MathEVmatics

Supercharge your learning

A2 Applied Mathematics for WJEC

Unit 3: The Normal Distribution

Worked Solutions

Please note: These solutions were generated with the help of ChatGPT.

They may therefore contain errors; however, statistically speaking, most should be correct…

Bah-dum-tssh.

Unit 3: Normal Distribution

Worked Solutions

Test Your Understanding 1

Question 1:

We use the **standard normal table** to find probabilities.

(a) $P(Z < 2)$

From Z-tables:

$$P(Z < 2) = 0.9772$$

(b) $P(Z \leq 0.75)$

From Z-tables:

$$P(Z \leq 0.75) = 0.7734$$

(c) $P(Z > 2.2)$

Using complement rule:

$$P(Z > 2.2) = 1 - P(Z \leq 2.2)$$

From tables:

$$P(Z \leq 2.2) = 0.9861$$

So,

$$P(Z > 2.2) = 1 - 0.9861 = 0.0139$$

(d) $P(Z \geq -1.81)$

Using complement rule:

$$P(Z \geq -1.81) = 1 - P(Z < -1.81)$$

From tables:

$$P(Z < -1.81) = 0.0351$$

So,

$$P(Z \geq -1.81) = 1 - 0.0351 = 0.9649$$

(e) $P(Z \leq -2.9)$

From Z-tables:

$$P(Z \leq -2.9) = 0.0019$$

(f) $P(Z > -0.3)$

Using complement rule:

$$P(Z > -0.3) = 1 - P(Z < -0.3)$$

From tables:

$$P(Z < -0.3) = 0.3821$$

So,

$$P(Z > -0.3) = 1 - 0.3821 = 0.6179$$

(g) $P(0.3 < Z < 1.1)$

From tables:

$$P(Z < 1.1) = 0.8643$$

$$P(Z < 0.3) = 0.6179$$

$$P(0.3 < Z < 1.1) = 0.8643 - 0.6179 = 0.2464$$

(h) $P(-0.4 \leq Z \leq 1)$

From tables:

$$P(Z < 1) = 0.8413$$

$$P(Z < -0.4) = 0.3446$$

$$P(-0.4 \leq Z \leq 1) = 0.8413 - 0.3446 = 0.4967$$

Question 3

We convert $X \sim N(124, 5^2)$ into a standard normal variable:

$$Z = \frac{X - 124}{5}$$

(a) $P(X \leq 130)$

$$Z = \frac{130 - 124}{5} = \frac{6}{5} = 1.2$$

From tables:

$$P(Z \leq 1.2) = 0.8849$$

(b) $P(X < 134)$

$$Z = \frac{134 - 124}{5} = \frac{10}{5} = 2$$

From tables:

$$P(Z < 2) = 0.9772$$

(c) $P(X \geq 127.5)$

$$Z = \frac{127.5 - 124}{5} = \frac{3.5}{5} = 0.7$$

$$P(Z \geq 0.7) = 1 - P(Z < 0.7)$$

From tables:

$$P(Z < 0.7) = 0.7580$$

$$P(Z \geq 0.7) = 1 - 0.7580 = 0.2420$$

(d) $P(X > 120)$

$$Z = \frac{120 - 124}{5} = \frac{-4}{5} = -0.8$$

$$P(Z > -0.8) = 1 - P(Z < -0.8)$$

From tables:

$$P(Z < -0.8) = 0.2119$$

$$P(Z > -0.8) = 1 - 0.2119 = 0.7881$$

(e) $P(X < 116)$

$$Z = \frac{116 - 124}{5} = \frac{-8}{5} = -1.6$$

From tables:

$$P(Z < -1.6) = 0.0548$$

(f) $P(122 \leq X \leq 126)$

Find Z-scores:

$$Z = \frac{122 - 124}{5} = -0.4, \quad Z = \frac{126 - 124}{5} = 0.4$$

$$P(Z < 0.4) = 0.6554, \quad P(Z < -0.4) = 0.3446$$

$$P(122 \leq X \leq 126) = 0.6554 - 0.3446 = 0.3108$$

Question 5

Given $X \sim N(100, 15^2)$:

(a) Two-thirds between 85 and 115

Find Z-scores:

$$Z = \frac{85 - 100}{15} = -1, \quad Z = \frac{115 - 100}{15} = 1$$

From tables:

$$P(-1 \leq Z \leq 1) = 0.8413 - 0.1587 = 0.6826 \approx 2/3$$

(b) 2% in excess of 130

$$Z = \frac{130 - 100}{15} = \frac{30}{15} = 2$$

$$P(Z > 2) = 1 - 0.9772 = 0.0228 \approx 2\%$$

(c) Probability 3 or more exceed 130

Each test is a **Bernoulli trial**:
$p = 0.0228$, $n = 30$, use **Binomial distribution** $X \sim \text{Bin}(30, 0.0228)$.

Find $P(X \geq 3)$ using binomial probabilities or Po↓n approximation.

Question 6

Given $X \sim N(30, 4^2)$:

Find Z-scores:

$$Z = \frac{26 - 30}{4} = -1, \quad Z = \frac{36 - 30}{4} = 1.5$$

$$P(X < 26) + P(X > 36) = P(Z < -1) + P(Z > 1.5)$$

From tables:

$$P(Z < -1) = 0.1587, \quad P(Z > 1.5) = 1 - 0.9332 = 0.0668$$

$$P(X < 26 \cup X > 36) = 0.1587 + 0.0668 = 0.2255$$

Question 7

Given that the volume V in each bottle follows:

$$V \sim N(1000, 25^2)$$

(a) Find $P(V < 940)$

Standardize V:

$$Z = \frac{940 \quad 1000}{25} = \frac{60}{25} = \quad 2.4$$

From standard normal tables:

$$P(Z < \quad 2.4) = 0.0082$$

(b) First, find $P(V > 1050)$:

$$Z = \frac{1050 \quad 1000}{25} = \frac{50}{25} = 2$$

From tables:

$$P(Z > 2) = 1 \quad P(Z < 2) = 1 \quad 0.9772 = 0.0228$$

Now, for 3 independent bottles, the probability that all **three** exceed 1050ml is:

$$(0.0228)^3 = 0.00001186$$

Question 8

Given:

$$W \sim N(85.4, 14^2)$$

(a) Find $P(W > 110)$

Standardize W:

$$Z = \frac{110 \quad 85.4}{14} = \frac{24.6}{14} = 1.757$$

Using normal tables:

$$P(Z < 1.76) \approx 0.9608$$

$$P(W > 110) = 1 \quad 0.9608 = 0.0392$$

$$\approx 3.92\%$$

Thus, about **3.92%** of men weigh more than 110 kg.

(b) Find the weight corresponding to the 95th percentile

We need to find w such that:

$$P(W \leq w) = 0.95$$

From normal tables:

$$Z_{0.95} = 1.645$$

Using the **inverse** Z-score formula:

$$w = \mu + Z\sigma$$

$$w = 85.4 + (1.645 \times 14)$$

$$w = 85.4 + 23.03 = 108.43$$

Thus, **to qualify for the trial, a man's weight must be at least** 108.43 **kg**.

Test Your Understanding 2

Question 1

We are given:

$$X \sim N(\mu, 3^2)$$

and

$$P(X > 30) = 0.15$$

Step 1: Find the corresponding Z-score

We need to find the Z-score for which the **upper tail probability** is (

From standard normal tables:

$$P(Z > 1.04) = 0.15$$

Thus:

$$Z = 1.04$$

Step 2: Use the standardization formula

The standard normal transformation formula is:

$$Z = \frac{X - \mu}{\sigma}$$

Substituting known values:

$$1.04 = \frac{30 - \mu}{3}$$

Step 3: Solve for μ

Rearrange:

$$30 - \mu = 1.04 \times 3$$

$$30 - \mu = 3.12$$

$$\mu = 30 - 3.12$$

$$\mu = 26.88$$

Question 2

Given:

$$Y \sim N(\mu, 5^2)$$

We convert to **Z-score**:

$$P(Y > 100) = 0.05$$

From Z-tables:

$$P(Z > 1.645) = 0.05$$

Using:

$$Z = \frac{Y - \mu}{\sigma}$$

$$1.645 = \frac{100 - \mu}{5}$$

Solving for μ:

$$\mu = 100 - (1.645 \times 5) = 100 - 8.225 = 91.775$$

$$\mu = 91.78$$

Question 3

Given:

$$W \sim N(\mu, 10^2)$$

We convert to **Z-score**:

$$P(W \leq 230) = 0.10$$

From Z-tables:

$$P(Z < -1.28) = 0.10$$

Using:

$$Z = \frac{W - \mu}{\sigma}$$

$$-1.28 = \frac{230 - \mu}{10}$$

Solving for μ:

$$\mu = 230 + (1.28 \times 10) = 230 + 12.8 = 242.8$$

$$\mu = 242.8$$

Question 4

Given:

$$X \sim N(50, \sigma^2)$$

We convert to **Z-score**:

$$P(X \leq 54) = 0.9$$

From Z-tables:

$$P(Z < 1.28) = 0.90$$

Using:

$$Z = \frac{X - \mu}{\sigma}$$

$$1.28 = \frac{54 - 50}{\sigma}$$

Solving for σ:

$$\sigma = \frac{4}{1.28} = 3.125$$

$$\sigma = 3.13$$

Question 5

Given:

$$L \sim N(3.5, \sigma^2)$$
$$P(L < 5) = 0.80$$

From Z-tables:

$$P(Z < 0.84) = 0.80$$

Using:

$$Z = \frac{5 - 3.5}{\sigma}$$
$$0.84 = \frac{1.5}{\sigma}$$

Solving for σ:

$$\sigma = \frac{1.5}{0.84} = 1.7857$$
$$\sigma = 1.79$$

Challenge

Step 1: Convert to Z-scores

From Z-tables:

$$P(Z < -1.28) = 0.10, \quad P(Z > 0.84) = 0.20$$

So,

$$Z_1 = -1.28, \quad Z_2 = 0.84$$

Using standardization:

$$Z_1 = \frac{20 - \mu}{\sigma} = -1.28$$
$$Z_2 = \frac{35 - \mu}{\sigma} = 0.84$$

Step 2: Solve for μ and σ

Rearrange:

$$\sigma = \frac{20 - \mu}{-1.28} = \frac{35 - \mu}{0.84}$$

Equating:

$$\frac{20 - \mu}{-1.28} = \frac{35 - \mu}{0.84}$$

Cross-multiplying:

$$(20 - \mu) \times 0.84 = (35 - \mu) \times (-1.28)$$
$$16.8 - 0.84\mu = -44.8 + 1.28\mu$$
$$16.8 + 44.8 = 1.28\mu + 0.84\mu$$
$$61.6 = 2.12\mu$$
$$\mu = \frac{61.6}{2.12} = 29.06$$

Now, solve for σ:

$$\sigma = \frac{20 - 29.06}{-1.28} = \frac{-9.06}{-1.28} = 7.08$$

Test Your Understanding 3

Question 1

Part (a): One-tailed test ($H_1 : \mu > 40$)

Given:

- $H_0 : \mu = 40$, $H_1 : \mu > 40$ (right-tailed test)
- $n = 20$, $\bar{x} = 44$, $\sigma = 1.5$
- Significance level: $\alpha = 0.05$

Step 1: Compute Z-score

$$Z = \frac{44 - 40}{\frac{1.5}{\sqrt{20}}}$$

$$Z = \frac{4}{\frac{1.5}{4.472}}$$

$$Z = \frac{4}{0.335} = 11.94$$

Step 2: Find critical value

For a **one-tailed test at** $\alpha = 0.05$:

$$Z_{0.05} = 1.645$$

Step 3: Decision

Since $Z = 11.94 > 1.645$, we **reject** H_0.

Conclusion: There is strong evidence to support that $\mu > 40$.

Part (b): One-tailed test ($H_1 : \mu < 30$)

Step 1: Compute the Z-score

We use the Z-test formula:

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

Substitute the given values:

$$Z = \frac{29.5 - 30}{\frac{3}{\sqrt{30}}}$$

First, calculate the denominator:

$$\frac{3}{\sqrt{30}} = \frac{3}{5.477} = 0.5477$$

Now, calculate the Z-score:

$$Z = \frac{-0.5}{0.5477} = -0.913$$

Step 2: Find the critical value

For a **left-tailed test at** $\alpha = 0.05$, we find the critical value Z_α from the Z-table:

$$Z_{\alpha=0.05} = -1.645$$

Step 3: Decision

Since $Z = -0.913$ is **greater than -1.645**, we **fail to reject** H_0.

Conclusion:

There is **not enough evidence** at the 5% level of sig↓:ance to support the claim that $\mu < 30$.

Part (c): Two-tailed test ($H_1 : \mu \neq 120$)

Step 1: Compute the Z-score

We use the Z-test formula for a population mean:

$$Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

Substitute the given values:

$$Z = \frac{122.5 - 120}{\frac{10}{\sqrt{100}}}$$

First, calculate the denominator:

$$\frac{10}{\sqrt{100}} = \frac{10}{10} = 1$$

Now, calculate the Z-score:

$$Z = \frac{2.5}{1} = 2.5$$

Step 2: Find the critical value

For a **two-tailed test at** $\alpha = 0.01$, the critical values correspond to $Z_{\alpha/2}$ and are found from the Z-table.

For $\alpha = 0.01$, the critical Z-values are:

$$Z_{\alpha/2} = \pm 2.576$$

Step 3: Decision

- The test is two-tailed, so we compare the absolute value of the calculated Z-score with the critical values.
- $|Z| = 2.5$, which is **less than 2.576**.

Conclusion:

Since $|Z| = 2.5$ is **less than the critical value** of 2.576, we **fail to reject** H_0.

Final Answer: There is **not enough evidence** at the 1% level of significance to conclude that $\mu \neq 120$.

Question 2

To determine the critical region(s) for the test statistic $\bar{X}$, we use the formula for the standardized test statistic in a hypothesis test:

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

where:

- $\bar{X}$ is the sample mean,
- μ_0 is the hypothesized population mean under H_0,
- σ is the population standard deviation,
- n is the sample size.

The critical region(s) depend on the significance level and whether the test is one-tailed or two-tailed.

(a) One-tailed test: $H_0 : \mu = 15$, $H_1 : \mu > 15$, $n = 30$, $\sigma = 2$, $\alpha = 0.05$

For a one-tailed test (right-tailed) at a 5% significance level, we find the critical value Z_α from the standard normal table:

$$Z_{0.05} = 1.645$$

The critical value for $\bar{X}$ is found by solving:

$$\bar{X} = \mu_0 + Z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} = 15 + (1.645) \cdot \frac{2}{\sqrt{30}}$$

$$\bar{X} = 15 + (1.645) \cdot (0.3651)$$

$$\bar{X} \approx 15.60$$

Critical region: $\bar{X} > 15.60$

(b) One-tailed test: $H_0 : \mu = 80$, $H_1 : \mu < 80$, $n = 40$, $\sigma = 4$, $\alpha = 0.10$

For a left-tailed test at a 10% significance level, we find the critical value Z_α:

$$Z_{0.10} = -1.28$$

The critical value for $\bar{X}$ is:

$$\bar{X} = \mu_0 + Z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} = 80 + (-1.28) \cdot \frac{4}{\sqrt{40}}$$

$$\bar{X} = 80 - (1.28) \cdot (0.6325)$$

$$\bar{X} \approx 80 - 0.81 = 79.19$$

Critical region: $\bar{X} < 79.19$

(c) Two-tailed test: $H_0 : \mu = 2.3$, $H_1 : \mu \neq 2.3$, $n = 50$, $\sigma = 0.2$, $\alpha = 0.01$

For a two-tailed test at a 1% significance level, we split α into two tails:

$$\alpha/2 = 0.005$$

From the standard normal table:

$$Z_{0.005} = 2.576$$

The critical values for $\bar{X}$ are:

$$\bar{X} = \mu_0 \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} = 2.3 \pm (2.576) \cdot \frac{0.2}{\sqrt{50}}$$

$$\bar{X} = 2.3 \pm (2.576) \cdot (0.0283)$$

$$\bar{X} = 2.3 \pm 0.073$$

Critical region: $\bar{X} < 2.227$ **or** $\bar{X} > 2.373$

Question 3

Given Information:

- Population mean: $\mu_0 = 80$
- Population standard deviation: $\sigma = 10$
- Sample size: $n = 50$
- Significance level: $\alpha = 0.05$
- One-tailed test (left-tailed), since we are testing if alcohol lowers the mean score.

(a) Finding the Critical Region

We are testing:

- **Null Hypothesis:** $H_0 : \mu = 80$ (No effect of alcohol)
- **Alternative Hypothesis:** $H_1 : \mu < 80$ (Alcohol lowers scores)

Since this is a **left-tailed test**, we find the critical value Z_α for $\alpha = 0.05$:

$$Z_{0.05} = -1.645$$

The critical value for $\bar{X}$ is:

$$\bar{X} = \mu_0 + Z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} = 80 + (-1.645) \cdot \frac{10}{\sqrt{50}}$$

$$\bar{X} = 80 - (1.645) \cdot (1.414)$$

$$\bar{X} = 80 - 2.33$$

$$\bar{X} \approx 77.67$$

Critical region: $\bar{X} < 77.67$

From part (a), the **critical region** for rejecting the null hypothesis is:

$$\bar{X} < 77.67$$

In part (b), the observed **sample mean is 77**, which is **less than 77.67** and falls within the critical region.

Since the sample mean is inside the critical region, we **reject the null hypothesis** at the 5% significance level. This suggests **strong evidence** that consuming 4 units of alcohol lowers cognitive test scores.

Question 4

Step 1: Define Hypotheses

We need to determine if the machine is incorrectly calibrated, meaning we conduct a **two-tailed test**:

- **Null Hypothesis (H_0)**: The machine is correctly calibrated.

$$H_0 : \mu = 5$$

- **Alternative Hypothesis (H_1)**: The machine is incorrectly calibrated.

$$H_1 : \mu \neq 5$$

Step 2: Compute the Test Statistic

The test statistic for the sample mean is:

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

Where:

- $\bar{X} = 5.1$ (sample mean),
- $\mu_0 = 5$ (hypothesized population mean),
- $\sigma = 0.5$ (population standard deviation),
- $n = 30$ (sample size).

Substituting the values:

$$Z = \frac{5.1 - 5}{0.5/\sqrt{30}}$$

$$Z = \frac{0.1}{0.0913}$$

$$Z \approx 1.096$$

Step 3: Determine the Critical Region

Since this is a **two-tailed test** at the **5% significance level**, we find the critical value for $\alpha/2 = 0.025$:

$$Z_{0.025} = 1.96$$

The rejection regions are:

$$Z < -1.96 \quad \text{or} \quad Z > 1.96$$

Step 4: Conclusion

The calculated test statistic is $Z \approx 1.096$, which **does not exceed** the critical value **1.96**.

Since Z **is within the acceptance region** ($-1.96 < 1.096 < 1.96$), we **fail to reject the null hypothesis**.

Step 5: Interpretation

At the **5% significance level**, there is **no sufficient evidence** to conclude that the machine is incorrectly calibrated. The sample mean of **5.1 cm** is within the expected variation, meaning the machine is likely producing screws with the correct length.

Question 5

(a) Probability that a randomly selected bag weighs less than 2 kg

We assume that the weight of porridge bags follows a normal distribution:

- Mean: $\mu = 2025$ g
- Standard deviation: $\sigma = 30$ g
- Threshold: 2 kg = **2000 g**

We need to find:

$$P(X < 2000)$$

Using the **Z-score formula**:

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = \frac{2000 - 2025}{30} = \frac{-25}{30} = -0.8333$$

Using the **Z-table**, the probability corresponding to $Z = -0.83$ is:

$$P(Z < -0.83) \approx 0.2033$$

Thus, the probability that a randomly selected bag weighs less than 2 kg is **0.2033 (or 20.33%)**.

Math EV matics

(b) Trading Standards Investigation

A sample of 36 bags is taken, and the sample mean is **2010 g**.

(i) Hypothesis Test at 5% Significance Level

We are testing whether the manufacturer's claim that $\mu = 2025$ g is correct.

- Given:
  - $\mu_0 = 2025$ g
  - $\sigma = 30$ g
  - $n = 36$
  - $\bar{X} = 2010$ g
  - $\alpha = 0.05$
- Hypotheses:
  - **Null Hypothesis** (H_0): The manufacturer's claim is correct.

$$H_0 : \mu = 2025$$

  - **Alternative Hypothesis** (H_1): The average weight is less than 2025 g (suggesting underfilled bags).

$$H_1 : \mu < 2025$$

  - This is a **left-tailed test**.

Step 1: Compute the Test Statistic

The test statistic is:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

$$Z = \frac{2010 - 2025}{30/\sqrt{36}}$$

$$Z = \frac{-15}{5} = -3.00$$

Step 2: Find the Critical Value

For a **5% left-tailed test**, the critical value from the **Z-table** is:

$$Z_{0.05} = -1.645$$

Step 3: Conclusion

Since $Z = -3.00$ **is less than** $Z_{0.05} = -1.645$, we **reject** H_0.

Interpretation:

At the 5% **significance level**, there is strong evidence that the average weight of the porridge bags is **less than** the claimed 2025 g. This supports the complaints that the manufacturer is overstating the weight.

(ii) Finding the Correct Value of σ

We now assume that the **manufacturer's claim about $\mu = 2025$ is correct**, but the **sample standard deviation σ might be different** from 30 g. Our goal is to find the correct value of σ that explains the sample mean of 2010 g.

Step 1: Use the Z-score Formula

From the hypothesis test, we used the formula:

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

Rearranging for σ:

$$\sigma = \frac{(\bar{X} - \mu) \times \sqrt{n}}{Z}$$

Step 2: Identify Given Values

- **Sample mean:** $\bar{X} = 2010$
- **Population mean (assumed correct):** $\mu = 2025$
- **Sample size:** $n = 36$
- **Significance level:** 5%
- **Critical value** for a **one-tailed test at** 5%: $Z_{0.05} = -1.645$ (since we initially rejected H_0, we should use a threshold that makes the sample just barely fall within an acceptable range)

Step 3: Compute the Correct σ

$$\sigma = \frac{(2010 - 2025) \times \sqrt{36}}{-1.645}$$

$$\sigma = \frac{(-15) \times 6}{-1.645}$$

$$\sigma = \frac{-90}{-1.645}$$

$$\sigma \approx 54.7$$

Conclusion

If the **mean truly is 2025 g**, then the correct standard deviation **must be about 54.7 g instead of 30 g**. This higher variability would explain why the sample mean was 2010 g while still being consistent with the assumed population mean.

Thus, the **manufacturer's claim about the mean** ($\mu = 2025$) **can only be correct if the actual standard deviation is significantly higher than initially claimed.**