MathEVmatics

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AS Mathematics for WJEC

Unit 4: Algebraic Division & The Factor Theorem

Examples and Practice Exercises

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Unit Learning Objectives

- Be able to perform algebraic division;
- Understand and use the Factor Theorem in both directions;
- Use the Factor Theorem to factorise and solve cubic equations.

We already have a range of tools for factorising and solving quadratic equations.

We will now extend this idea to higher-order polynomials.

Definition: A polynomial is an algebraic expression consisting of two or more terms with positive integer powers.

The degree of the polynomial is the highest power of the expression; i.e. a quadratic expression is a polynomial of degree 2, a cubic expression is a polynomial of degree 3 and so on.

Algebraic Division

We need to be able to divide polynomials algebraically. This is an important skill which needs to be practiced until you are able to perform it fluently and confidently.

Example 1: Divide $x^3 + 10x^2 + 31x + 30$ by $(x + 5)$.

As always, we should take special care when negatives are involved!

Example 2: Divide $2x^3 - 3x^2 - 5x + 6$ by $(x - 2)$, and hence fully factorise the cubic.

We can use this method with quartics and beyond, too!

Example 3: Divide $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $(x - 1)$.

Examiner Tip: Watch out for questions where one of the terms of the polynomial appears to be 'missing' – these are a classic examiner favourite!

Example 4: Divide $x^3 + 4x^2 - 3$ by $(x + 1)$.

Not everything we divide by will be a factor! We can use this method to find remainders too!

Example 5: Find the remainder when we divide $2x^3 + x^2 - 16x - 17$ by $(x - 3)$.

Bonus Task: Substitute x = 3 into the original function – what do you notice?

Test Your Understanding 1

Question 1

Write the following polynomials in for the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a) $x^3 + 5x^2 + 11x + 10$ by $(x + 2)$.

b) $x^3 + x^2 - 3x - 3$ by $(x + 1)$.

c) $2x^3 + 6x^2 + 9x + 5$ by $(x + 1)$.

d) $4x^3 - 5x^2 - 23x + 6$ by $(x - 3)$.

e) $-3x^3 + 13x^2 + 15x - 25$ by $(x - 5)$.

f*) $x^3 + x + 10$ by $(x + 2)$.

Question 2

Given that $(x + 3)$ is a factor of each of the following cubics, factorise each cubic fully.

a) $x^3 + 6x^2 + 11x + 6$

b) $-x^3 - 3x^2 + 25x + 75$

Question 3

Divide:

a) $x^4 + 6x^3 + 5x^2 - 16x - 16$ by $(x + 4)$.

b) $x^4 - 9x^3 + 18x^2 + 9x - 4$ by $(x - 4)$.

c) $6x^4 + x^3 + 4x^2 + 19x + 10$ by $(3x + 2)$.

Question 4

a) Divide $x^3 + x^2 - 12$ by $(x - 2)$.

b) Hence, show that $x = 2$ is the only solution to the cubic equation $x^3 + x^2 - 12 = 0$.

Question 5

Show that $x^3 - 2x^2 + 5x - 10 \equiv (x - 2)(x^2 + 5)$ by division.

Question 6

Find the remainder when:

a) $x^3 + x^2 - 5x + 2$ is divided by $(x - 2)$

b) $2x^3 + 5x^2 - 4x - 35$ is divided by $(x + 3)$

c) $-6x^3 + x^2 + 38x + 19$ is divided by $(3x - 2)$

Question 7

Show that the remainder is 5 when $3x^3 - 2x^2 + 4$ is divided by $(x - 1)$.

Question 8

Show that, when $4x^4 - 12x^3 - 21x^2 + 17x + 14$ is divided by $(2x + 1)$, the remainder is 2.

Question 9

Simplify $\frac{4x^3-39x-9}{x+3}$

Question 10

Divide $x^4 - 81$ by $(x - 3)$

Question 11

$$f(x) = 2x^3 + 5x^2 - x - 6$$

a) Find the remainder when $f(x)$ is divided by $(x + 1)$

b) Given that $(x - 1)$ is a factor of $f(x)$, fully factorise $f(x)$.

c) Hence write down all of the real roots of the equation $f(x) = 0$, and sketch $y = f(x)$.

CHALLENGE QUESTION

Divide $x^4 - x^2 - 12$ by $(x^2 + 3)$ and hence fully factorise the expression.

Solutions on page 20.

The Factor Theorem

Quick Task:

$f(x) = x^2 + 4x - 12$

a) By factorising, solve $f(x) = 0$

b) Substitute each of your x-values into the original $f(x)$ – what do you notice?

The **Factor Theorem** states that, for a polynomial $f(x)$,

- If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$;
- If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

This is an incredibly powerful result which allows us to factorise and solve cubic (and higher degree) polynomials.

If given a cubic function $f(x) = ax^3 + bx^2 + cx + d$

1. We can fnd a factor of $f(x)$ by substituting values into it until we find a value, p, such that $f(p) = 0$.
2. We can then divide $f(x)$ by the factor, $(x-p)$, which will give us a quotient of the form $ax^2 + bx + c$ (and no remainder, since $(x - p)$ is a factor).
3. We can then write $f(x)$ in the form $(x - p)(ax^2 + bx + c)$ and, if possible, factorise the quadratic to write $f(x)$ as the product of three linear factors.

Example 1:

a) By the factor theorem, show that (x-1) is a factor of $x^3 + 4x^2 + x - 6$.

(2 marks)

b) Hence, fully factorise $x^3 + 4x^2 + x - 6$.

(3 marks)

Examiner Tip: For part a), we MUST give the conclusion and it must contain both 'ingredients' in order to score the mark. In other words we must state that $f(a) = 0$, hence $(x - a)$ is a factor.

THE EXAMINER WILL NOT BE GENEROUS ON THIS MARK!!

WJEC love to ask candidates to factorise/solve a cubic **without** giving them the factor. Sneaky Mr/Mrs WJEC! However, we can easily overcome this problem.

Example 2

$$f(x) = 2x^3 - 7x^2 - 17x + 10$$

a) Solve the cubic equation $f(x) = 0$ **(6 marks)**

b) Hence sketch the graph of $y = f(x)$, showing clearly the coordinates of each point of intersection with the coordinate axes. **(3 marks)**

PRO examiner tip: *Most textbooks will tell you to randomly 'substitute' values into f(x) until you find a factor, however we can be cleverer than this!*

If f(x) was in factorised form, i.e. $(x + a)(x + b)(x + c)$, *then* $a \times b \times c$ *would have to multiply to 10 – in other words, we should only start off by trying values which are factors of the constant, i.e. 1, -1, 2, -2, 5, -5 etc.*

If all of these fail, we can then try each of them divided by factors of the x^3 *coefficient – this is extremely rare, however. We will look at this in the next section.*

Space for additional notes:

Worked Examination Question

The polynomial $f(x) = 2x^3 - 3x^2 + ax + 6$ is divisible by $(x + 2)$, where a is a real constant.

i) Find the value of a. **(3 marks)**

ii) Showing all your working, solve the equation $f(x) = 0$. **(4 marks)**

Test your Understanding 2

Question 1

Use the factor theorem to show that:

a) $(x-4)$ is a factor of $x^3-2x^2-11x+12$

b) $(x+3)$ is a factor of $3x^4+13x^3-5x^2-57x-18$

c) $(x-2)$ is a factor of $-2x^3+7x^2-5x-2$

Question 2

Show that $(x-5)$ is a factor of $x^3-21x-20$ and hence factorise the expression completely.

Question 3

Show that $(x+3)$ is a factor of $4x^3+4x^2-29x-15$ and hence factorise the expression completely.

Question 4

Given that each of these expressions has a factor of the form $(x \pm p)$, find the value of p and hence factorise the expression completely.

a) x^3-7x^2-x+7

b) $2x^3+3x^2-23x-12$

c) $x^3-6x^2+12x-8$

Question 5

By fully factorising the right-hand side of each equation, sketch the following graphs (showing clearly all points of intersection with the axes).

a) $y=2x^3-x^2-5x-2$

b) $y=6x^3+x^2-42x-45$

Question 6

Given that $(x-1)$ is a factor of $6x^3 - x^2 - 11x + a$, find the value of a.

Question 7

Given that $(x+2)$ is a factor of $4x^3 - mx^2 - 12$, find the value of m.

Question 8

Given that $(x+3)$ and $(x-2)$ are factors of $cx^3 + dx^2 - 9x - 18$, find the values of c and d.

Question 9

Given that $(x+5)$ and $(x-2)$ are factors of $px^3 + qx^2 - 67x - 10$, find the values of p and q.

Question 10

$g(x) = 3x^3 + 4x^2 + 13x + 12$

a) Use the factor theorem to show that $(x+1)$ is a factor of $g(x)$

b) Hence show that -1 is the only real root of the equation $g(x) = 0$

CHALLENGE QUESTION

Fully factorise $6x^4 + 7x^3 - 23x^2 - 14x + 24$, showing all of your working clearly.

Solutions on page 22.

Extending the Factor Theorem

So far we have looked at only factors of the form $(x \pm p)$, but we can extend this to find more complicated factors of the form $(px \pm q)$.

Let's consider a quadratic, e.g. $f(x) = 6x^2 + 11x - 10$.

Let's factorise this expression, then solve $f(x) = 0$ and substitute the roots into $f(x)$.

Thus, the **Factor Theorem** can be extended such that, for a polynomial $f(x)$,

- If $f\left(\frac{q}{p}\right) = 0$, then $(px - q)$ is a factor of $f(x)$;
- If $(px - q)$ is a factor of $f(x)$, then $f(\frac{q}{p}) = 0$.

This *looks* more complicated, but in reality we're just solving the bracket equal to zero, like we've always done!

Example 1

Show that $(3x - 2)$ is a factor of $3x^3 + 7x^2 - 4$.

This is simple enough! Much trickier, however, is where we have to find the factor for ourselves. This would really be the ‘A-grade’ standard on this topic, as in the next example.

Example 2

Fully factorise $8x^3 + 12x^2 - 2x - 3$.

Examiner commentary: Eek! A quick mental check shows us that neither f(1) or f(-1) will give us zero here, nor do f(3) or f(-3) help us.

This means the factorised form will look like (ax+b)(cx+d)(ex+f), so we will have to try any factors of 3 ***divided by*** *any factors of the x^3 term. In other words, we may need to try* $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$ *and so on.*

This looks daunting, but remember, we only need to find ONE factor! Once we find this, we can use long division and reduce the question to factorising a quadratic.

Test your Understanding 3

Question 1

Show that $(2x - 1)$ is a factor of $6x^3 - 5x^2 - 3x + 2$.

Question 2

Show that $(4x - 3)$ is a factor of $16x^3 + 20x^2 - 44x + 15$.

Question 3

Show that $(3x + 2)$ is a factor of $12x^3 - 16x^2 - 31x - 10$ and hence fully factorise the expression.

Question 4

$f(x) = 3x^3 + 5x^2 + 40x - 14$ has a root at $x = \frac{1}{3}$

Show that this is the only real root of $f(x)$.

Question 5

Given that $(3x - 1)$ and $(3x - 2)$ are factors of $px^3 - qx^2 - 19x + 6$, find the values of p and q.

Solutions on page 24.

Solutions – Test Your Understanding 1

Question 1

a) $(x+2)(x^2+3x+5)$

b) $(x+1)(x^2-3)$

c) $(x+1)(2x^2+4x+5)$

d) $(x-3)(4x^2+7x-2)$

e) $(x-5)(5-2x-3x^2)$

f) $(x+2)(x^2-2x+5)$

Question 2

a) $(x+3)(x+1)(x+2)$

b) $(x+3)(x+5)(5-x)$

Question 3

a) $(x+4)(x^3+2x^2-3x-4)$

b) $(x-4)(x^3-5x^2-2x+1)$

c) $(3x+2)(2x^3-x^2+2x+5)$

Question 4

a) $(x-2)(x^2+3x+6)$

b) Any valid method to show that $x^2+3x+6=0$ has no solutions, i.e. discriminant = -15 which is <0, or attempting to use the formula leads to $x=\frac{-3\pm\sqrt{-15}}{2}$ which means no real roots. Alternatively, completing the square leads to $(x+1.5)^2+3.75$, showing that the quadratic has a minimum point at (-1.5, 3.75) and thus no roots. Finally, a method of differentiation and finding the stationary point leads to this same conclusion.

Question 5

Division by (x – 2) leads to the required answer.

Question 6

a) 4 b) -32 c) 43

Question 7 and Question 8

Division leads to the required answer.

Question 9

$4x^2 - 12x - 3$

Question 10

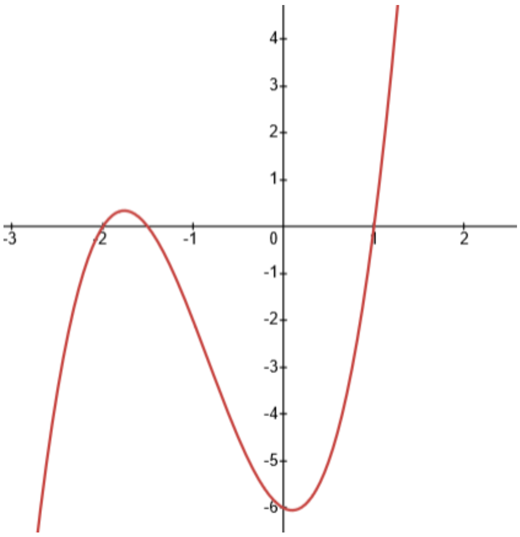
$x^3 + 3x^2 + 9x + 27$

Question 11

a) -2

b) $(x-1)(x+2)(2x+3)$

c) x = 1, x = -2, x = -3/2



CHALLENGE QUESTION

$(x^2+3)(x+2)(x-2)$

Solutions – Test Your Understanding 2

Question 1

a) Substitute x = 4, and achieve f(x) = 0 with conclusion.

b) Substitute x = -3, and achieve f(x) = 0 with conclusion.

c) Substitute x = 2, and achieve f(x) = 0 with conclusion.

Question 2

$f(5) = 0$, therefore $(x - 5)$ is a factor, and $f(x) = (x - 5)(x + 1)(x + 4)$

Question 3

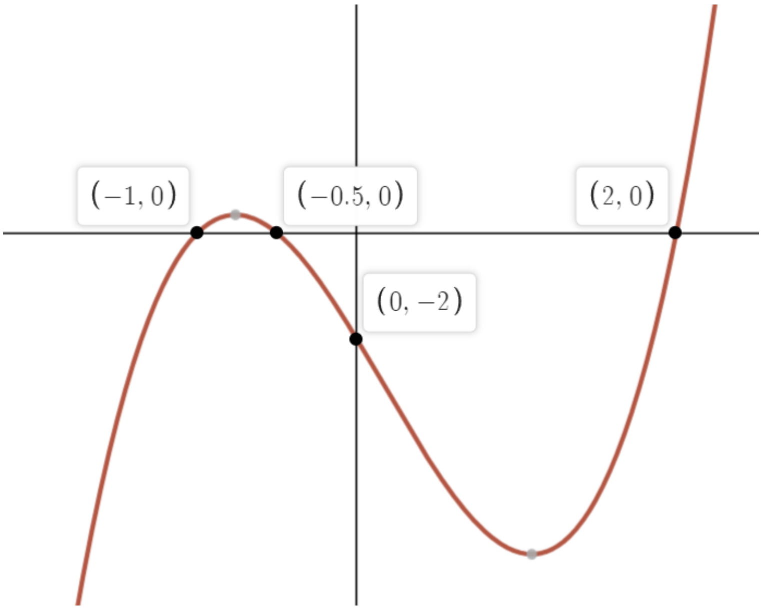
$f(-3) = 0$, therefore $(x + 3)$ is a factor, and $f(x) = (x + 3)(2x + 1)(2x - 5)$

Question 4

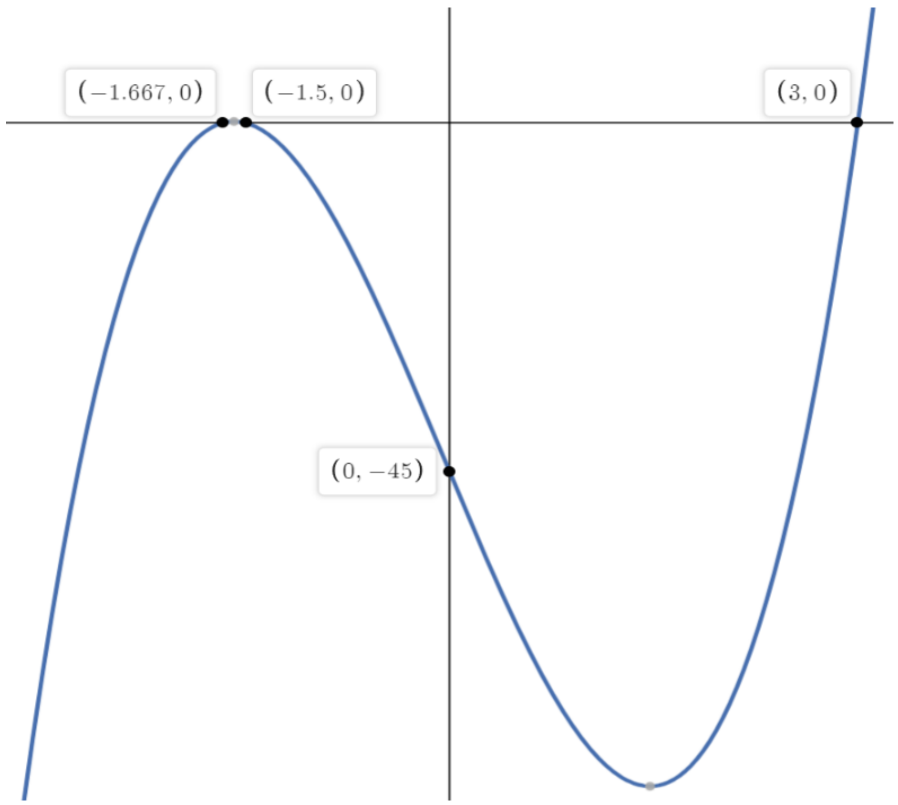
a) $(x - 7)(x + 1)(x - 1)$ b) $(x + 4)(x - 3)(2x + 1)$ c) $(x - 2)^3$

Question 5

a) $(2x + 1)(x + 1)(x - 2)$



b) $(3x + 5)(2x + 3)(x - 3)$



Question 6 $a = 6$

Question 7 $m = -11$

Question 8 $c = 2$, d= 5

Question 9 $p = 7, q = 22$

Question 10

a) Clearly shows $g(-1)$, and states conclusion "Since $g(-1) = 0$ therefore $(x + 1)$ is a factor."

b) $g(x) = (x + 1)(3x^2 + x + 12)$ followed by a valid method to show that the quadratic factor has no roots. E.g. $b^2 - 4ac = -143$ which is < 0.

CHALLENGE QUESTION

$(x - 1)(x + 2)(2x + 3)(3x - 4)$

Solutions – Test Your Understanding 3

Question 1

Show $f\left(\frac{1}{2}\right) = 0$ and give a suitable conclusion.

Question 2

Show $f\left(\frac{3}{4}\right) = 0$ and gives a suitable conclusion.

Question 3

Show $f\left(-\frac{2}{3}\right) = 0$ and gives a suitable conclusion.

Expression factorises to $(2x + 1)(3x + 2)(2x - 5)$

Question 4

If $x = \frac{1}{3}$ is a root, then $(3x - 1)$ is a factor.

Thus, $f(x)$ can be written in the form $(3x - 1)(x^2 + 2x + 14)$ (by e.g. long division)

Then, considering $x^2 + 2x + 14 = 0$, e.g. the discriminant is equal to -52, which is less than zero, thus the quadratic has no roots and therefore the given root is the only one.

Question 5

$p = 36$ and $q = 9$

Extension Material

The Remainder Theorem

This is not required knowledge by WJEC for AS or A2 mathematics, but it is a useful trick to quickly check that you have performed long division correctly!

By long division, find the remainder when $f(x) = 3x^3 + 14x^2 + 13x - 1$ is divided by $(x + 2)$.

Now, substitute $x = -2$ into $f(x)$ *(remember, when we did this with a factor, we got zero!) – what do you get?*

If you have done this correctly, you should have obtained the same remainder. Congratulations – you have discovered the Remainder Theorem! It states:

- When we divide a function $f(x)$ by $(x - a)$, the remainder is given by $f(a)$.
- When we divide a function $f(x)$ by $(px - q)$, the remainder is given by $f(\frac{q}{p})$.

You can use this result to quickly check (e.g. on your calculator) that an obtained remainder is the correct one.

Extension – Test Your Understanding

Question 1

Use the remainder theorem to determine the remainder when dividing:

a) $x^3 + 8x^2 + 17x + 16$ by $(x + 5)$

b) $x^3 - 15x^2 + 61x - 48$ by $(x - 7)$

c) $1 - 22x^2 - 6x^3$ by $(x + 2)$

In each case, verify your answer by long division.

Question 2

Given that when $x^3 - 4x^2 + 5x + c$ is divided by $(x - 2)$ the remainder is 5, find the value of the constant c.

Question 3

Given that when $2x^3 - 9x^2 + kx + 5$ is divided by $(2x - 1)$ the remainder is -2, find the value of the constant k.

Question 4

Given that the remainder is 22 when $2x^3 + ax^2 + 13$ is divided by $(x + 3)$, find the remainder when $2x^3 + ax^2 + 13$ is divided by $(x - 4)$.

Question 5

$f(x) \equiv px^3 + qx^2 + qx + 3$

a) Given that $(x + 1)$ is a factor of $f(x)$, find the value of the constant p.

b) Given further that when $f(x)$ is divided by $(x - 2)$ the remainder is 15, find the value of the constant q.

Solutions to Extension Test Your Understanding

1a) 6 b) -13 c) -39

2) c = 3

3) k = -10

4) a = 7 and thus f(4) = 253

5a) p = 3 b) q = -2

Grade Enhancer™ - Apply your Knowledge!

These past-paper or past-paper-style questions are designed to test and develop your understanding of the content learnt.

These should be completed and submitted within one week of the end of the unit.

Question 1

You are given that $f(x) = 3x^3 + 2px^2 - 4x + 5p$, where $p \in \mathbb{R}$.

Given that $(x+3)$ is a factor of $f(x)$, find the value of the constant p. [3]

Question 2

$$f(x) = 4x^3 - 12x^2 + 2x - 6$$

(a) Find the value of $f(3)$.

[1]

(b) Hence show that there is only one real solution to the equation $f(x) = 0$

[5]

Question 3

$f(x) = (x-4)(x^2 - 3x + k) - 42$ where k is a constant.

Given that $(x+2)$ is a factor of $f(x)$, find the value of *k*. [3]

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Question 4

(a) Given that $x - 3$ is a factor of $px^3 - 13x^2 - 19x + 12$, write down an equation satisfied by p. Hence show that $p = 6$. [2]

(b) Solve the equation $6x^3 - 13x^2 - 19x + 12 = 0$. [4]

Question 5

Given that $(x - 2)$ and $(x + 2)$ are factors of the polynomial $2x^3 + px^2 + qx - 12$,

a) find the values of p and q, [4]

b) determine the other factor of the polynomial. [1]

Question 6

(a) Given that $x - 2$ is a factor of $kx^3 + 2x^2 - 41x + 10$, write down an equation satisfied by k. Hence show that $k = 8$. [2]

(b) Factorise $8x^3 + 2x^2 - 41x + 10$. [3]

(c) Find the remainder when $8x^3 + 2x^2 - 41x + 10$ is divided by $2x + 1$. [2]

Question 7

Solve the equation $6x^3 + 13x^2 - 10x - 24 = 0$. [6]

Question 8

Use an **algebraic method** to solve the equation $12x^3 - 29x^2 + 7x + 6 = 0$. Show all your working. [6]

Question 9

$$f(x) = 2x^3 + 5x^2 + 2x + 15$$

(a) Verify that $(x + 3)$ is a factor of $f(x)$.

[2]

(b) Hence write $f(x)$ as the product of a linear and quadratic factor, and hence show that $f(x) = 0$ only has one real root.

[4]

(c) Hence, or otherwise, write down the real root of the equation $f(x - 3) = 0$

[1]

Question 10

$$f(x) = 4x^3 + 5x^2 - 10x + 4k, \;\; x \in \mathbb{R}, k > 0.$$

Given $(x - k)$ is a factor of $f(x)$,

(a) show that

$$k(4k^2 + 5k - 6) = 0$$

[2]

(b) Hence find the value of k, and solve the equation $f(x) = 3$

[4]

TOTAL AVAILABLE: 55 MARKS

Now you have completed the unit…

Objective	Met	Know	Mastered
Be able to perform algebraic division.			
Understand and use the Factor Theorem in both directions.			
Use the Factor Theorem to factorise and solve cubic equations.			

Notes/Areas to Develop: