MathEVmatics

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AS Mathematics for WJEC

Proof

Examples and Practice Exercises

Learning Objectives

- Understand how to disprove a statement by counter-example;
- Understand what is meant by **proof by deduction**, and be able to complete simple proofs by deduction using standard algebraic techniques;
- Understand what is meant by **proof by exhaustion**, and be able to complete simple proofs by exhaustion using standard algebraic techniques;
- Understand how to represent specific types of number (e.g. even and odd, consecutive numbers etc) using algebra.

You can show me a million examples of something working, and I still don't know it will work for EVERY value.

You only need show me one counter-example for me to know a statement is false!

This is the power of mathematical proof. It is absolute – true for every infinitely conceivable value we can imagine.

Disproof by Counter-Example

This is a reasonably simple concept to understand. You will be given a statement, and asked to show it isn't true by finding an example which 'breaks' the statement. You have met this, in simple cases, at GCSE already.

Example 1 -By means of a counter-example, disprove the statement:

"The sum of two distinct irrational numbers is irrational."

Example 2 – By means of a counter-example, disprove the statement:

"The product of two distinct irrational numbers is irrational."

Example 3 – By means of a counter-example, disprove the statement:

"The sum of two consecutive primes is always even."

Proof by Exhaustion

This is another reasonably simple idea to understand.

Given a result to prove for a **finite** number of values, we can verify the result by checking every case.

Example 4

Show that the sum of two consecutive square numbers between 100 and 200 inclusive is an odd number.

Example 5

Show that, for integer values of n such that $2 \leq n \leq 7$, m = n + 1 is not divisible by 10.

Example 6

a) Prove that, for integer values of n such that $1 \leq n \leq 5$, $n^2 + n + 11$ is prime.

b) By means of a counter-example, show that $n^2 + n + 11$ is not prime for all integer values of n.

Proof by Deduction

You have already met several examples of this without realising. For example, the proofs of the laws of logarithms are proofs by deduction.

All a proof by deduction really means is that, from a logical starting point, we complete a series of mathematically valid steps to arrive at the answer we wish to prove.

Example 7 – Prove that $(x+1)(x+2)(x+3) \equiv x^3 + 6x^2 + 11x + 6$ for all values of x.

Example 8 – Prove that $x^2 - 2x + 6$ is positive for all values of x.

Some slightly more challenging proofs can require us to write our own algebra. We should be prepared for anything they can throw at us!

An even number	
An odd number	
Two consecutive numbers	
Two consecutive odd numbers	
The square of an odd number	
The sum of two consecutive integers	
A multiple of 4	
One less than a multiple of 5	
The sum of the squares of two consecutive integers	
A power of 2	
Two distinct odd numbers	
The sum of the squares of two consecutive odd integers	

Example 9 – Prove that the sum of three consecutive integers is always a multiple of 3.

Example 10 – Prove that the sum of the squares of two consecutive odd integers is always 2 more than a multiple of 8.

Test Your Understanding – Exercise

Question 1

Disprove the statement "$n^2 - n + 3$ is prime for all integers n".

Question 2

Prove that $(2x + 1)(x + 6)(x - 5) \equiv 2x^3 + 3x^2 - 59x - 30$.

Question 3

Prove that every odd integer between 2 and 20 is either prime, or a product of no more than three primes.

Question 4

Show that, for $1 \leq n \leq 3$, $2^n - 1$ is not always prime.

Question 5

Give a counter-example to disprove the statement "An irrational number divided by a different irrational number is irrational."

Question 6

Prove that $x^2 - 2x + 10 > 0$ for all real values of x.

Question 7

Prove that the sum of two consecutive odd integers is always a multiple of 4.

Question 8

Prove that $(2n + 3)^2 - (2n - 3)^2$ is always a multiple of 6 for all integer values of n.

Question 9

Prove that, for any positive integer n, $n^3 + n$ is an even integer.

Question 10

Prove that $\left(a^3-b^3\right)=\left(a-b\right)\left(a^2+ab+b^2\right)$ for all values of a and b.

Question 11

Let p be a prime number such that $2<p<25$. Prove, by exhaustion, that for all such p, $\left(p-1\right)\left(p+1\right)$ is divisible by 8.

Question 12 *(WJEC 2017)*

Show, by counter-example, that the statement

"*If x is an acute angle than $\sin(x+30)>\sin x$*"

is false.

Question 13

Disprove by counter-example the statement: "If two functions f(x) and g(x) are such that f'(x) = g'(x), then f(x) = g(x)".

Question 14

Find a counter-example to show that the following statement is false:

"*If a is rational and b is irrational, then $\log_a b$ is irrational.*"

Question 15

Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

Challenge for the Criminally Insane

If the circle $\left(x-a\right)^2+\left(y-b\right)^2=r^2$ and the line $y=mx+c$ <u>do not</u> meet, prove that

$$m^2\left(r^2-a^2\right)+2am\left(b-c\right)+2bc-b^2-c^2+r^2<0.$$

End of Topic Skill Check

Question 1 *(WJEC 2018)*

In each of the two statements below, c and d are real numbers. One of the statements is true, while the other is false.

A : $(2c - d)^2 = 4c^2 - d^2$, for all values of c and d.

B : $8c^3 - d^3 = (2c - d)(4c^2 + 2cd + d^2)$, for all values of c and d.

a) Identify the statement which is false. Show, by counter example, that this statement is in fact false. [2]

b) Identify the statement which is true. Give a proof to show that this statement is in fact true. [2]

Question 2 *(WJEC 2017)*

Show, by counter-example, that the following statement is false.

'If $\dfrac{7x - 200}{x} > 5$, then $x > 100$.' [2]

Question 3 *(WJEC 2016)*

Show, by counter-example, that the following statement is false.

'If the integers a, b, c, d are such that a is a factor of c and b is a factor of d, then $(a + b)$ is a factor of $(c + d)$.' [2]

Question 4 *(WJEC 2023)*

Show, by counter example, that the following statement is false.

"For all positive integer values of n, $n^2 + 1$ is a prime number." [3]

Question 5 *(WJEC 2022)*

In each of the two statements below, x and y are real numbers. One of the statements is true while the other is false.

A : $x^2 + y^2 \geqslant 2xy$, for all real values of x and y.

B : $x + y \geqslant 2\sqrt{xy}$, for all real values of x and y.

a) Identify the statement which is false. Find a counter example to show that this statement is in fact false. [3]

b) Identify the statement which is true. Give a proof to show that this statement is in fact true. [2]

Question 6 *(WJEC 2019)*

Given that n is an integer such that $1 \leqslant n \leqslant 4$, prove that $2n^2 + 5$ is a prime number. [3]

Now you have completed the unit…

Objective	Met	Know	Mastered
• *Disproof by counter-example.*			
• *Understand and use simple proofs by exhaustion.*			
• *Be able to express certain types of number (e.g. odd, one less than a square) algebraically.*			
• *Understand and create simple proofs by deduction.*			