MathEVmatics

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A2 Mathematics for WJEC

Unit 12: Integration

Examples and Practice Exercises

Unit Learning Objectives

- ***To recap on the idea of finding basic integrals, understanding this as an area under the curve;***
- ***To understand the idea of integration as the limit of a summation;***
- ***To understand and use the standard results for integration, including those given in the formula booklet;***
- ***To be able to use the reverse chain rule to integrate the form*** $\boldsymbol{f(ax+b)}$***;***
- ***To be able to integrate expressions requiring partial fractions;***
- ***To be able to integrate using trigonometric identities;***
- ***To understand and use integration by parts;***
- ***To understand and use integration by substitution;***
- ***To be able to integrate functions defined parametrically;***
- ***To be able to numerically estimate integrals using the trapezium rule***
- ***To be able to find the area enclosed between two curves;***

Prior Learning Atoms:

- *Trigonometry and Radians*
- *Differentiation*
- *Partial Fractions*
- *Parametric Equations*

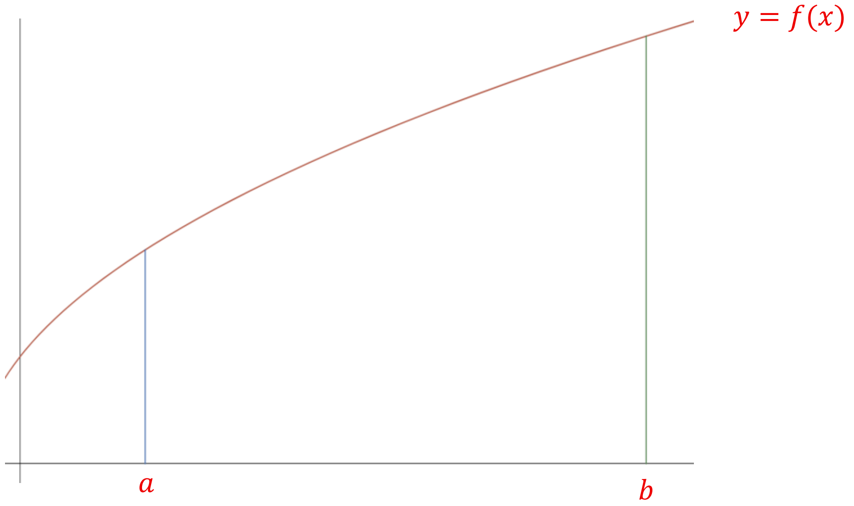
Now you have completed the unit…

Objective	Met	Know	Mastered
I understand the idea of integration as a limit of summation.			
I can use the standard results for integration.			
I can integrate functions of the form $f(ax+b)$.			
I can integrate the result of partial fractions.			
I can integrate where I require trigonometric identities.			
I can integrate by parts, including where parts is required more than once.			
I can integrate by substitution.			
I can integrate functions defined parametrically.			
I can use the trapezium rule to evaluate an integral numerically.			
I can find the area between two curves.			

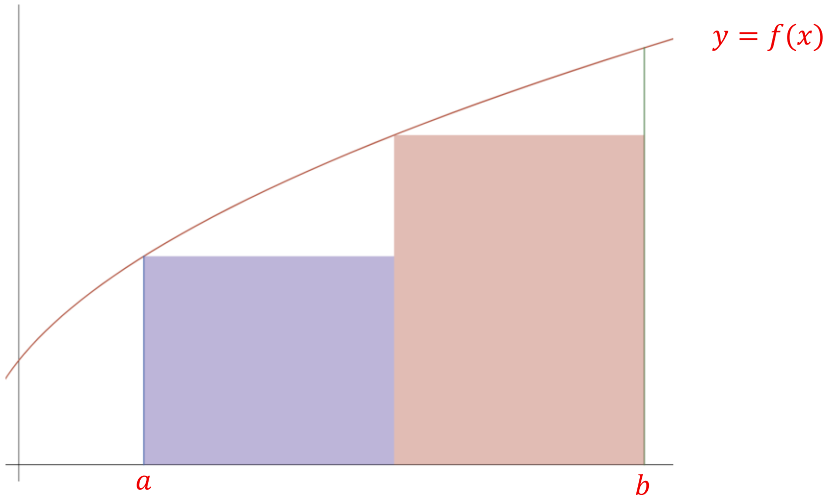
Notes/Areas to Develop:

Integration as a limit of summation

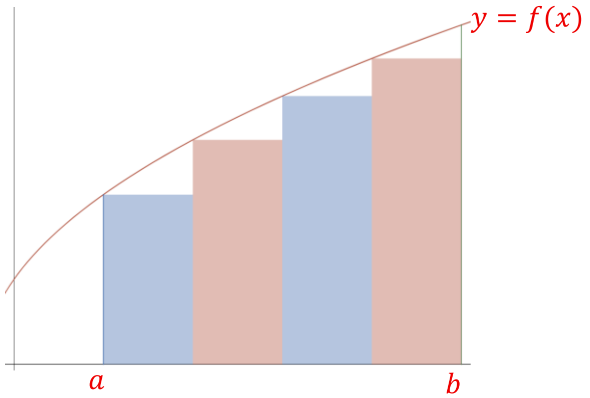
Consider finding the area underneath the curve, between the limits x = a and x = b, as shown below:

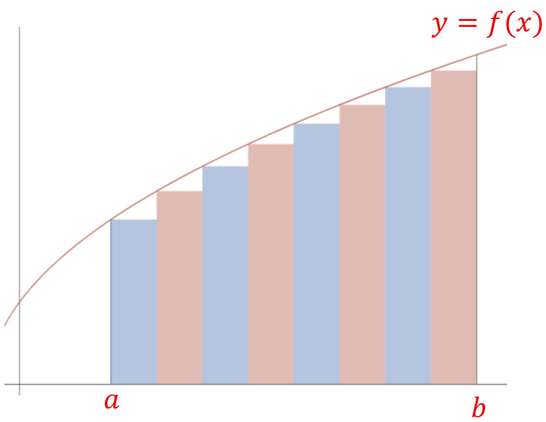


We could perhaps split into two rectangles to estimate the area:

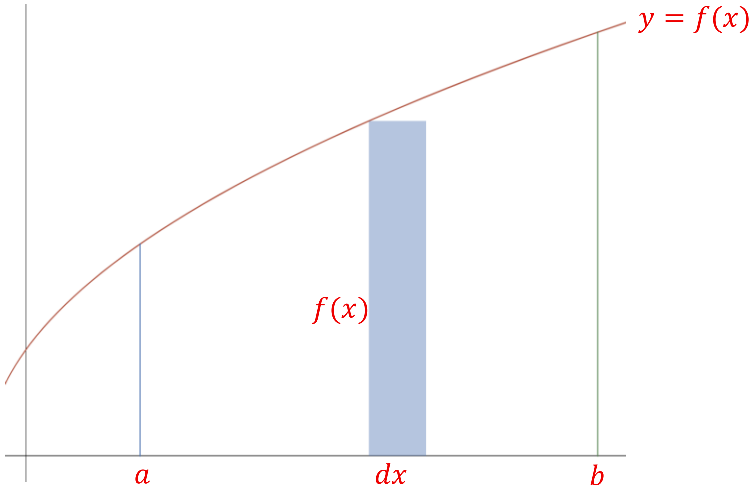


Using more rectangles would give a more accurate approximation:





We can extend this idea to consider what would happen if we take infinitely more rectangles, each with an infinitely thin width:



Each rectangle would have a height equal to the y-coordinate on the curve, $f(x)$

Each rectangle would have a width corresponding to a very small change in x, i.e. dx.

The area of each rectangle would be therefore given by $f(x)dx$.

If we had infinitely many infinitely thin rectangles, then the sum of their areas would approach the area under the curve.

Therefore,

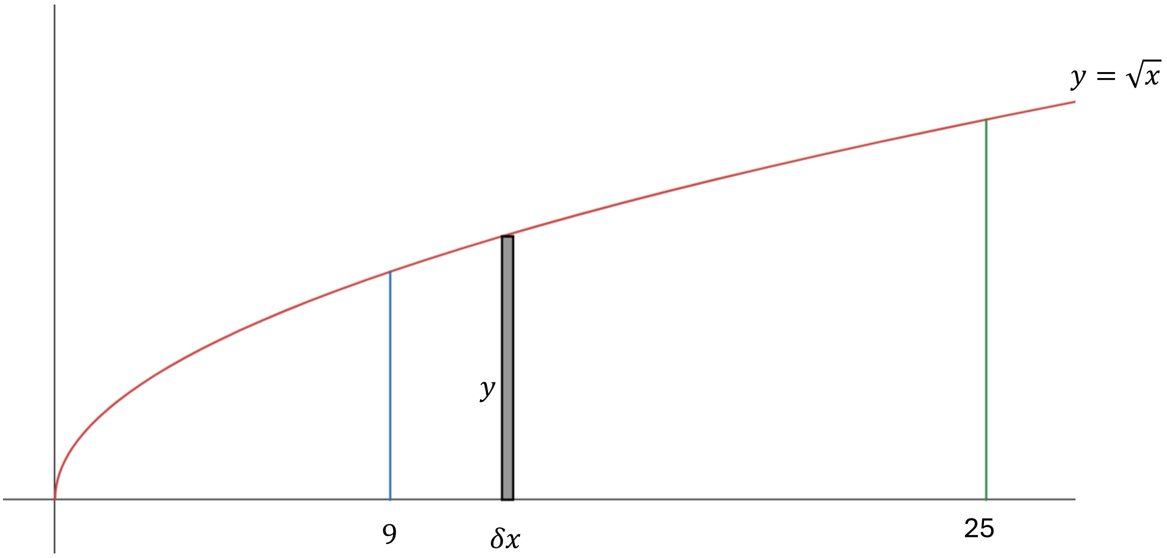
$$Area\ under\ curve\ between\ x = a\ and\ x = b$$

$$= \int_a^b f(x)dx$$

$$= \lim_{dx \to 0} \sum_{x=a}^{x=b} f(x)dx$$

The integral symbol is really an elongated 'S' to represent this limit of summation!

Example 1:



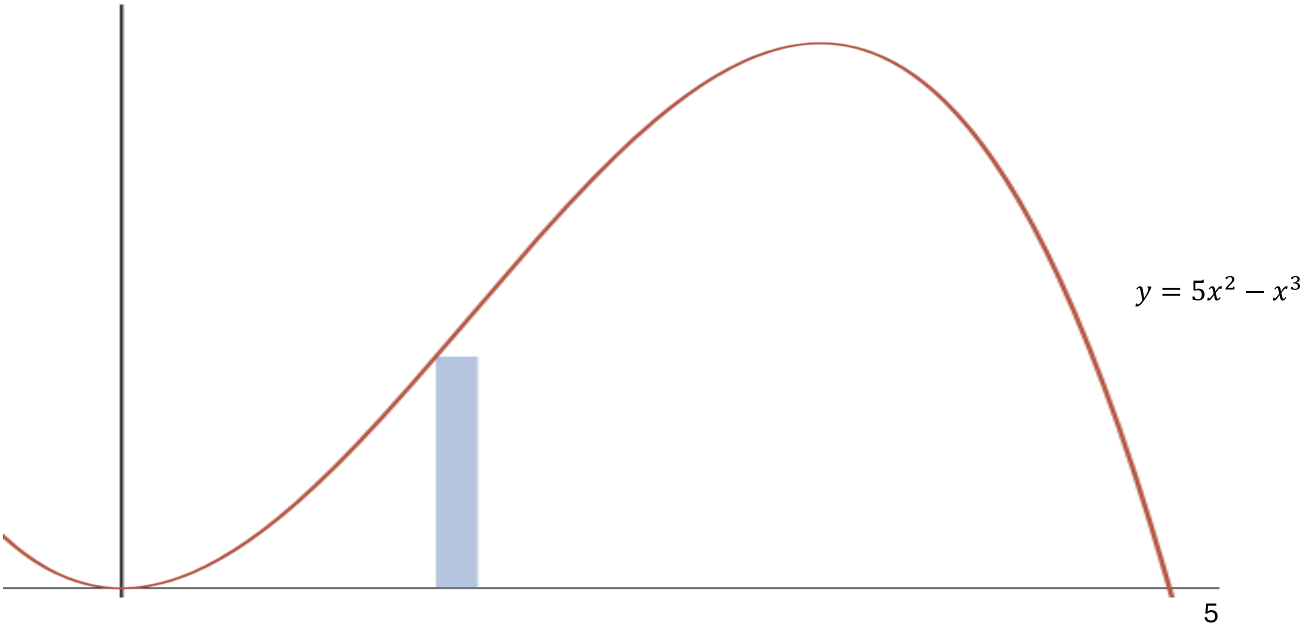
The diagram above shows the curve $y = \sqrt{x}$.

The shaded rectangle has height y and width δx. Find:

$$\lim_{\delta x \to 0} \sum_{x=9}^{x=25} \sqrt{x}\,\delta x$$

Note that, as of the current date (March 2025 at time of writing) WJEC have not examined this understanding. Still, it's mentioned in the spec, so… better safe than sorry!

Task 1:



The diagram above shows the curve $y = 5x^2 - x^3$

The shaded rectangle has height y and width δx. Find:

$$\lim_{\delta x \to 0} \sum_{x=0}^{x=5} (5x^2 - x^3)\, \delta x$$

Integration using Standard Results/Techniques

We now get into the real 'meat and potatoes' – finding integrals from all sorts of functions!

First, it is useful to note that:

$$\int ax^n \, dx = a \int x^n \, dx$$

i.e. if we have a constant in front of a term we wish to integrate, we can take that constant 'outside' the integral and simply multiply by it. This can be a very useful trick!

Another useful trick is that:

$$\int \big(f(x) + g(x)\big) dx = \int f(x) dx + \int g(x) dx$$

i.e. we can 'split' an integral with multiple terms into separate parts to deal with each one separately.

During AS mathematics, we only had one general integration result, which was that:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

i.e. to integrate a standard polynomial term we added one to the power and then divided by the new power.

However, we have now learnt to differentiate a wide array of other functions (such as exponential and logarithmic functions, and many trigonometric functions). As such, we have a number of new integration results to learn and use!

Standard results:

Must be learnt:	**Given in formula booklet:**
$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$	$\int \sec^2 x \, dx = \tan x + c$
$\int e^x \, dx = e^x + c$	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
$\int \frac{1}{x} dx = \ln\lvert x \rvert + c$	$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
$\int \sin x \, dx = -\cos x + c$	$\int \sec x \tan x \, dx = \sec x + c$
$\int \cos x \, dx = \sin x + c$	*...and many more!*

It's really worth noting that many of the ones given in the formula booklet are actually listed in the ***differentiation*** *results (i.e. they are the other way around from above), but we should be aware that we can use them as integration results also. Work smarter, not harder!*

Overall, the formula booklet gives us the following usable results:

Differentiation Results:

$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$

Integration Results

$\tan x$	$\ln\lvert\sec x\rvert$	
$\cot x$	$\ln\lvert\sin x\rvert$	
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert = \ln\lvert\tan\left(\frac{1}{2}x\right)\rvert$	
$\sec x$	$\ln\lvert\sec x+\tan x\rvert = \ln\lvert\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\rvert$	
$\sec^2 x$	$\tan x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	

Example 1: Integrate the following with respect to x:

a) $\int (3\cos x + \sqrt{x} + e^x)\, dx$

b) $\int \left(\frac{2}{x} + \frac{\cos x}{\sin^2 x}\right) dx$

Task 1: Find $\int 3\sec x(\sec x - 2\tan x)\,dx$

Task 2: Given that $n > 0$ and that $\int_{n}^{2n} \frac{3x+1}{x}\,dx = \ln 54$, find the exact value of n, giving your answer as simply as possible.

Now: Complete Test Your Understanding 1

Test Your Understanding 1

Question 1

Find:

$$\lim_{\delta x \to 0} \sum_{x=1}^{8} x^2 \, \delta x$$

Question 2

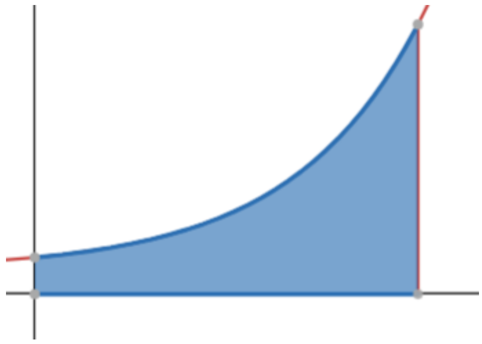
Use standard results to find:

a) $\int 3e^x dx$ b) $\int \frac{1}{x} dx$ c) $\int \sin x \, dx$ d) $\int (3 + x^{-1}) dx$

e) $\int (\frac{2}{x} + \frac{1}{x^2}) \, dx$ f) $\int \frac{2x+3}{x}$ g) $\int 2sec^2 x dx$ h) $\int (e^x - 1)^2 \, dx$

Question 3

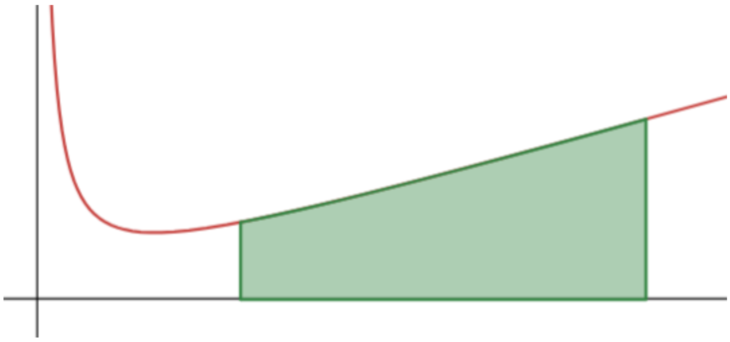
The diagram below shows the curve with equation $y = e^x + 2$, and the line $x = 3$.



Find the exact area of the shaded region.

Question 4

The diagram below shows the curve $y = 3x + \frac{1}{x}$, the lines $x = 1$ and $x = 3$. Find the exact area of the shaded region.



Question 5

Find $\int \left(\frac{1}{cos^2 x} + \frac{1}{x}\right) dx$

Question 6

Find $\int \left(\frac{x+1}{x^2} - \frac{\cos x - 1}{sin^2 x}\right) dx$

Challenge

Given that $\int_p^{3p} \frac{4x-1}{x} dx = 2 - \ln 3$, $p > 0,$ find the value of p.

Integrating $f(ax+b)$

We are able to use the chain rule to quickly differentiate terms such as $\sin(5x+2)$.

In general, we differentiate the function normally, and multiply by the derivative of the bracket.

In the same way, since integration is the reverse of differentiation, we can 'reverse' the chain rule to integrate terms of the form $f(ax+b)$. We will basically identify the integral of the function, and then **divide** by the derivative of the bracket. So, e.g.:

$$\int \sec 3x \tan 3x \, dx = \frac{1}{3}\sec 3x + c$$

Another way of thinking of this is that we know that the derivative of $\sec 3x$ *is* $3\sec 3x \tan 3x$, *so we divide the result by 3 to adjust it to the integral we require.*

Example 1: Find the following integrals.

a) $\int \cos 3x \, dx$ b) $\int e^{5x-2} \, dx$ c) $\int cosec^2(3x-2) \, dx$

Example 2: Find

a) $\int \frac{2}{3x+2} dx$

b) $\int \frac{1}{(2x-3)^2} dx$

*Examiner Pro Tip: The WJEC **really** like integrals of this form. Like, like-like them. We'll see why in the next section!*

Example 3: Find $\int (3x+5)^4 dx$

Task 1: Find the following integrals:

a) $\int \sin\left(2x + \frac{\pi}{3}\right) dx$

b) $\int \frac{1}{7x+1} dx$

c) $\int sec^2(x-2)\, dx$

Now: Complete Test Your Understanding 2.

Test Your Understanding 2

Question 1

Work out:

a) $\int \cos(5x - \pi)\, dx$

b) $\int 5e^{2x}\, dx$

c) $\int sec2xtan2xdx$

d) $\int cosec^2(4x)dx$

e) $\int \frac{1}{3x-2}dx$

f) $\int 5\sin(2x+3)\, dx$

g) $\int \frac{3}{5x-1}dx$

h) $\int \frac{1}{(2x+1)^2}dx$

i) $\int (3x-4)^5\, dx$

j) $\int 6e^{-x}dx$

k) $\int sec^2(\pi x - 2)dx$

l) $\int cosec(mx)\cot(mx)\, dx$

Question 2

Find $\int (e^{3x} - \frac{2}{3x-1} + \sin 2x)dx$

Question 3

Evaluate $\int_1^3 (2x-3)^3 dx$

Question 4

Evaluate $\int_1^4 (x - \mathrm{e}^{2x+1})dx$

Question 5

Given that $\int_1^n (3x-1)^2 dx = 56$, find the value of n.

Challenge

Given that $\int_e^{e^3} \frac{1}{kx}dx = \frac{1}{8}$, find the value of k.

Integration involving Partial Fractions

This is **by far** the most common setting for a partial fractions question in Unit 3.

Example 1: You are given that

$$f(x) = \frac{25x - 5}{(3x+2)(2x-3)(x+1)}$$

a) Express $f(x)$ in partial fractions.

b) Hence, find $\int f(x)\, dx$.

There's not much more to be said – it is mechanical, technical algebra. It could be worth 8-10 marks in your examination – so practise it!

Task 1: Find $\int \frac{9}{(1-x)(1+2x)^2}\,dx$

Integration requiring Trigonometric Identities

Many of the most challenging A2 integrations require the use of trigonometric identities in order to rewrite the integrand in a suitable form.

Example 1: Find $\int \tan^2 x \, dx$

Suspicious, there doesn't seem to be a result for this in the formula booklet… I wonder if there's an identity?

Task 1: Find $\int (\sec x - \tan x)^2 \, dx$

Mega Tip: *We have seen that we can easily integrate double angle functions such as* $\cos 2x$. *Powers of* $\sin x$ *and* $\cos x$ *are a nuisance – we usually have to use identities linking them to* $\cos 2x$.

Example 2:

a) Find $\int sin^2 x\, dx$

b) Further, show that $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} sin^2 x\, dx = \frac{2\pi + 3\sqrt{3}}{24}$

Task 2: Find $\int \sin 4x \cos 4x \, dx$

Hint: We have a product of $\sin x$ *and* $\cos x$ *– do we have an identity for that?*

Task 3: Find $\int cos^2 x \, dx$

Now: Complete Test Your Understanding 3

Test Your Understanding 3

Question 1

By first splitting into partial fractions, integrate the following:

a) $\int \frac{1}{(x-2)(2x-3)} dx$ b) $\int \frac{5}{(3x-1)(2x+1)} dx$ c) $\int \frac{60x}{(x-1)(2x+1)(2x+3)} dx$

Question 2

a) Show that $\frac{x+4}{(x+1)(2x-1)^2} = \frac{1}{3(x+1)} - \frac{2}{3(2x-1)} + \frac{3}{(2x-1)^2}$

b) Hence, show that $\int_1^2 \frac{x+4}{(x+1)(2x-1)^2} = 1 - \ln(\sqrt[3]{2})$

Question 3

Find

$$\int \frac{1-2x}{(x+3)(2x+1)} dx$$

Question 4

Find the following:

a) $\int (1 + cot^2 x) dx$ b) $\int cos^2 x dx$ c) $\int 4 sinx cosx\, dx$

d) $\int 2tan^2 2x\, dx$ e) $\int (1 - sinx)^2 dx$ f) $\int cos^2 2x\, dx$

Question 5

Show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin^2 x$$

can be written in the form $a\pi + b$ where a, b are rational numbers.

Question 6

Find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (cosx - secx)^2\, dx$$

Question 7

Find the exact value of $\int_0^{\frac{\pi}{3}} f(x)dx$, where $f(x) = 3sin^2x + 5cos^2x$.

Question 8

Evaluate

$$\int_0^1 \frac{3}{2x^2 - 32} dx$$

Challenge Question

a) Use the identities for $\sin{(A+B)}$ and $\sin{(A-B)}$ to show that

$$\sin A \cos B \equiv \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

b) Hence, find

$$\int \sin 3x \cos x \, dx$$

Finding the area enclosed between two curves

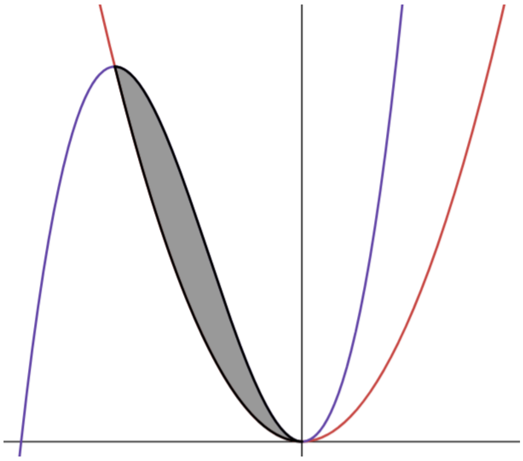
If we want to find the area bounded between two curves with equations $y = f(x)$ and $y = g(x)$ where $g(x) \geq f(x)$ throughout the interval $[a, b]$ of integration, we can use the result

$$A = \int_a^b \big(g(x) - f(x)\big)dx$$

(Note: We sometimes call $g(x)$ the 'upper' function and $f(x)$ the 'lower' function.)

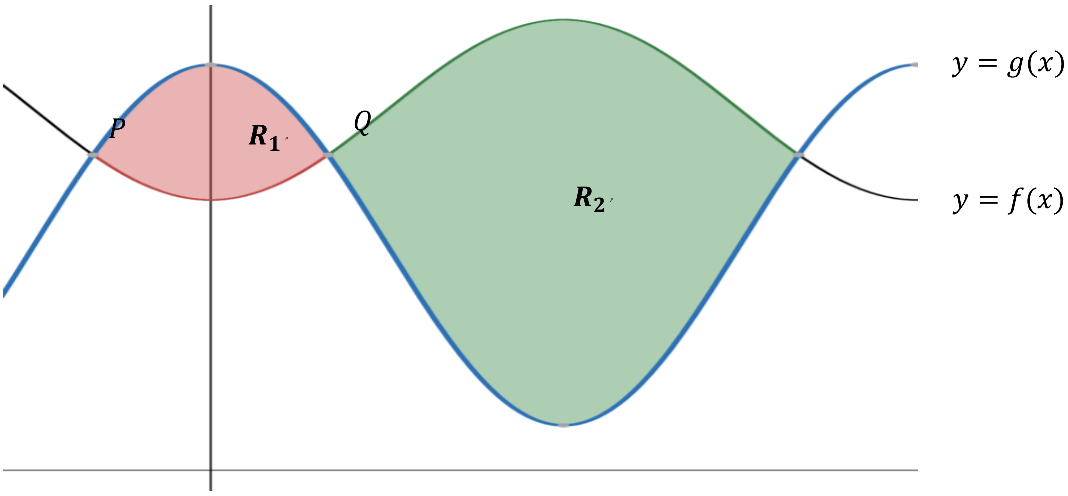
Example 1:

$f(x) = x^2$ and $g(x) = x^3 + 3x^2$. The graph below shows a sketch of $y = f(x)$ and $y = g(x)$.



Find the area of the shaded region bounded by the two curves.

Task 1:



The graph above shows the curves $y = f(x)$ and $y = g(x)$, where $f(x) = -2\cos x + 8$ and $g(x) = 4cosx + 5$.

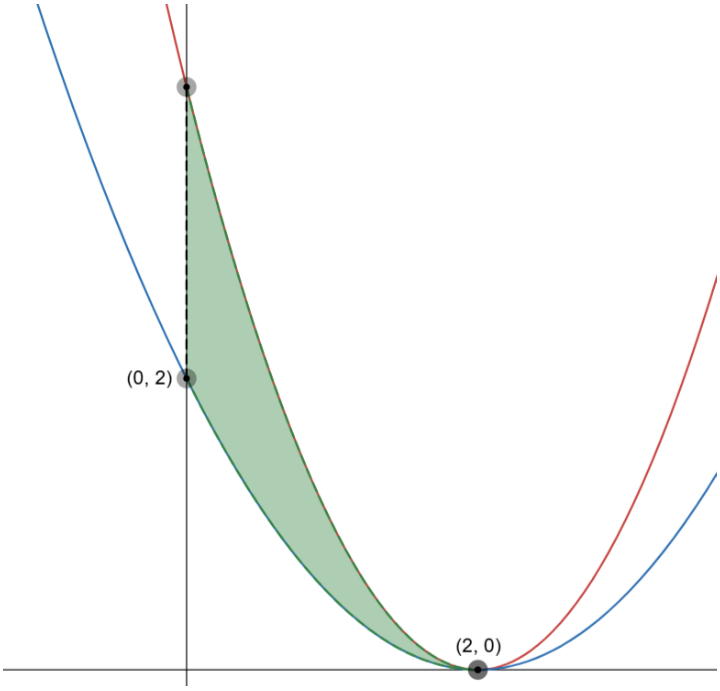
a) Find the coordinates of P, Q and R.

b) Hence, find the exact areas of the two regions R_1 and R_2.

Now: Complete Test Your Understanding 4

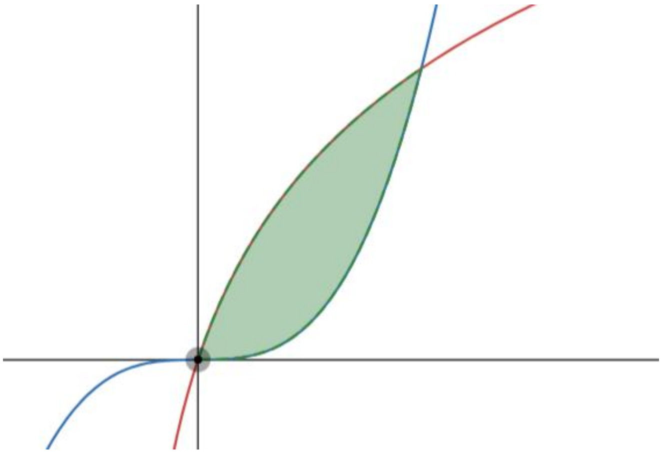
Test Your Understanding 4

Question 1



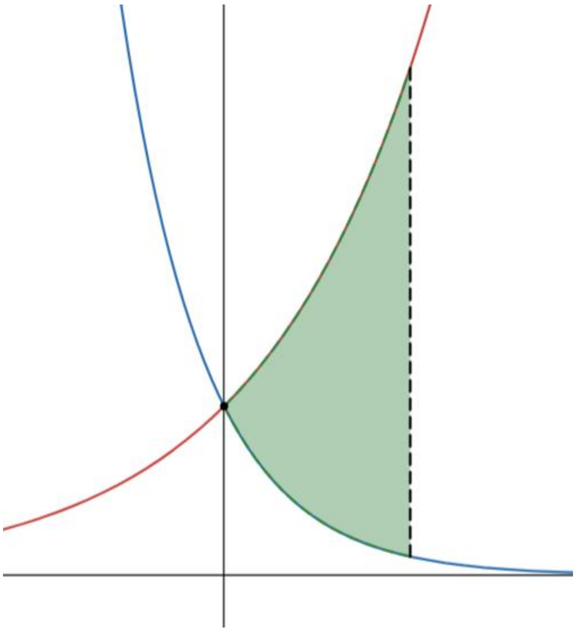
The diagram above shows the curve $f(x) = (x-2)^2$ in red, and $g(x) = \frac{1}{2}(x-2)^2$ in blue. Show that the shaded area is equal to $\frac{4}{3}$.

Question 2



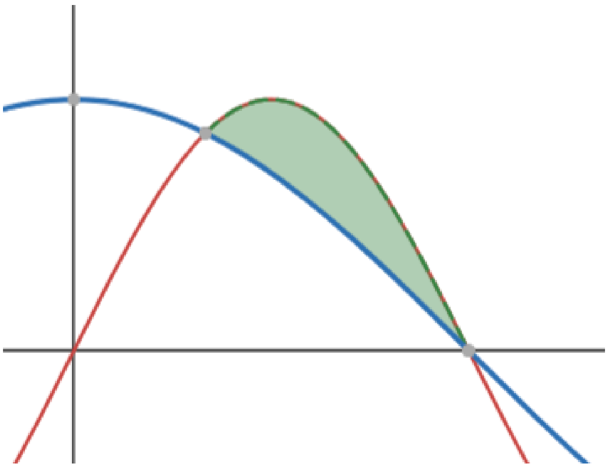
The diagram above shows the curve $f(x) = \ln(3x+1)$ in red, and $g(x) = x^3$ in blue. Given that the two curves intersect at $x = 0$ and again when $x = \sqrt{2}$, show that the shaded area is equal to 0.48117 to 5 decimal places.

Question 3



The diagram above shows the curve $f(x) = e^x$ in red and $g(x) = e^{-2x}$ in blue, and the line $x = \ln 3$. Find the area of the region bounded by the two curves and the line.

Question 4



The diagram above shows the curve $f(x) = \sin 2x$ in red, and $g(x) = \cos x$ in blue.

a) Show that the first two positive solutions to $f(x) = g(x)$ are $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

b) Hence, find the exact value of the shaded area in the diagram.

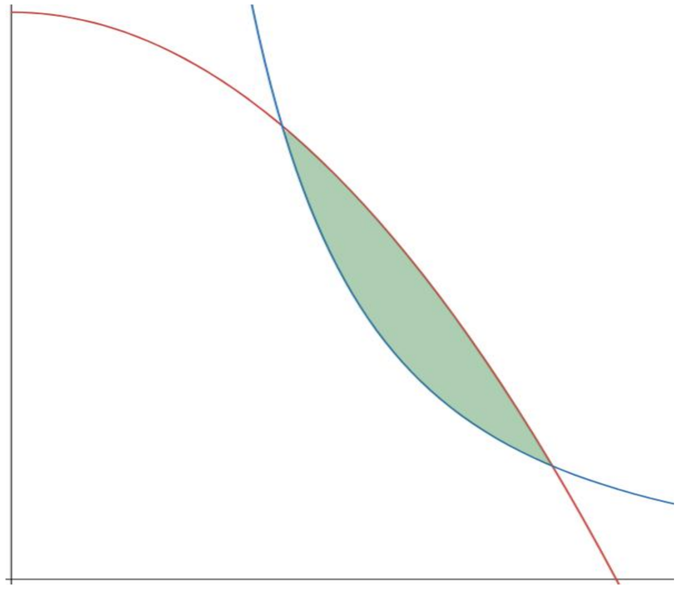
Question 5

Find the area of the region enclosed by the two curves with equations $y = x^2$ and $y = 8 - x^2$.

Question 6

a) Using an appropriate algebraic method, show that the curves C_1 and C_2, with equations $y = x^3 + 9x^2 + 18x + 11$ and $y = 3x^2 + 7x + 5$ respectively, intersect at $x = -2$ and $x = -1$.
b) Find the area enclosed between the two curves between $x = -2$ and $x = -1$.

Question 7



The above diagram shows the curves $y = 5 - x^2$ and $y = \frac{4}{x^2}$. Find the area of the region enclosed between the two curves. (You should not require a calculator to solve for the points of intersection).

CHALLENGE QUESTION

A line L_1 has equation $y = x - 1$, and two curves C_1 and C_2 have curves $y = x^2 - 6x + 9$ and $y = 2 - (3 - x)^2$ respectively.

a) Find the points A and B of intersection of C_1 and C_2, and show that L_1 also meets the curve at A.
b) Find the other point of intersection, D, of L_1 and C_2.
c) The curve C_1 meets the x-axis at E. Find the area of the region $ABDE$.

A diagram may help to tackle this problem!

Integration by Parts

Think "Integration's answer to the product rule."

In unit 6 we learnt the product rule to find derivatives of functions of the form uv where u and v were functions of x.

We found that:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

By integrating and rearranging, we can easily prove the formula for integration by parts:

Key Point:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

This result is given in the formula booklet. It looks scarier than it is – I promise!

Example 1: Find $\int x \cos x \, dx$.

Examiner tip: In the large majority of cases, if one of the two 'parts' is of the form x^n *then we will let this be our* "u" *– as differentiating it will make it 'simpler'.*

Task 1: Find $\int_1^2 xe^x dx$

There is one big exception to the previous tip. If our integral contains $\ln x$, since we do not have a known/given result for integrating it, we will **always** have to make this our u.

Example 2: Find $\int 2x \ln x \, dx$

Task 2: Find $\int \ln x \, dx$

Hint: Start by writing $\ln x$ *as* $1 \ln x$.

We can need to use parts twice, as in the following example:

Example 3: Find $\int x^2\, e^x dx$

Now: Complete Test Your Understanding 5

Test Your Understanding 5

Question 1

Use integration by parts to find the following:

a) $\int x\sin x \, dx$

b) $\int xe^{3x} dx$

c) $\int 2x\sec^2 x \, dx$

d) $\int (3x-1)\, e^x \, dx$

e) $\int (x+2)\sin x \, dx$

f) $\int x\operatorname{cosec}^2 x \, dx$

g) $\int x\cos 3x \, dx$

h) $\int \frac{x}{e^{2x}} dx$

i) $\int x\sqrt{x+2}\, dx$

Question 2

Find the following integrals:

a) $\int \ln x \, dx$

b) $\int 3x\ln x \, dx$

c) $\int x^3 \ln x \, dx$

Question 3

Evaluate the following integrals exactly:

a) $\int_0^1 xe^x \, dx$

b) $\int_0^{\frac{\pi}{4}} x\sin x \, \mathrm{d}x$

c) $\int_1^2 \ln x \, \mathrm{d}x$

d) $\int_0^{\ln 2} xe^x \mathrm{d}x$

e) $\int_0^{\pi} x\sin\left(\frac{x}{3}\right)\mathrm{d}x$

f) $\int_0^1 2x(1+x)^4 \mathrm{d}x$

Question 4

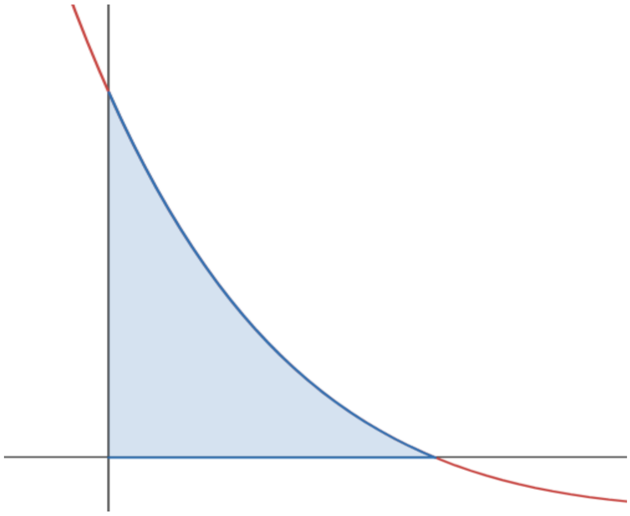
Integrate the following:

a) $\int x^2 \sin x \, dx$ b) $\int x^2 e^{3x} dx$ c) $\int 3x^2 (\cos 2x)$

Question 5

Show that $\int_1^3 (x+1)\, e^{2x} dx = ae^6 - be^2$ where a, b are rational numbers to be found.

Question 6



The diagram above shows the graph of $y = (6-3x)e^{-0.5x}$.

Find the exact shaded area shown on the diagram.

Challenge

Use integration by parts to find I, where $I = \int e^x \sin x \, dx$.

Integration by Substitution

I flipping love a 'u-sub'. This is one of the most powerful ways to tackle an integration – but, this comes with challenge. You like a challenge, right?

Let's illustrate the idea with an integral we can already find.

Example 1: Use the substitution $u = 2x + 3$ to find $\int (2x+3)^4 \, dx$.

Task 1: Use the substitution $u = 3x - 2$ to find $\int e^{3x-2} \, dx$.

They allow us to transform unpleasant products into nice ones…

Example 2: Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\,dx$

Task 2: Use the substitution $u = \sin x - 1$ to find $\int \cos x \sin x\,(\sin x - 1)^4\,dx$.

Now THIS looks like real mathematics!!

Now: Complete Test Your Understanding 6, Page 38, Question 1.

Integration by Substitution – use of Limits

Using limits of integration with substitution is not a problem. We have two choices!

1. Turn the integration back into the initial variable (usually x) and use the original limits.

OR…

2. Transform the x-limits into u-limits – we can save some time this way!

Example 1: Use integration by substitution, with $u = x - 1$, to evaluate

$$\int_1^3 x(x-1)^3 dx$$

Task 1: Use the substitution $u = x^2 - 3$ to evaluate

$$\int_2^3 \frac{2x}{x^2 - 3}\,dx$$

Now: Complete Test Your Understanding 6, Pages 38-40, Questions 2-6.

Finding a Substitution

Rarely, we can be required to find a suitable substitution. Usually, this is when we have an integral with a product/quotient where one part is the derivative of the other (or a multiple thereof). Especially with fractions, it also usually allows us to turn multiple terms in the denominator into a single u^n term.

Example 1: Find $\int \frac{4x}{1+x^2}\, dx$.

Example 2: Write down a suitable substitution for each of the following integrals and thus transform the integration into an integration in terms of u. *(You do not need to perform the integration.)*

a) $\int \frac{\cos x}{3\sin x+2}\, dx$ b) $\int \frac{2x}{(x^2+3)^3}\, dx$ c) $\int \cos x \sin^3 x\, dx$.

Task 1: For each of the following, state a suitable u-substitution.

a) $\int 4x(x^2+2)^4\,dx$ b) $\int \frac{2\cos x}{\sin^4 x}\,dx$ c) $\int x^2\,(x^3+3)^4 dx$

Task 2: By first choosing a suitable substitution, find

$$\int \frac{cosec^2 x}{(2+\cot x)^3}\,dx$$

A Special Substitution

This is a very rare guest to A-Level – I can only find one PPQ where it comes up! Let's consider the following integral:

$$\int_0^{3/2} \frac{\sqrt{3}-x}{\sqrt{3-x^2}}\,dx$$

No 'nice' u-substitution seems to work here – we always end up either with another thing we can't integrate, or a horrific mess with repeated substitutions. Parts also won't work as we end up with terms that are beyond A-level to integrate!

The substitution here is … a trigonometric one! We in fact substitute $x = \sqrt{3}sin\theta$ – the reason being that, when we substitute this in for x in the denominator, via trig identities this simplifies down to $\sqrt{3}cos\theta$ – which happens also to be $\frac{dx}{d\theta}$!

One of my students asked me, "But how would I know to do this?!" The answer is simple – experience and preparation. There are **very** few candidates who would ever make such a conceptual leap without having seen it before – perhaps a few rogue geniuses. This sort of substitution is 'more' frequent in Further Mathematics circles, but is still a tough spot. So this type of question, as unlikely as it is, would be very much an A/A* discriminator question.

Now: Complete Test Your Understanding 6, Page 38-40, remaining questions.

Test Your Understanding 6

Question 1

Use the given substitutions to find the following integrals:

a) $\int 2x(1+x^2)^4 dx, \quad u = 1+x^2$

b) $\int \frac{3x^2}{x^3-1} dx, \quad u = x^3 - 1$

c) $\int 2xe^{x^2} dx, \quad u = x^2$

d) $\int \cos x(1+\sin x)^3 dx, \quad u = 1+\sin x$

e) $\int x(x^2-4)\, dx, \quad u = x^2 - 4$

f) $\int \frac{4x}{x^2-1} dx, \quad u = x^2 - 1$

g) $\int 9x^2(x^3-1)\, dx, \quad u = x^3 - 1$

h) $\int e^{2x}(1+e^{2x})^3 dx, \quad u = 1+e^{2x}$

i) $\int \cos^3 2x \sin 2x\, dx, \quad u = \cos 2x$

j) $\int sec^3 x \tan x\, dx, \quad u = secx$

k) $\int \frac{3x}{x^2+1} dx, \quad u = x^2 + 1$

l) $\int 5xe^{1-x^2} dx, \quad u = 1 - x^2$

m) $\int \frac{\sin x}{1-\cos x} dx, \quad u = 1 - \cos x$

n) $\int \cos x e^{\sin x} dx, \quad u = \sin x$

Question 2

By using the substitution $u = x^2 + x$, show that $\int_1^3 (2x+1)(x^2+x)^3 \mathrm{d}x = 5180$

Question 3

Evaluate the following integrals given the following substitutions:

a) $\int_0^1 \frac{6x}{x^2+1} dx, \quad u = x^2 + 1$

b) $\int_0^1 4xe^{x^2} dx, \quad u = x^2$

c) $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x \, dx, \quad u = \sin x$

d) $\int_1^2 t^2 (t^3 + 1) \, dt, \quad u = t^3 + 1$

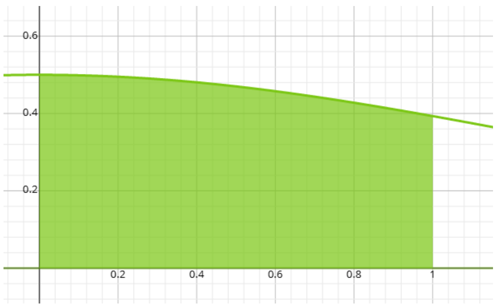
e) $\int_0^{\frac{\pi}{2}} \cos x \sqrt{2 + \sin x} \, dx, \quad u = 2 + \sin x$

f) $\int_{\frac{\pi}{6}}^{\frac{pi}{3}} cot x \, dx, \quad u = \sin x$

Question 4

Using the substitution $u = \cos x$, show that $\int \tan x dx = \ln|sec x| + c$.

Question 5



The diagram above shows the curve with equation $y = \frac{2e^x}{(1+e^x)^2}$.

Show that the shaded area is $\frac{e-1}{1+e}$.

Question 6

Given that $\int_0^k 2x^2 e^{x^3} dx = \frac{2}{3}(e^{64} - 1)$, use the substitution $u = x^3$ to find the value of k.

Question 7

Use a suitable substitution to find the following:

a) $\int 2x(1+x^2)^5 dx$

b) $\int \frac{4e^x}{(e^x-1)^2} dx$

c) $\int \frac{4x}{(x^2-1)^2} dx$

d) $\int \frac{\cos 2x}{3+\sin 2x} dx$

e) $\int x(3x^2+1)^2 dx$

f) $\int 4e^{2x}(1+e^{2x})^3 dx$

Question 8

Using an appropriate substitution, show that

$$\int_1^2 \frac{\ln x}{x} \mathrm{d}x = \frac{\ln^2(2)}{2}$$

Hint: We <u>never</u> want to ***integrate*** $\ln x$ *if we can help it!*

Question 9

Using an appropriate substitution, show that

$$\int_1^2 \frac{-e^{2x}}{(1-e^{2x})} \mathrm{d}x = \sqrt{\left(\frac{e^4-1}{e^2-1}\right)}$$

Challenge

Using the substitution $x = sin\theta$, prove that $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$.

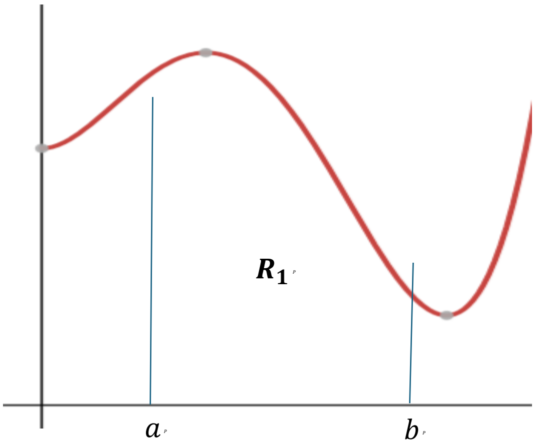
Try to find suitable substitutions to prove the results:

- $\int -\frac{1}{\sqrt{1-x^2}} dx = \arccos x + c$
- $\int \frac{1}{1+x^2} dx = \arctan x + c$

Numerical Methods for Integration:

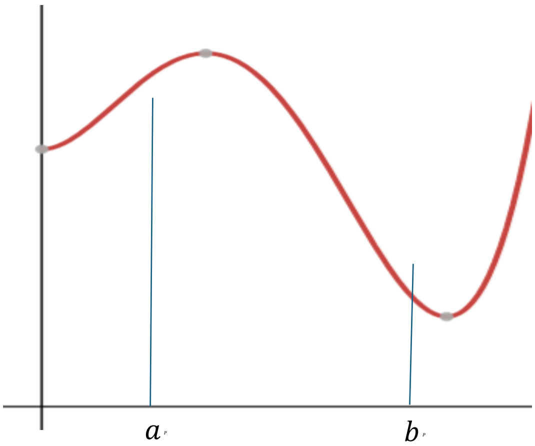
The Trapezium Rule

Integration is hard. There, I said it! As such, there are many times we can not integrate a function algebraically. However, we can use a numerical method to approximate the area beneath a curve.



What on earth is this curve?!

We can divide the area up into n strips of equal width h:



There are n *strips of equal width, with* $(n+1)$ *ordinates.*

We can then find the y-values for each x-ordinate (which we tend to label y_0, y_1 and so on up to the final one y_n. Joining the tops of these y-values we create n trapeziums.

From here, it really isn't too hard to derive the trapezium rule…

The trapezium rule states that

$$\int_a^b y\,dx \approx \frac{h}{2}(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1}), \text{ where } h = \frac{b-a}{n}.$$

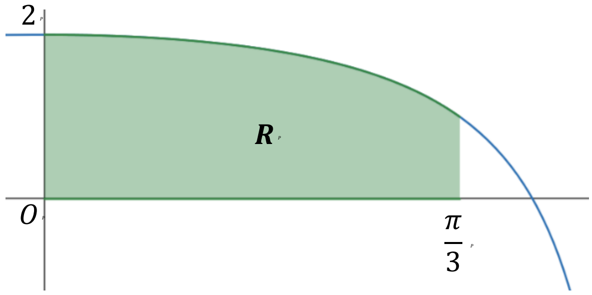
This isn't the most frequent visitor to A2 papers, but they should be nice bonus marks when it does come up – this used to be on AS mathematics!!

Example 1:

The diagram shows the curve with equation

$y = 3 - \sec x$.

The region R is bounded by the curve, the x-axis, the y-axis and the line $x = \frac{\pi}{3}$.



a) Use the trapezium rule with five ordinates (four strips) to estimate the area of R.

b) State, giving a reason, whether your estimate is an underestimate or an overestimate.

Useful note: The trapezium rule underestimates if the curve is concave for the interval of x-values being considered. If the curve is convex, the trapezium rule will overestimate the area. Usually we can see this by imagining drawing the tops of the trapezia on our curve.

Now: Complete Test Your Understanding 7

Test Your Understanding 7

Question 1

For each of the following curves, copy and complete the table and hence find an estimate for each area, stating the number of ordinates/strips used. Give all values to 4 s.f.

a) $y = (\ln x)^2, x > 0$

x	2	2.25	2.5	2.75	3
y	0.4805		0.8396		

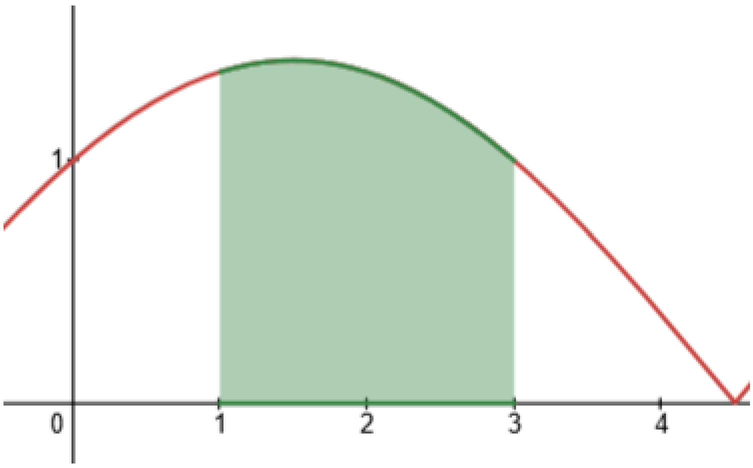
b) $y = x \log_5 x$

x	3	3.5	4	4.5
y	2.048	2.724	3.445	

c) $y = \sqrt{\frac{10-x}{x}}$

x	8	8.4	8.8	9.2	9.6	10
y	0.5			0.2949		0

Question 2



The diagram above shows the graph of $y = \sqrt{1 + \sin\left(\frac{\pi x}{3}\right)}$.

The table below gives corresponding values x and y, the latter given to 4 significant figures:

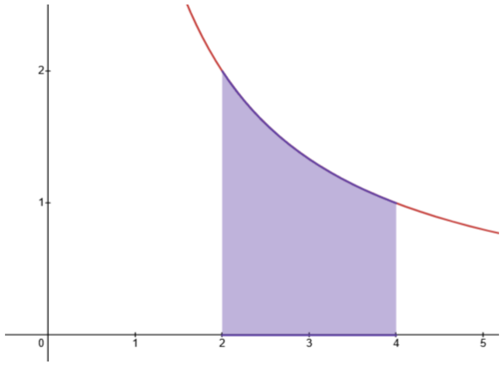
x	1	1.5	2	2.5	3
y	1.366	1.414	1.366	1.225	

a) Find the value of y when $x = 3$.

b) Use the trapezium rule with five ordinates to find an estimate for the area shaded.

c) State, giving a reason, whether your answer is an overestimate or an underestimate.

Question 3



The diagram shows the graph of the curve with equation $y = \frac{4}{x}$.

a) Copy and complete the following table, giving the exact value of y for each corresponding x-value.

x	2	2.5	3	3.5	4
y	2				

b) Hence find an estimate for the shaded area to 3 significant figures.

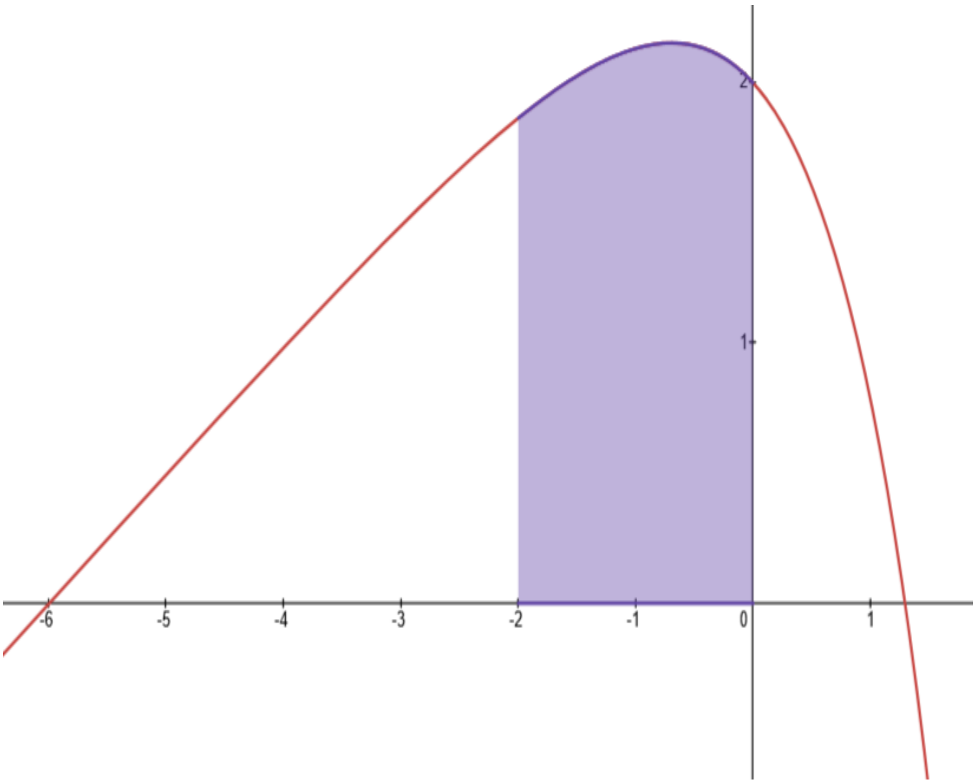
c) Show that the exact area is $4\ln 2$.

Question 4

The diagram shows the curve with equation $y = 3 + 0.5x - e^x$. The shaded region is bounded by the curve, the x-axis and the lines $x = -2$ and $x = 0$.

a) Using four strips of equal width, estimate the shaded area using the trapezium rule. Show clearly your values used.

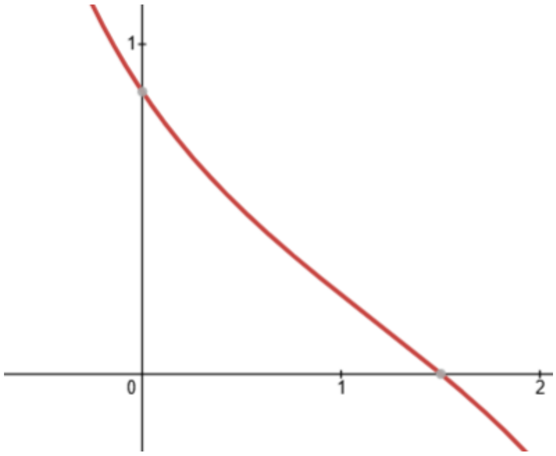
b) Explain how you could find a more accurate estimate for the area.



Question 5

The diagram shows the graph of the curve

$$y = \frac{12x - 18}{(2x + 3)(2x - 7)}.$$



a) Show that the curve intersects the x-axis when $x = 1.5$.

b) Use the trapezium rule with three strips (four ordinates) in order to estimate the area enclosed by the curve, the positive x-axis and the positive y-axis. Give all values to 4 significant figures.

c) Use integration with partial fractions to find the actual area of the curve to 4 significant figures.

d) Calculate the percentage error of the estimate from the actual area.

Now: You are ready to face the Grade Enhancer™, Page 51.

Integration of functions defined parametrically – EXTENSION MATERIAL

Note: This is not explicitly listed in the specification; however it's also not explicitly stated that it is NOT in the specification. Therefore, for a full coverage comparable to the English A2 specifications I include it here, but you may wish to ignore it!

When we integrate normally for a function y with respect to x, we are finding

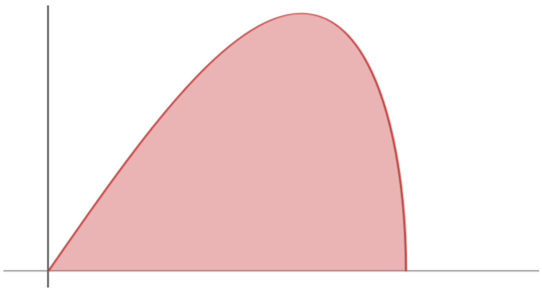
$$\int y\,dx.$$

When we have our functions y and x written parametrically in terms of, say, t, we can use our 'cancelling trick' again:

$$\int y\frac{dx}{dt}dt$$

If we are finding an area, then we use the t-limits that correspond to the x-values we wish to integrate between.

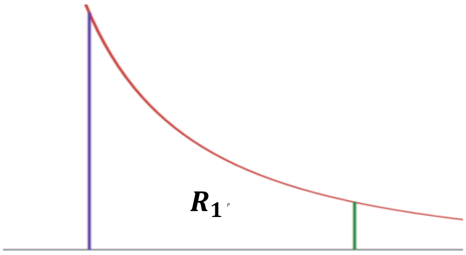
Example: The diagram below shows the curve defined parametrically by $x = 2\cos t$ and $y = 3\sin 2t$ for $0 \leq t \leq \frac{\pi}{2}$.



a) Find the coordinates of the points where the curve intersects the x-axis.

b) Hence, find the area bounded by the curve and the x-axis.

Task 1: The diagram shows a part of the curve given parametrically by $x = \ln(t+3)\,, y = \frac{2}{t+1}\,,$ where $t > 3, t \neq -1$, and the lines $x = \ln 3$ and $x = \ln 7$.



a) Show that the area R_1 bounded by the curve, x-axis and the lines $x = \ln 3$ and $x = \ln 7$ is given by

$$\int_0^4 \frac{2}{(t+1)(t+3)}\,dt$$

b) Hence find the exact area, giving your answer in the form $\ln\frac{p}{q}$.

Test Your Understanding (Optional – not on WJEC syllabus)

Question 1

Integrate the following pairs of parametric equations:

a) $x = 3t - 1,\ y = t^2$

b) $x = t^3,\ y = 1 - t$

c) $x = ln|t|, y = t^2 (t \neq 0)$

d) $x = sint,\ y = t$

e) $x = t^3, y = \frac{1}{t} + 1 (t \neq 0)$

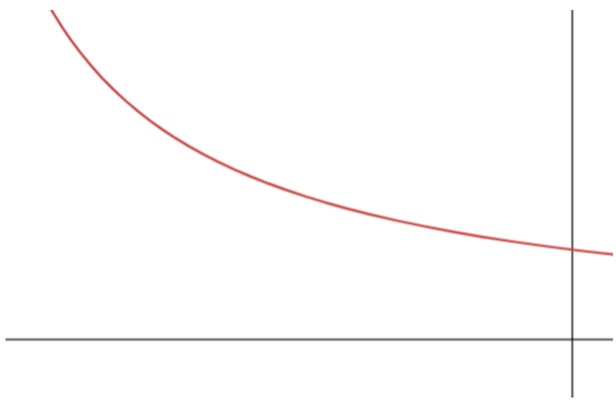
f) $x = \sin t\,,\ y = \cos t$

g) $x = \tan t\,,\ y = \cot^2 t$

h) $x = e^{2t},\ y = t$

Hint: You may need to use ANY of the methods of integration met previously! Bon appetit!

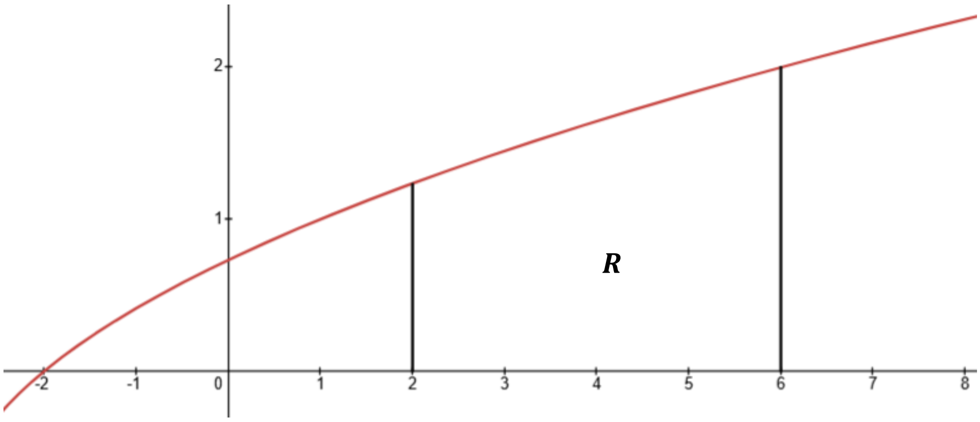
Question 2



The diagram above shows the curve defined by parametric equations $x = 2t - 4, y = \frac{1}{t}$, where $t \neq 0$.

Find the area bounded by the curve, the x- and y-axes and the line $x = -2$.

Question 3



The diagram above shows the curve defined parametrically by equations $x = t^2 - 3$ and $y = t - 1$.

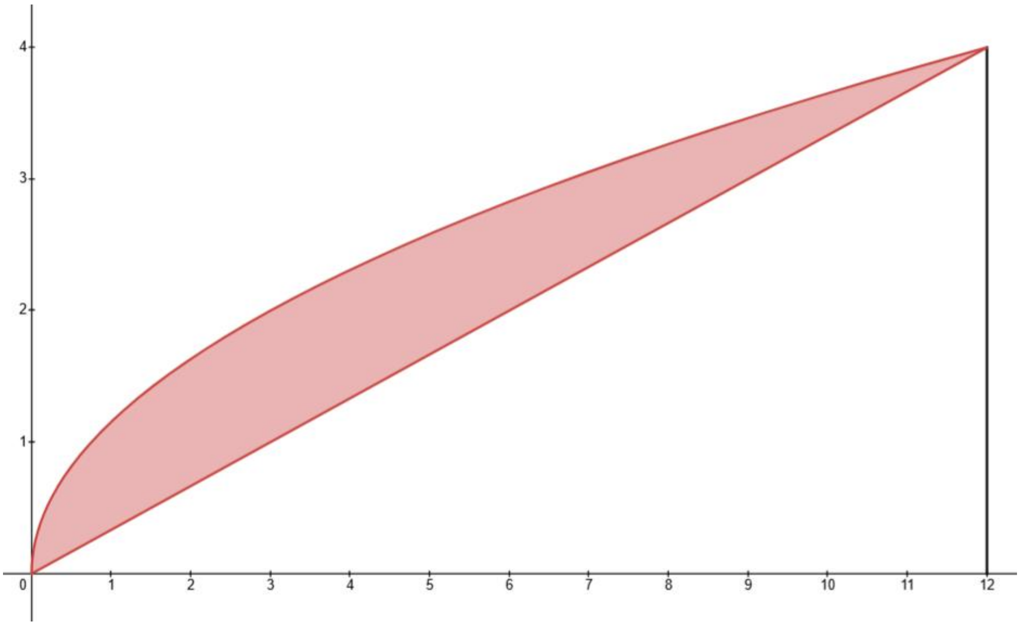
Find the area of the region bounded by the curve, the x-axis and the lines $x = 2$ and $x = 6$.

Question 4

(a) Using the substitution $u = sinx$, find $\int \sin^2 x \, cosx \, dx$.

(b) The diagram below shows the graph of the curve with parametric equations

$$x = 12\sin^2 t, \; y = 4\sin t, \; 0 \leqslant t \leqslant \frac{\pi}{2}$$



Find the area of the shaded region.

Grade Enhancer™ - Apply your knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(WJEC 2016)*

(a) Find each of the following, simplifying your answer wherever possible.

(i) $\int 7e^{5-\frac{3}{4}x}\,dx$ (ii) $\int \sin\left(\frac{2x}{3}+5\right)dx$ (iii) $\int \frac{8}{(9-10x)^3}\,dx$ [6]

(b) Given that $a > 0$ and that

$$\int_a^6 \frac{1}{4x+3}\,dx = 0{\cdot}1986,$$

find the value of the constant a. Give your answer correct to one decimal place. [5]

Question 2 *(WJEC 2019)*

(a) Find $\int x\sin 2x\,dx$. [4]

(b) Use the substitution $u = 5 - x^2$ to evaluate

$$\int_0^2 \frac{x}{(5-x^2)^3}\,dx.$$ [4]

Question 3 *(WJEC 2019)*

a) Express $\frac{9}{(x-1)(x+2)^2}$ in terms of partial fractions. [4]

b) Find $\int \frac{9}{(x-1)(x+2)^2}\,dx$. [3]

Question 4 *(WJEC 2019)*

a) Find $\int (e^{2x} + 6\sin 3x)\,dx$. [2]

b) Find $\int 7(x^2 + \sin x)^6(2x + \cos x)\,dx$. [1]

c) Find $\int \frac{1}{x^2}\ln x\,dx$. [4]

d) Use the substitution $u = 2\cos x + 1$ to evaluate

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{(2\cos x + 1)^2}\,dx \ .$$ [4]

Question 5 *(WJEC 2023)*

The function f is given by

$$f(x) = \frac{25x + 32}{(2x-5)(x+1)(x+2)} \ .$$

(a) Express $f(x)$ in terms of partial fractions. [4]

(b) Show that $\int_1^2 f(x)\,dx = -\ln P$, where P is an integer whose value is to be found. [5]

Question 6 *(WJEC 2018)*

Evaluate

a) $\int_1^2 x^3 \ln x\,dx$. [6]

b) $\int_0^1 \frac{2+x}{\sqrt{4-x^2}}\,dx$. [6]

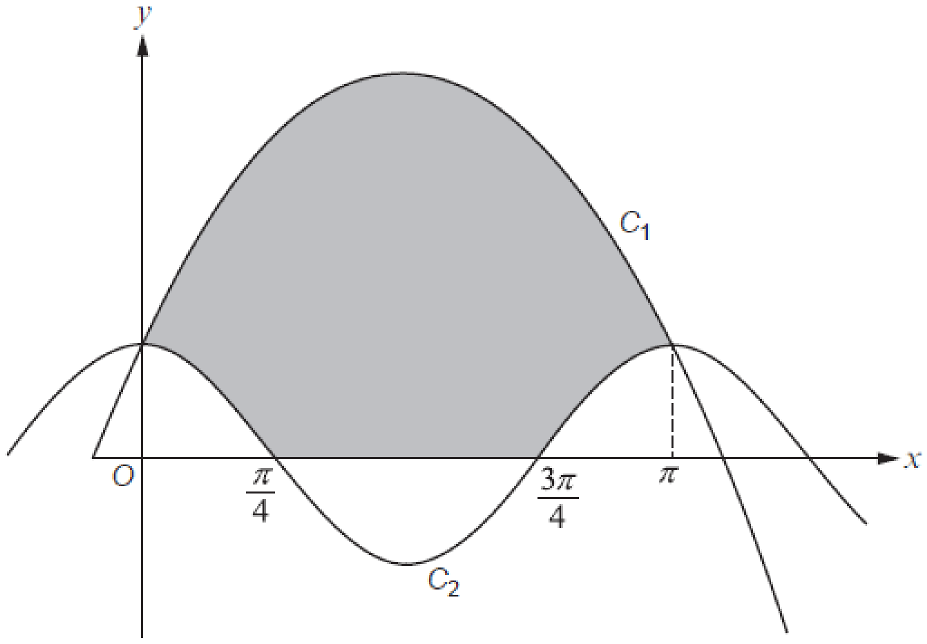
Math EV matics

Question 7 *(WJEC 2024)*

Find $\int x^2 \sin 2x \, dx$. [5]

Question 8 *(WJEC 2024)*

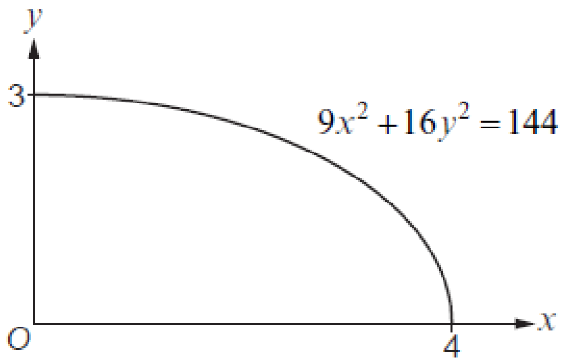
The diagram below shows a sketch of the curve C_1 with equation $y = -x^2 + \pi x + 1$ and a sketch of the curve C_2 with equation $y = \cos 2x$. The curves intersect at the points where $x = 0$ and $x = \pi$.



Calculate the area of the shaded region enclosed by C_1, C_2 and the x-axis. Give your answer in terms of π. [9]

Question 9 (W*JEC 2023)*

The aerial view of a patio under construction is shown below.



The curved edge of the patio is described by the equation $9x^2+16y^2=144$, where x and y are measured in metres.

To construct the patio, the area enclosed by the curve and the coordinate axes is to be covered with a layer of concrete of depth 0·06 m.

a) Show that the volume of concrete required for the construction of the patio is given by $0{\cdot}015\int_0^4 \sqrt{144-9x^2}\,\mathrm{d}x$. [3]

b) Use the trapezium rule with six ordinates to estimate the volume of concrete required. [4]

c) State whether your answer in part (b) is an overestimate or an underestimate of the volume required. Give a reason for your answer. [1]

Question 10 (*WJEC 2022)*

a) Use a suitable substitution to find

$$\int \frac{x^2}{(x+3)^4}\,\mathrm{d}x.$$ [5]

b) Hence evaluate $\int_0^1 \frac{x^2}{(x+3)^4}\,\mathrm{d}x$. [2]

TOTAL MARKS AVAILABLE: 76 MARKS.