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A2 Mathematics for WJEC

Unit 14 - Proof by Contradiction

Examples and Practice Exercises

Learning Objectives

- Understand what is meant by 'proof by contradiction';
- Be able to perform proof by contradiction in various cases where it is specified as the method required.

Proof by contradiction is, possibly, the second-most powerful mathematical technique to prove statements.

The logic is as follows: Let's suppose that we want to prove a statement, A.

We start off by assuming the **negation** of A.

We show, that by following sound mathematical steps, this leads to an absurdity/falsehood.

Since all of the steps were sound, the only thing which could be wrong is the assumption itself.

Since the negation was wrong, that means that the original statement A had to be true.

Definition: The ***negation*** *of a statement A is an alternative statement which asserts the falsehood of A, e.g., if A was the statement "all doors are wooden," the negation would be "There exists at least one door that is not wooden.".*

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Example 1 - By contradiction, prove the statement:

"There is no largest odd integer."

Example 2 – By contradiction, prove the statement:

"For any integer n, if n^2 is even then n must also be even."

Example 3 – By contradiction, prove the statement:

"There are an infinite number of primes."

Example 4 – By contradiction, prove the statement:

"If x is real and positive, then $x + \frac{49}{x} \geq 14$***."*** *(WJEC 2008)*

You will see from the Test Your Understanding that WJEC love this type of question!

One of the examiner's favourite questions at A2 level relates to proving the irrationality of surds.

To complete this proof, we have to understand two important things:

- A ***rational*** number is a number which can be written in the form $\frac{a}{b}$ where a and b are integers.
- All rational numbers have a simplest form, i.e. where the numerator and denominator share no common factors.

Example 5 – By contradiction, prove the statement:

"$\sqrt{2}$ is irrational."

This is the most likely thing they will examine on this topic. They might make it $\sqrt{3}, \sqrt{5}$ etc – but the method is identical each time!

Test Your Understanding – Exercise

Question 1

Write down the negation of the statement: “All multiples of 2 are even.”

Question 2

Prove by contradiction the statement: “There is no largest even integer.”

Question 3

Prove by contradiction the statement: “For all $n \in \mathbb{Z}$, if n^3 is even then n must be even.”

Question 4

Prove by contradiction the statement: “For all integers $n \in \mathbb{Z}$**,** if n^2 is odd, then n is odd.”

Question 5

Prove by contradiction the statement: “$\sqrt{3}$ is irrational.”

Question 6

Prove by contradiction that, for all values of $\emptyset$,

$$sin\emptyset \ + \ cos\emptyset \leq 2$$

Question 6 *(WJEC 2010)*

Prove by contradiction the following proposition.

If a, b are positive real numbers, then $a + b \geqslant 2\sqrt{ab}$.

The first line of the proof is given below.

Assume that positive real numbers a, b exist such that $a + b < 2\sqrt{ab}$. [3]

Question 6 *(WJEC 2011)*

Prove by contradiction that, when x is real and positive,

$$4x + \frac{9}{x} \geq 12$$

Question 7 *(WJEC 2018)*

Prove by contradiction that, for every real number x such that $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\sin x + \cos x \geqslant 1.$$ [4]

Question 8 *(WJEC 2019)*

Use proof by contradiction to show that $\sqrt{6}$ is irrational. [5]

Question 9 *(WJEC 2012)*

Complete the following proof by contradiction to show that $\sqrt{5}$ is irrational.

Assume that $\sqrt{5}$ is rational. Then $\sqrt{5}$ may be written in the form $\frac{a}{b}$, where a, b are integers having no common factors.

$\therefore$ $a^2 = 5b^2$.
$\therefore$ *a^2 has a factor 5.*
$\therefore$ *a has a factor 5 so that $a = 5k$, where k is an integer.* [3]

Question 10 *(WJEC 2024)*

Prove by contradiction the following proposition:

When x is real and positive, $x + \frac{81}{x} \geqslant 18$. [4]

Now you have completed the unit…

Objective	Met	Know	Mastered
• *Understand what a "Proof by Contradiction" involves.*			
• *Understand and be able to write a suitable negation to a statement.*			
• *Perform simple proofs by contradiction.*			
• *Perform exam-style proofs by contradiction, e.g. to show irrationality of a surd.*			

Space for notes and/or things to develop: