.15 + \frac{1}{12}$$
$$= \frac{3}{20} + \frac{1}{12}$$
$$= \frac{9}{60} + \frac{5}{60} = \frac{14}{60}$$
$$= \frac{7}{30}$$

Part (d): Conditional Probability

We need the probability that he catches the first train, **given** that he waits between 12 and 20 minutes.

Let A be the event "Catches first train".

Let B be the event "Waits between 12 and 20 minutes" ($12 < T < 20$).

We require $P(A \mid B)$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- **Numerator** $P(A \cap B)$**:** This is the probability that he waits between 12 and 20 minutes AND catches the first train. This is exactly **Scenario 1** calculated in part (c).

$$P(A \cap B) = 0.15 = \frac{3}{20}$$

- **Denominator** $P(B)$**:** This is the total probability calculated in part (c).

$$P(B) = \frac{7}{30}$$

Calculation:

$$P(A \mid B) = \frac{3/20}{7/30}$$

$$= \frac{9}{14}$$

Test Your Understanding 2 – Worked Solutions

1.

(a) $X \sim U[2,6]$. PDF height

$$= \frac{1}{6-2} = \frac{1}{4}$$

$$P(2.5 < X < 4.5) = \frac{4.5 - 2.5}{4} = \frac{2}{4} = 0.5$$

(b)

$$3X < k - X \Longrightarrow 4X < k \Longrightarrow X < \frac{k}{4}$$

$$P\left(X < \frac{k}{4}\right) = \frac{\frac{k}{4} - 2}{4} = 0.3$$

$$\frac{k}{4} - 2 = 1.2 \Longrightarrow \frac{k}{4} = 3.2 \Longrightarrow k = 12.8$$

2.

(a) $X \sim U[2,8]$. Denominator

$$8 - 2 = 6$$

$$P(3.5 < X < 6) = \frac{6 - 3.5}{6} = \frac{2.5}{6} = \frac{5}{12}$$

(b)

$$P(X > 5 \mid X > 3) = \frac{P(X > 5)}{P(X > 3)} = \frac{\frac{8-5}{6}}{\frac{8-3}{6}} = \frac{3}{5} = 0.6$$

(c) Let k be the 80th percentile.

$$\frac{k-2}{6} = 0.8 \Longrightarrow k - 2 = 4.8 \Longrightarrow k = 6.8$$

3.

(a) $X \sim U[10,20]$.

Mean

$$= \frac{10 + 20}{2} = 15$$

Variance

$$= \frac{(20-10)^2}{12} = \frac{100}{12} = \frac{25}{3} \approx 8.33$$

(b) IQR $= Q_3 - Q_1$.

$$Q_1 = 10 + 0.25(10) = 12.5$$

$$Q_3 = 10 + 0.75(10) = 17.5$$

IQR

$$= 17.5 - 12.5 = 5$$

(c)

$$P(12 < X < 16 \mid X < 18) = \frac{P(12 < X < 16)}{P(X < 18)}$$

(since range $12 - 16$ is inside $10 - 18$).

$$= \frac{\frac{16-12}{10}}{\frac{18-10}{10}} = \frac{4}{8} = 0.5$$

4.

PDF height is $\frac{1}{b-a}$.

$$P(Y < 8) = \frac{8-a}{b-a} = 0.2$$

$$\Longrightarrow 8 - a = 0.2b - 0.2a$$

$$\Longrightarrow 0.2b + 0.8a = 8$$

$$\Longrightarrow b + 4a = 40$$

(Eq 1).

$$P(Y > 12) = \frac{b-12}{b-a} = 0.4$$

$$\Longrightarrow b - 12 = 0.4b - 0.4a$$

$$\Longrightarrow 0.6b + 0.4a = 12$$

$$\Longrightarrow 3b + 2a = 60$$

(Eq 2).

From Eq 1: $b = 40 - 4a$. Substitute into Eq 2:

$$3(40 - 4a) + 2a = 60$$

$$\Longrightarrow 120 - 12a + 2a = 60$$

$$\Longrightarrow 10a = 60$$

$$\Longrightarrow a = 6$$

$$b = 40 - 24 = 16$$

$$\therefore a = 6, b = 16$$

5.
Mean

$$\mu = \frac{a+b}{2} = 4 \Longrightarrow a + b = 8$$

Variance

$$\sigma^2 = \frac{(b-a)^2}{12} = 3 \Longrightarrow (b-a)^2 = 36$$

Since $b > a$,

$$b - a = 6$$

Adding equations:

$$(a+b) + (b-a) = 8 + 6 \Longrightarrow 2b = 14 \Longrightarrow b = 7$$

Substituting back:

$$a + 7 = 8 \Longrightarrow a = 1$$

6.
(a) $R \sim U[2,6]$.
Mean

$$= \frac{2+6}{2} = 4$$

Variance

$$= \frac{(6-2)^2}{12} = \frac{16}{12} = \frac{4}{3}$$

Standard Deviation

$$= \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \approx 1.155$$

(b) $A = \pi R^2$

$$P(A \leq 16\pi) = P(\pi R^2 \leq 16\pi) = P(R^2 \leq 16) = P(R \leq 4)$$
$$P(R \leq 4) = \frac{4-2}{6-2} = \frac{2}{4} = 0.5$$

7.
(a)

$$T \sim U[1,9]$$

$$P(T > 6) = \frac{9 - 6}{8} = \frac{3}{8}$$

(b)

$$P(T < 3) = \frac{3 - 1}{8} = \frac{2}{8} = \frac{1}{4}$$

Probability for 3 successive calls

$$= \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

(c)

$$P(T < 5 \mid T > 2) = \frac{P(2 < T < 5)}{P(T > 2)}$$

$$= \frac{3}{7}$$

8.

(a)

$$X \sim U[40,60]$$

(b) (i)

$$P(X < 45) = \frac{45 - 40}{20} = \frac{5}{20} = 0.25$$

(ii)

$$P(X = 45) = 0$$

(c)

$$IQR = (60 - 40) \times 0.5 = 10$$

(Or calculate $Q_3 = 55, Q_1 = 45$).

(d)

$$P(X \geq x) = \frac{60 - x}{20} = 0.8$$

$$60 - x = 16 \Longrightarrow x = 44$$

e) Let Y be the number of bowls <45g.

$$Y \sim B(5,0.25)$$

$$P(Y = 2) = \binom{5}{2}(0.25)^2(0.75)^3 = 10 \times 0.0625 \times 0.421875 \approx 0.2637$$

9. Real roots implies discriminant $D \geq 0$.

$$y^2 + ky + (k+3) = 0$$

$$D = k^2 - 4(1)(k+3) \geq 0 \Longrightarrow k^2 - 4k - 12 \geq 0$$

$$(k-6)(k+2) \geq 0$$

Critical values $6, -2$
Condition: $k \leq -2$ or $k \geq 6$

Given $k \sim U[-4,4]$, possible valid range is $[-4,-2]$. (Note: $k \geq 6$ is outside the distribution).
Length of valid range $= (-2) - (-4) = 2$. Total length $= 4 - (-4) = 8$.
Probability

$$= \frac{2}{8} = 0.25$$

10.
(a) $W \sim U[0,20]$. W is waiting time for the *next scheduled* bus.
(b) Mean $= 10$. Variance

$$= \frac{20^2}{12} = \frac{400}{12} = \frac{100}{3} \approx 33.3$$

(c) Assuming e.g. tourist arrival is truly random and independent of bus schedule.
(d) (i) Let T be total wait time.

Scenario A: Catches 1st bus ($p = 0.9$). Wait $W \sim U[0,20]$. Need

$$15 < W < 25$$

Valid range in $[0,20]$ is $15 < \mathrm{W} < 20$
Probability is $0.9 \times \frac{5}{20} = 0.9 \times 0.25 = 0.225$.

Scenario B: Misses 1st bus ($p = 0.1$). In this case $0 < W < 5$ is his wait time for the next bus.

Length 5.
Probability is $0.1 \times \frac{5}{20} = 0.1 \times 0.25 = 0.025$.

Total Prob $= 0.225 + 0.025 = 0.25$

(ii)

$$P(\text{Missed 1st} \mid 15 < T < 25) = \frac{P(\text{Missed 1st} \cap \text{Wait 15-25})}{P(\text{Wait 15-25})}$$

numerator is Scenario B result 0.025

denominator is Total Prob 0.25

Result $= \frac{0.025}{0.25} = 0.1$

11.

(a) $X \sim U[0,12]$.

(b) Rectangle from $x = 0$ to $x = 12$ with height $\frac{1}{12}$.

(c) Shorter piece <4. This happens if $PC < 4$ OR $PC > 8$ (so $CQ < 4$).

Range

$$[0,4]$$

and

$$[8,12]$$

Total length 8.

Prob

$$= \frac{8}{12} = \frac{2}{3}$$

(d) Area

$$A = X(12 - X)$$

Need

$$12X - X^2 > 32 \Longrightarrow X^2 - 12X + 32 < 0$$

$$(X - 4)(X - 8) < 0$$

$$4 < X < 8$$

Length of interval $= 4$

Prob

$$= \frac{4}{12} = \frac{1}{3}$$

12.

(a) Split isosceles triangle down the middle. Hypotenuse 5, angle $\frac{\theta}{2}$. Opposite side is $\frac{X}{2}$.

$$\sin\left(\frac{\theta}{2}\right) = \frac{\frac{X}{2}}{5} \Longrightarrow \frac{X}{10} = \sin\left(\frac{\theta}{2}\right) \Longrightarrow X = 10\sin\left(\frac{\theta}{2}\right)$$

(b)

$$X \leq 5 \Longrightarrow 10\sin\left(\frac{\theta}{2}\right) \leq 5 \Longrightarrow \sin\left(\frac{\theta}{2}\right) \leq 0.5$$
$$\frac{\theta}{2} \leq \frac{\pi}{6}$$
$$\Longrightarrow \theta \leq \frac{\pi}{3}$$

The length of this valid interval is $\frac{\pi}{3}$.

The length of the total interval is $\frac{5\pi}{6}$.

$$P(X \leq 5) = \frac{\text{Valid Length}}{\text{Total Length}} = \frac{\pi/3}{5\pi/6} = \frac{2\pi/6}{5\pi/6} = \frac{2}{5} \text{ (or 0.4)}$$

(c) Area of triangle

$$= \frac{1}{2}ab\sin C = \frac{1}{2}(5)(5)\sin\theta = 12.5\sin\theta$$

We require the Area > 6.25.

$$12.5\sin\theta > 6.25$$
$$\sin\ \theta > 0.5$$

We look for solutions for θ within the distribution domain

$$\left[0, \frac{5\pi}{6}\right]$$

$$\sin\ \theta = 0.5$$

at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

The sine function is greater than 0.5 between these values:

$$\frac{\pi}{6} < \theta < \frac{5\pi}{6}$$

This entire range lies within (or is exactly bounded by) the distribution interval

$$\left[0, \frac{5\pi}{6}\right]$$

Length of valid interval $= \frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

$$P(\text{Area} > 6.25) = \frac{2\pi/3}{5\pi/6} = \frac{4\pi/6}{5\pi/6} = \frac{4}{5} \text{ (or 0.8)}$$

13.

(a) L is total length. Cut at random. Longer piece X must be between $L/2$ and L. $X \sim U[L/2, L]$. (PDF height $\frac{1}{L/2} = \frac{2}{L}$).

(b) Shorter piece $= L - X$.

(c) Area $A = X(L - X)$.

$$X(L - X) > \frac{2L^2}{9} \Rightarrow LX - X^2 - \frac{2L^2}{9} > 0$$

Multiply by -1:

$$X^2 - LX + \frac{2L^2}{9} < 0$$

(d) Roots of

$$X^2 - LX + \frac{2L^2}{9} = 0$$

$$X = \frac{L \pm \sqrt{L^2 - 4\left(\frac{2L^2}{9}\right)}}{2}$$

$$= \frac{L \pm \sqrt{L^2 - \frac{8L^2}{9}}}{2}$$

$$= \frac{L \pm \sqrt{\frac{L^2}{9}}}{2}$$

$$= \frac{L \pm L/3}{2}$$

$$X_1 = \frac{4L/3}{2} = \frac{2L}{3}$$

$$X_2 = \frac{2L/3}{2} = \frac{L}{3}$$

Inequality holds for

$$\frac{L}{3} < X < \frac{2L}{3}$$

Intersect with distribution domain

$$X \in [L/2, L]$$

Valid range:

$$\left[L/2, \frac{2L}{3}\right]$$

Length of valid range

$$= \frac{2L}{3} - \frac{L}{2} = \frac{4L - 3L}{6} = \frac{L}{6}$$

Probability

$$= \frac{\text{Valid Length}}{\text{Total Length}} = \frac{L/6}{L/2} = \frac{1}{3}$$

14.

(a): Find $P(X < -0.75)$

The random variable X follows a continuous uniform distribution on the interval $[-2,2]$.

$$X \sim U[-2,2]$$

$$\begin{aligned} P(X < -0.75) \quad &= \frac{-0.75 - (-2)}{4} \\ &= \frac{1.25}{4} \\ &= 0.3125 \left(\text{or } \frac{5}{16}\right) \end{aligned}$$

(b): Find $P(X \leq 1.8)$

The inequality $|X| \leq 1.8$ is equivalent to:

$$-1.8 \leq X \leq 1.8$$

$$\begin{aligned} P(|X| \leq 1.8) \quad &= \frac{1.8 - (-1.8)}{4} \\ &= \frac{3.6}{4} \\ &= 0.9 \end{aligned}$$

(c): Probability that more than half of the sample are within 1.8mm

Let Y be the random variable representing the number of cuts in the sample that are within 1.8mm of the target.
From part (b), the probability of a single cut being within 1.8mm is $p = 0.9$.
The sample size is $n = 10$.

We model this with a Binomial distribution: $Y \sim B(10,0.9)$

"More than half" of a sample of 10 means 6, 7, 8, 9, or 10.

$$P(Y > 5) = P(Y \geq 6)$$

It is computationally efficient to assume the variable F represents the "failures" (cuts not within limits), where $F \sim B(10{,}0.1)$.
The condition "at least 6 successes" ($Y \geq 6$) is equivalent to "at most 4 failures" ($F \leq 4$).

Using cumulative binomial tables or a calculator for $B(10{,}0.1)$:

$$\begin{aligned} P(Y \geq 6) &= P(F \leq 4) \\ &= P(F = 0) + P(F = 1) + P(F = 2) + P(F = 3) + P(F = 4) \\ &\approx 0.3487 + 0.3874 + 0.1937 + 0.0574 + 0.0112 \\ &= 0.9984 \end{aligned}$$

Part (d): Poisson Approximation for Rejected Cuts

Let R be the random variable representing the number of rejected cuts in a randomly selected hour.

- The machine produces $n = 600$ cuts per hour.
- A cut is rejected if it is "1.8mm or more outside the target length".
- The probability of rejection, p, is $1 - P(\text{within } 1.8) = 1 - 0.9 = 0.1$

1. Establish the Poisson Model:

$$R \sim Po(60)$$

2. Calculate the Probability:
We require the probability that at least 90 cuts are rejected:

$$P(R \geq 90)$$

$$= 1 - P(R \leq 89)$$

$$\approx 0.0181 \approx 0.02\%$$