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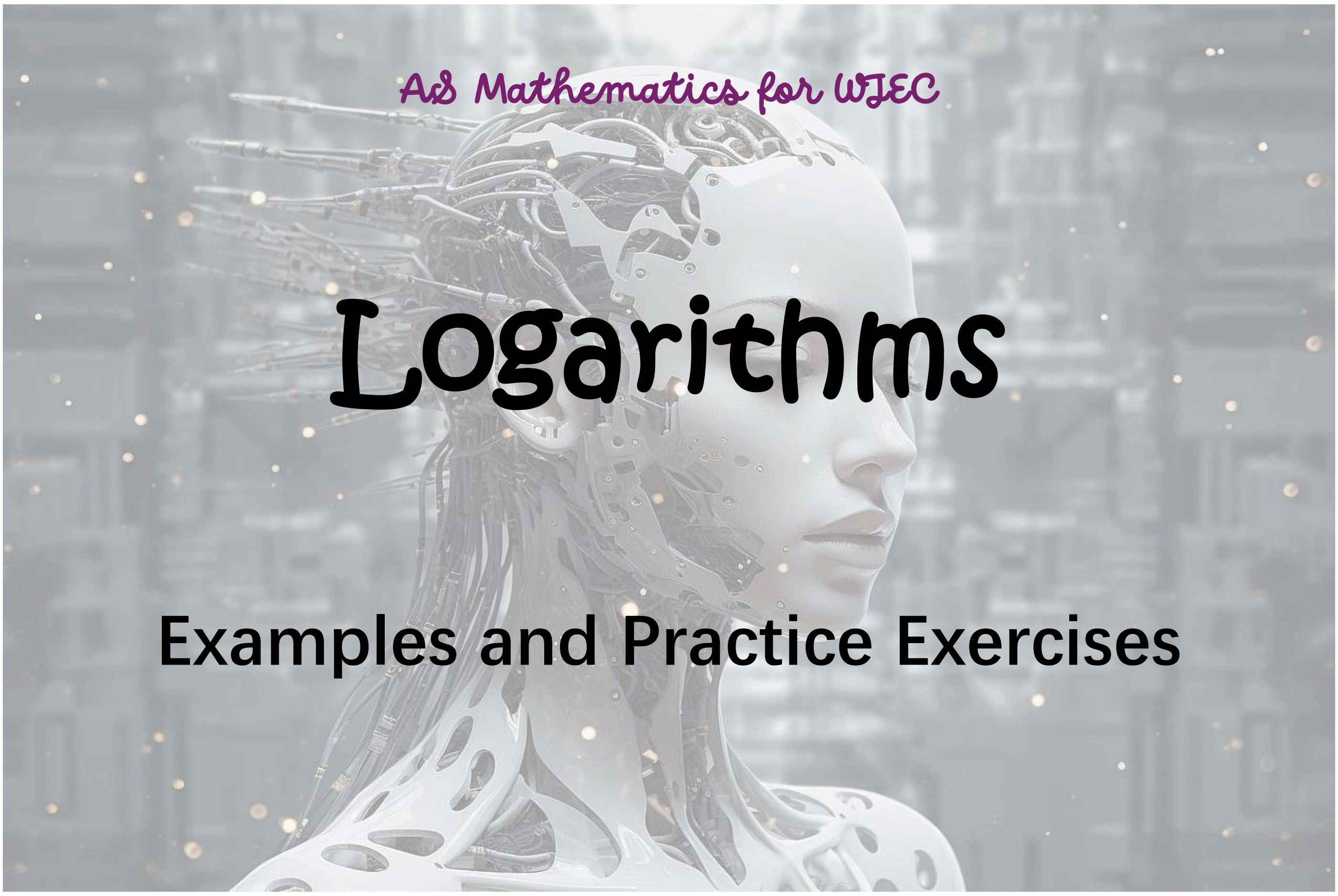


AS Mathematics for WJEC

Logarithms

Examples and Practice Exercises

Unit Learning Objectives

- To understand logarithms as the inverse (undoing) function of exponentials.
- To know, understand and use the relationship that, if $a^x = b$, then $\log_a b = x$.
- Know, use and prove the laws of logarithms.
- Be able to solve exponential and logarithmic equations, and use logarithms in the solving of exponential modelling problems.
- Understand that logarithms can be used to analyse non-linear data, and to be able to convert between logarithmic equations and equations of the form $Y = mX + c$.

Before you start this unit, it is vital that you have completed the previous unit on exponential functions.

Logarithms are vital components of pure and applied mathematics. In the real world, they are used for measuring some of the most dramatic and dangerous natural disasters such as earthquakes and volcanoes. They are also among the most desirable functions in computer science due to their ability to convert multiplication into addition, drastically improving AI efficiency and processing power in cutting-edge fields such as machine learning and artificial intelligence development.

Introducing Logarithms

Logarithms (invented by Scotsman John Napier in the early 1600s) are a tool for solving exponential problems – i.e. problems where the unknown is a power. Logarithms are, in fact, the inverse of exponentials, and this means that any relationship that can be written in terms of an exponent can also be written in a logarithmic form.

For example, something as simple as $2^x = 5$ would currently be beyond us other than by use of graphical tools or trial and improvement, yet can be solved in seconds once logarithms have been met and understood.

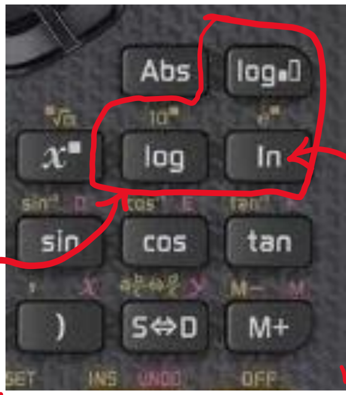
Key Relationship:

- If $a^x = b$, then $\log_a b = x$ (a must be positive and $a \neq 1$)

In other words, the 'answer' to a logarithm is the power we would need in the exponential relationship.

Going back to our earlier question, this means that if $2^x = 5$, then $\log_2 5 = x$, which you can quickly type into your calculator to find that x = __________ (to 3 d.p.).

On the 'new' Casio



This is specifically $\log_{10}$

On the 'old' Casio

This is the 'natural' logarithm – we will use this lots in Y13.

Example 1 – Write the following as logarithmic relationships.

a) $4^2 = 16$ b) $3^5 = 243$ c) $9^{\frac{1}{2}} = 3$

Example 2 – Write each of the following as exponential relationships.

a) $log_3 27 = 3$ b) $log_5 3125 = 5$ c) $log_2 \left(\frac{1}{4}\right) = -2$

Example 3 – Without using a calculator, find the value of:

a) $log_2 16$ b) $log_4 0.25$ c) $log_{0.5} 8$ d) $log_a(a^3)$

Example 4 – Use your calculator to evaluate, to 3 decimal places:

a) $log_2 15$ b) $log_4 22$ c) $log_\pi e$ d) $log_{10} 80$

Test Your Understanding – Exercise 1

Question 1

Rewrite each of the following using logarithms.

a) $3^4 = 81$ b) $2^{-3} = \frac{1}{8}$ c) $2^{10} = 1024$

d) $8^1 = 8$ e) $10^4 = 10000$ f) $(0.5)^4 = 0.0625$

Question 2

Rewrite each of the following using powers.

a) $log_2 8 = 3$ b) $log_9 27 = \frac{3}{2}$ c) $log_{10} 0.001 = -3$

d) $log_{0.2}(0.008) = 3$

Question 3

Evaluate the following **without a calculator**.

a) $log_2 32$ b) $log_5 125$ c) $log_6 6$

d) $log_8 2\sqrt{2}$ e) $log_{0.5} 4$ f) $log_k(k^5)$

Question 4

Find the value of x in each of the following, **without a calculator**.

a) $log_3 x = 2$ b) $log_x 5 = 1$ c) $log_x 64 = 3$

d) $log_3(x-2) = 2$ e) $log_2(3x+5) = 5$

Question 5

Evaluate the following to 3 d.p. using a calculator.

a) $log_8 318$ b) $log_4 71$ c) $log_e 2$

Question 6

a) Without using a calculator, explain why $log_3 20$ must take a value between 2 and 3.

b) Use your calculator to find the value of $log_3 20$ to 4 significant figures.

Question 7

a) Find the values of the following:

i) $log_2 2$ ii) $log_5 5$ iii) $log_{10} 10$

b) Explain what you notice about these answers, and why this must always be true.

Question 8

a) Find the values of the following:

i) $log_4 1$ ii) $log_7 1$ iii) $log_{10} 1$

b) Explain what you notice about these answers, and why this must always be true.

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Laws of Logarithms

We have learnt many rules (laws) for indices (powers).

Since logarithms are related to powers, it should be no surprise that there are a number of laws that we can learn and use.

The laws of logarithms:

- $\log_a x + \log_a y = \log_a(xy)$
- $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$
- $\log_a(x^n) = n\log_a x$

Some special cases we should know:

- $\log_a\left(\frac{1}{x}\right) = -\log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$

Remember that $a>0$, and $a \neq 1$

Also, x, y etc must be positive - we cannot take the log of a negative.

Examiner Tip: WJEC also expects you to be able to prove the three laws of logarithms – I have included these proofs on the next pages. Learn them!

- Proof 1: $\log_a x + \log_a y = \log_a(xy)$

- Proof 2: $log_a x - log_a y = log_a \left(\frac{x}{y}\right)$

- Proof 3: $\log_a(x^n) = n \log_a x$

Examiner tip: I know it seems daunting to 'learn' these – but you should notice that the three proofs are actually very similar. If you understand and can construct the first proof accurately, you likely can do all three!

These proofs are all provided in the solution pack.

Example 1 – Write as single logarithms:

a) $log_2 5 + log_2 8$

b) $log_3 45 - log_3 9$

c) $2\, log_3 4 + 3\, log_3 2$

d) $log_{10} 2 - 2\, log_{10}\left(\frac{1}{3}\right)$

Example 2 **–** Write in terms of $log_a p, log_a q,$ and $log_a r$

a) $log_a(pq^3r^2)$

b) $log_a\left(\frac{p^2}{q^5}\right)$

c) $log_a\left(\frac{p\sqrt{q}}{r}\right)$

d) $log_a\left(\frac{r^3}{a^2}\right)$

Additional space if required:

Example 3 – Solve the equation $log_2 4 + 2\, log_2 x = 6$

Pro tip: Remember that $\log_a x$ is only defined for positive values of x, so we need to reject any negative values that we reach.

Example 4 – Solve the equation $log_2(x + 10) - log_2(x - 3) = 3$

Test your Understanding – Exercise 2

Question 1

Write each of the following as a single logarithm, simplifying your answer if possible.

a) $log_2 3 + log_2 7$

b) $log_3 28 - log_3 4$

c) $4\, log_3 2 + log_3 10$

d) $log_4 80 - 2\, log_4 5$

e) $log_{10} 20 + 2\, log_{10} 3 - log_{10} 60$

f) $2\, log_{10} 3 + 4\, log_{10} 2$

g) $log_3 25 + log_3 10 - 3\, log_3 5$

Question 2

Write in terms of $log_a p, log_a q,$ and $log_a r$

a) $log_a(p^2qr^5)$

b) $log_a\left(\frac{q^4}{\sqrt{p}}\right)$

c) $log_a(a^3p^4)$

Question 3

Solve the following equations:

a) $log_2 5 + log_2 x = 3$

b) $log_4 20 - log_4 x = 5$

c) $2\, log_2 x = 10 - log_2 4$

d) $2\, log_9(x + 1) = 2\, log_9(2x - 3) + 1$

Question 4

a) Given that $log_3(x + 1) = 1 + 2\, log_3(x - 1)$, show that $3x^2 - 7x + 2 = 0$.

b) Hence, solve $log_3(x + 1) = 1 + 2\, log_3(x - 1)$.

Solving Equations using Logarithms

Example 1 – Solve the following, giving your answers to 3 decimal places.

a) $5^x = 30$ b) $2^{5x+3} = 41$

Example 2 – Solve the equation $3^{2x} - 7(3^x) + 12 = 0$, giving your answers to 3 significant figures.

Examiner tip: This is called a ***hidden quadratic*** *– look out for these tough questions!*

Example 3 – Solve the equation $2^x = 3^{x+2}$, giving your answer to 3 sig. figs.

This step of 'taking logs of both sides' (or TLOBS) is extremely common at both AS and A2!

Test Your Understanding – Exercise 3

Question 1

Solve the following equations, giving your answers to 2 decimal places.

a) $3^x = 30$ b) $2^x = 10$ c) $5^x = 3$

d) $3^{2x} = 80$ e) $5^{x+2} = 40$ f) $12^{3-x} = 214$

Question 2

Solve, giving your answers to 3 significant figures where appropriate.

a) $2^{2x} - 7(2^x) + 10 = 0$ b) $3^{2x} - 5(3^x) + 6 = 0$ c) $7^{2x} - 10(7^x) + 21 = 0$

d) $4^x - 3(2^x) + 2 = 0$ e) $3^{2x+1} - 7(3^x) + 2 = 0$

Question 3

Solve the following equations, giving your answers to 3 decimal places.

a) $5^x = 2^{x+3}$ b) $4^{2x+1} = 3^x$ c) $7^{x-2} = 3^{2-x}$

WJEC May 2023: Solve the following equations for values of x.

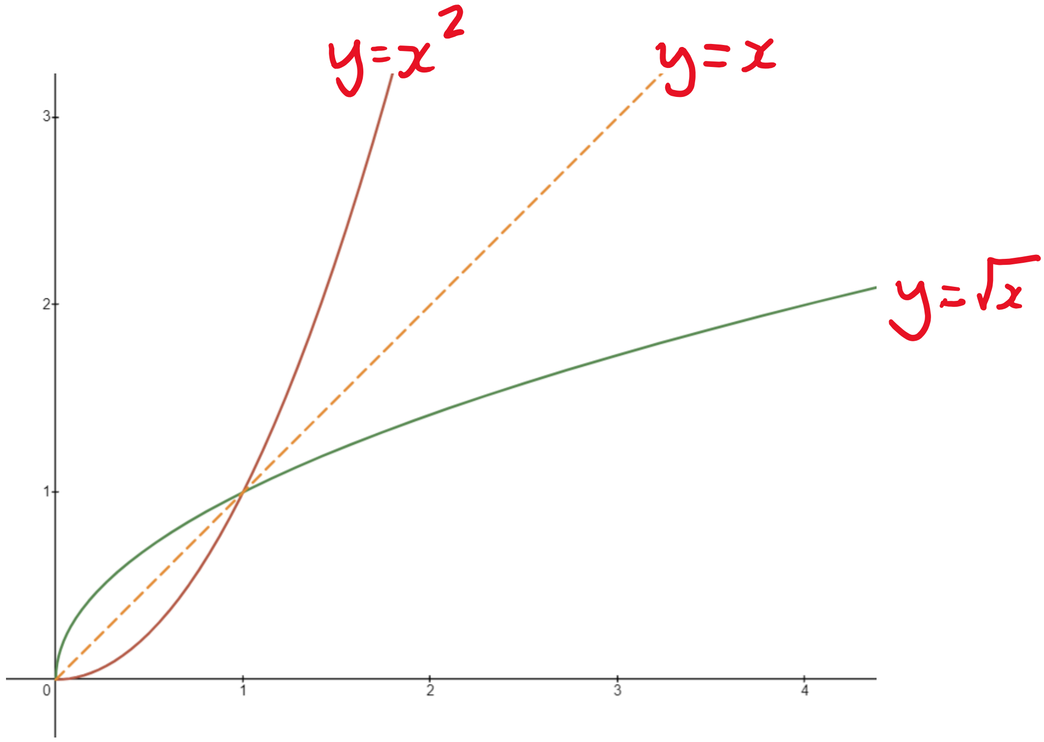
$5^{2x+1} = 14$ [3]

$3\log_7(2x) - \log_7(8x^2) + \log_7 x = \log_3 81$ [6]

Introducing the natural logarithm, $\ln x$

But first, a little pre-amble. For any graph representing a function y = f(x), the graph of the inverse function is a reflection in the line y = x.

For example, $y = x^2$ *(restricted to* $x > 0$*) and* $y = \sqrt{x}$ *are inverses of each other. Graphically:*



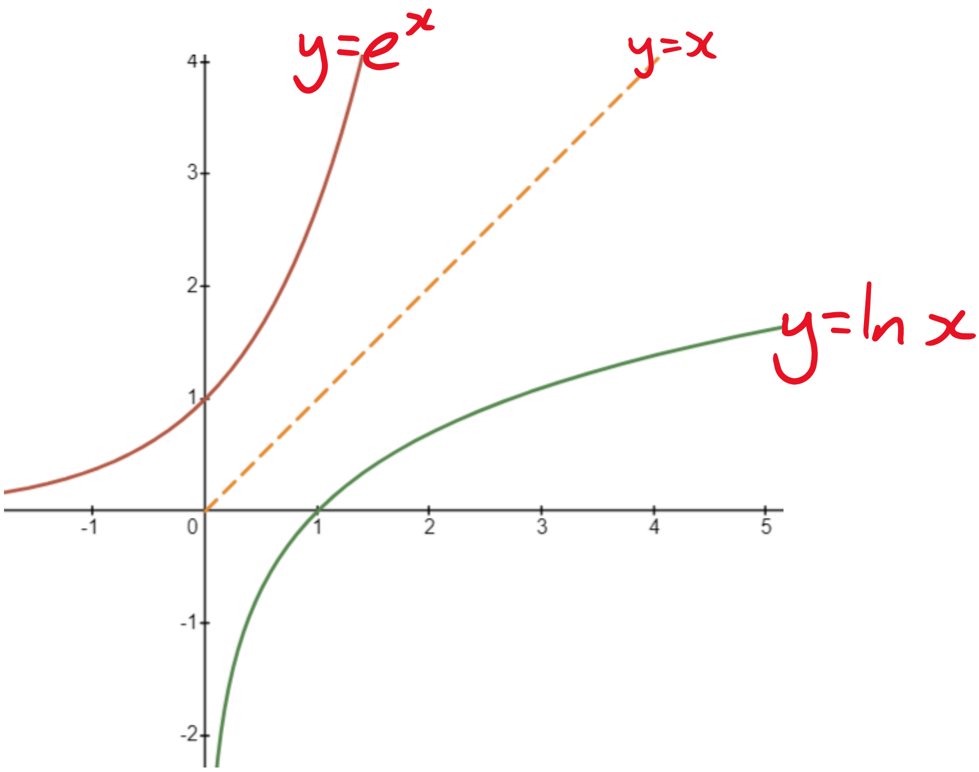
This is true for ALL functions and their inverses, and it is an important property of inverses that we will study more deeply at A2.

Because Euler's constant e is so important and useful, it is often a good idea to work with natural logarithms, that is, logarithms with a base of e. This means that the inverse function of $y = e^x$ is $y = \ln x$

Notation: The natural logarithm, $\log_e x$, is written as $\ln x$.

We are not 100% sure on the origins of 'ln' as the notation, but it is likely due to many of the finest mathematicians being speakers of French/German/Latin during the 18th and 19th centuries where these techniques were being developed.

Recall that the graph of $y = e^x$ passed through $(0,1)$ and did not cross the x-axis (i.e. y = 0 was an asymptote).
Since $y = \ln x$ is the inverse of $y = e^x$, it is a reflection in the line $y = x$:



This means that the graph of $y = \ln x$:

- Does not cross the y-axis (i.e. there is an asymptote at $x = 0$, and there is no y-intercept);
- Cannot take negative values of x. (That is, $x > 0$)
- Passes through the point $(1,0)$.

The graph continues growing forever without limit – but extremely slowly!!

This is what makes logarithmic-order algorithms incredibly powerful in computer science – we can increase the number of inputs (x) hugely without significantly increasing the run-time of an algorithm.

Since e^x and $\ln x$ are inverses, they 'undo' each other. That is:

- $e^{\ln x} = x$, and
- $\ln e^x = x$

If $\ln x = k$, then $x = e^k$ (This is just our definition of logarithms!)

Example 1 – Solve the following equations, giving your answers in exact form.

a) $\mathrm{e}^{\mathrm{x}} = 7$ b) $ln\, x = 5$

Example 2 – Solve these equations, giving your answers in exact form.

a) $e^{3x+2} = 11$ b) $2\, ln\, x + 1 = 7$ c) $e^{2x} + 4e^x - 5 = 0$

Test your Understanding – Exercise 4

Question 1

Solve the following equations, giving exact answers in terms of e or ln as appropriate.

a) $e^x = 4$ b) $ln\,x = 5$ c) $e^{3x} = 10$ d) $ln(3x) = 2$

e) $5e^{2-x} = 20$ f) $ln(3x - 5) = 4$ g) $7e^{3x} - 1 = 343$ h) $ln(12 - x) = 2$

Question 2

Solve the following equations, giving exact answers in terms of e or ln where appropriate.

a) $e^{2x} - 8e^x + 15 = 0$ b) $5e^{2x} + 3 = 16e^x$

c) $(ln\,x)^2 + 2\,ln\,x - 8 = 0$ d) $(2\,ln\,x)^2 = 13\,ln\,x + 12$

Question 3

Show that the solution to the equation $2^x e^{3x+2} = 5$ is of the form $\frac{ln\,a+b}{ln\,c+d}$ where a, b, c, d are integers to be determined.

Question 4

The concentration of a radioactive substance in a beaker of water is modelled by the equation $C = 4e^{-0.08t}$, where C is the concentration of the substance in grams per litre, and t is the time in minutes since the substance was introduced to the beaker of water.

a) Interpret the meaning of the '4' in this model.

b) Find the concentration of the radioactive substance after 10 minutes.

c) Find, to the nearest second, how long it takes for the concentration of substance to reduce to 0.25 grams per litre.

d) Find the rate of change of the decay of substance over time.

Using Logarithms to analyse non-linear data

We can use logarithms to convert exponential relationships into functions which behave in an approximately linear way on a logarithmic scale.

This is exactly how the Richter magnitude scale (used for measuring the severity of an earthquake) works. Each increase of 1 on the Richter scale (e.g. from 4 to 5) would correspond with approximately a 10-fold increase in the severity.

Example 1 – Show that an equation of the form $y = ax^n$ can be written in the form $\log y = m \log x + c$

This means that, for a graph $y = ax^n$*, the graph of log y against log x will be a straight line.*

Example 2 – Show that an equation of the form $y = ka^x$ can be written in the form $\log y = mx + c$.

This means that, for a graph $y = ka^x$*, the graph of log y against x will be a straight line.*

Example 1

A biologist monitored the growth of an algae on an exposed rock face over a 3 year period.

The area, A m^2, of the rock face covered by algae t months after the start of monitoring is modelled by the equation

$$\log_{10} A = 0.0518t + 1.0374.$$

a) Write this equation in the form $A = pq^t$, where p and q are constants to be found to 3 significant figures.

b) Use the model to find the area of rock face covered by the algae at the end of the 3-year monitoring period. Give your answer to 3 significant figures.

Example 2 (EdExcel iAL 2021, adapted)

The percentage, P, of the population of a small country who have access to the internet, is modelled by the equation

$$P = ab^t$$

where a and b are constants and t is the number of years after the start of 2005.

a) Show that this model can be written in the form $\log_{10} P = \log_{10} a + t\log_{10} b$.

Using data gathered over 5 years, a graph of $\log_{10} P$ against t was plotted. It was found to be approximately linear, with gradient 0.09 and $\log_{10} P$-intercept 0.68.

b) Use this information to find, according to the model, the values of a and b to 2 decimal places.

c) Interpret the meaning of the constant a in the model.

d) Use the model to estimate the percentage of the population with access to the internet at the start of 2015.

End of Unit Assessment

Solutions for these questions should be completed and submitted within one week of the end of this module. Remember to show sufficient workings to make your method clear.

There are 56 marks available in total; use this as a guide for how long the work should take.

Question 1

a) Solve the equation

$$7^{5-4x} = 11.$$

Show your working and give your answer correct to three decimal places. [3]

b) Solve the equation

$$\log_8 x = -\frac{1}{3}.$$ [2]

(WJEC 2014)

Question 2

a) Express

$$\frac{1}{3}\log_b x^{15} - \log_b 27x + 4\log_b \frac{3}{x}$$

as a single logarithm in its simplest form. [4]

b) Given that $\log_d 5 = \frac{1}{3}$, find the value of d. [2]

(WJEC 2017)

Question 3

a) Solve the equation

$$4^{3x+1} = 22.$$

Show your working and give your answer correct to two decimal places. [3]

b) Given that

$$\log_d z = 2\log_d 6 - \log_d 9 - 1,$$

express z in terms of d, giving your answer in a form **not** involving logarithms. [4]

(WJEC 2016)

Question 4

(a) Solve the equation

$$3^{\frac{5x}{4}-2} = 7.$$

Show your working and give your answer correct to three decimal places. [3]

(b) The positive numbers a and b are such that

$$\log_a b = 5.$$

(i) Express b as a power of a.

(ii) **Using your answer to part (i)**, evaluate $\log_b a$. [3]

(WJEC 2014)

Question 5

a) Solve $2\log_{10}x = 1 + \log_{10}5 - \log_{10}2$. [4]

b) Solve $3 = 2e^{0\cdot 5x}$. [2]

c) Express $4^x - 10 \times 2^x$ in terms of y, where $y = 2^x$.

Hence solve the equation $4^x - 10 \times 2^x = -16$. [5]

(WJEC 2018)

Question 6

Given that $x > 0$, show that

$$\log_a x^n = n \log_a x.$$ [3]

(WJEC 2016)

Question 7

Find all values of x satisfying the equation

$$\log_a(6x^2 + 9x + 2) - \log_a x = 4\log_a 2.$$ [5]

(WJEC 2015)

Question 8

Given that $x > 0$, $y > 0$, show that

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$ [3]

(WJEC 2015)

Question 9

Two quantities are related by the equation $Q = 1{\cdot}25P^3$. Explain why the graph of $\log_{10}Q$ against $\log_{10}P$ is a straight line. State the gradient of the straight line and the intercept on the $\log_{10}Q$ axis of the graph. [4]

(WJEC 2019)

Question 10

The value of a car, £V, may be modelled as a continuous variable. At time t years, the value of the car is given by $V = Ae^{kt}$, where A and k are constants. When the car is new, it is worth £30 000. When the car is two years old, it is worth £20 000. Determine the value of the car when it is six years old, giving your answer correct to the nearest £100. [6]

(WJEC 2018)

Now you have completed the unit...

Objective	Met	Know	Mastered
• *To understand logarithms as the inverse (undoing) function of exponentials.*			
• *To know, understand and use the relationship that, if* $a^x = b$, *then* $\log_a b = x$.			
• *Know, use and prove the laws of logarithms.*			
• *Be able to solve exponential and logarithmic equations, and use logarithms in the solving of exponential modelling problems.*			
• *Understand that logarithms can be used to analyse non-linear data, and to be able to convert between logarithmic equations and equations of the form* $Y = mX + c$.			

Notes/Areas to Develop: