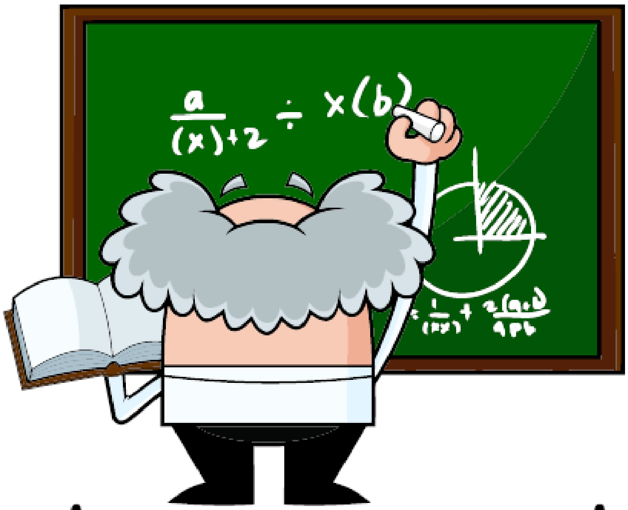

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AS Mathematics for WJEC

Unit 5:

Graphs and Transformations

Examples and Practice Exercises

Unit Learning Objectives

- ***Understand how to sketch cubic and quartic functions, including considering their points of intersection with axes and behaviour at repeated roots;***
- ***Understand and sketch the reciprocal graphs*** $y = \frac{a}{x}$ ***and*** $y = \frac{a}{x^2}$***, including their asymptotes;***
- ***Understand and apply transformations to graphs of the form*** $y = f(x)$***.***

Prior Learning Atoms:

- *Shapes of Cubic Graphs (GCSE)*
- *Quadratic Equations and Graphs (AS Unit 2)*
- *Solving Simultaneous Equations (AS Unit 3)*
- *Polynomial Division (AS Unit 4)*

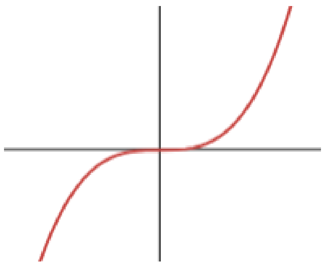
Now you have completed the unit...

Objective	Met	Know	Mastered
I can sketch cubic and quartic graphs.			
I can sketch reciprocal graphs.			
I can apply transformations to graphs, understanding the effect on specific points.			

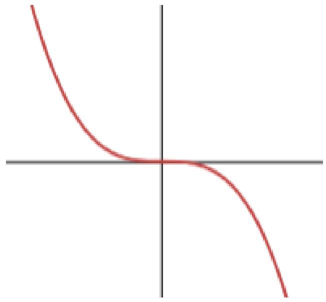
Notes/Areas to Develop:

Sketching Cubic Graphs

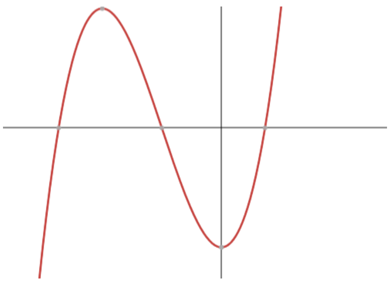
From GCSE, we should have met the basic shape of cubic graphs via plotting from tables:

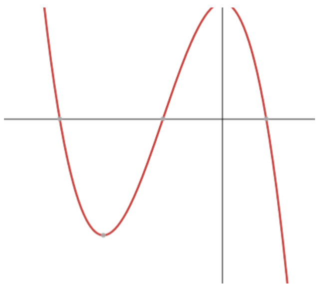


Positive x^3 coefficient



Negative x^3 coefficient





In this section we will look more closely at sketching graphs of the form

$y = ax^3 + bx^2 + cx + d$, understanding:

- The y-intercept;
- How to find the x-intercept(s);
- The maximum number of roots/turning points for a cubic;
- What happens at a repeated root.

The majority of examples will be pre-factorised to speed things up; however, now we have met the factor theorem and polynomial division, we should be prepared to factorise a cubic if required!

To sketch cubics of the form $y = ax^3 + bx^2 + cx + d$, we go through the following steps:

- Note whether it is a 'positive' or 'negative' x^3 coefficient for the correct shape.
- Understand the y-intercept occurs when $x = 0$, i.e. $y = d$
- Factorise (if needed) to find the roots (up to 3 for a cubic) – these are the points where the curve hits the x-axis.

Note that these are basically the same steps as with a quadratic!

Example 1: Sketch the following graphs, showing all points of intersection with the axes:

a) $y = (x + 2)(x + 1)(1 - x)$

b) $y = (x - 1)^2(x + 2)$

Task 1: Sketch the following graphs, showing all points of intersection with the axes:

a) $y = x(x-2)(x+1)$

b) $y = x^3 - 3x$

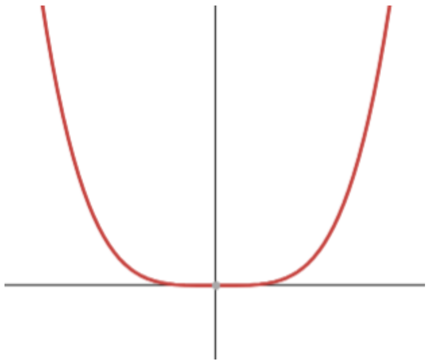
Example 2: Sketch $y = (x+2)^3$

Sketching Quartic Graphs

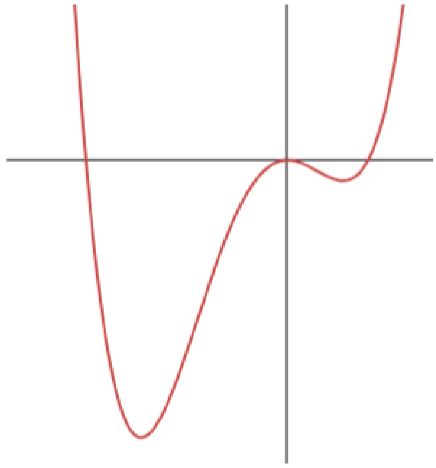
Anakin Skywalker: "This is where the fun begins."

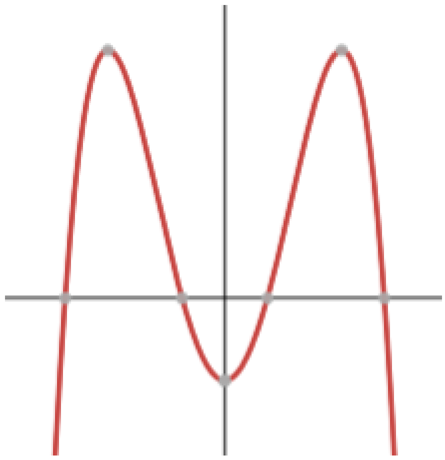
A quartic graph will (when expanded) have the form $y = ax^4 + bx^3 + cx^2 + dx + e$, where the coefficients a, b, c, d, e are real numbers, and $a \neq 0$.

There is much more variation on the possible shape of a quartic graph:



Positive x^4 coefficients





Negative x^4 coefficient

The same principles for sketching apply as for cubics; consider the 'shape', let $x = 0$ to find the y-intercept, find the roots for the x-intercepts, and be careful!

It is worth noting that WJEC rarely ever ask about quartics – but as always it pays to be prepared for the worst!

Example 3: Sketch the following graphs.

a) $y = (x+1)(x-1)(x+3)(x-2)$

b) $y = x(x+2)^2(2-x)$

Task 2: Sketch the following graphs:

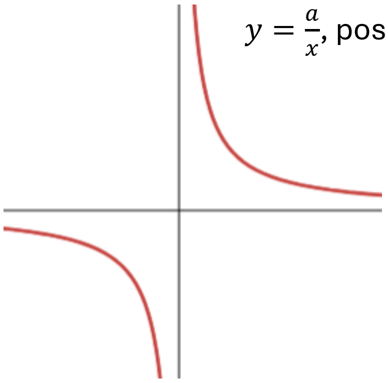
a) $y = (x+1)^2(x-2)^2$

b) $y = x^3(x+2)$

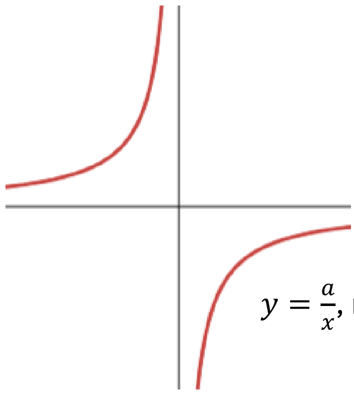
Now: Complete Test Your Understanding 1, Page 23.

Sketching Reciprocal Graphs

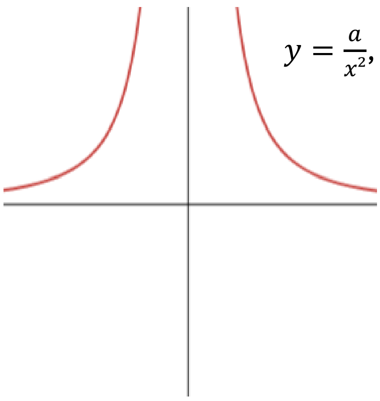
From GCSE we have briefly met graphs of the form $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$, both from plotting and from our study of inverse proportion. These graphs are wonderfully odd, and are basically just variations on each other:



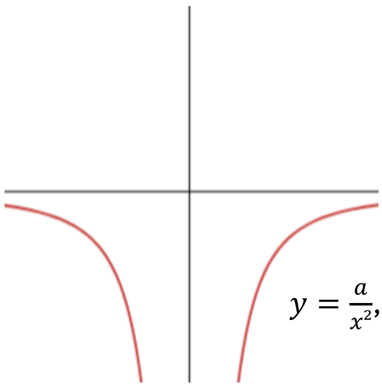
$y = \frac{a}{x}$, positive a



$y = \frac{a}{x}$, negative a



$y = \frac{a}{x^2}$, positive a



$y = \frac{a}{x^2}$, negative a

Increasing the value of a only serves to 'stretch' the graph.

Example 1: On the same set of axes, sketch $y = \frac{2}{x}$ and $y = \frac{6}{x}$.

Task 1: On the same set of axes, sketch $y = \frac{2}{x^2}$ and $y = -\frac{2}{x^2}$.

Intersections of Curves

We can be asked to consider how many times a pair of curves intersect, or even to find the point(s) of intersection of two curves. These are just extensions of skills we have already met!

Example 2: On the same diagram, sketch the curves $y = \frac{p}{x}$ and $y = x^3 - qx$, where p, q are positive constants. Hence, state the number of real solutions given by the equation

$$x^3 - qx - \frac{p}{x} = 0$$

Task 2: By sketching the graphs of $y = x^3$ and $y = \frac{1}{x^2}$ on the same axes, and giving a reason, state the number of real solutions given by the equation

$$x^3 - \frac{1}{x^2} = 0$$

Example 3:

a) Sketch, on the same axes, the graphs $y = x^2 - 2x$ and $y = x^2 - x^3$.

b) Find the coordinates of each of the points of intersection.

Space for additional notes:

Now: Complete Test Your Understanding 2, Page xx.

Transformations of Graphs

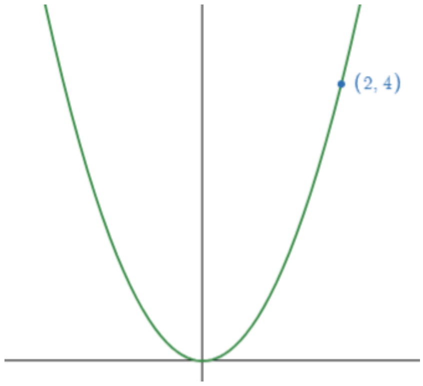
"This is where the fun ends..." – Anakin (rumoured) before losing multiple limbs.

You have met the different transformations at GCSE. If given a graph of $y = f(x)$, there are six types of transformation for us to understand (well, three, but they can be in either the x- or y-direction).

Translations

- $y = f(x) + a$ is a translation of a units in the positive y-direction.
- $y = f(x + a)$ is a translation of a units in the negative x-direction.

To try and understand these, let's consider $y = f(x)$ where $f(x) = x^2$:



The graph of $y = x^2$

$y = f(x) + 1$

$y = f(x + 1)$

Task 1: Given that $f(x) = x^3$, sketch the graphs o $y = f(x) - 1$ and $y = f(x - 1)$ on separate diagrams.

Task 2: Sketch the graph of $y = \frac{1}{x} + 1$, giving the equation of any asymptotes.

Stretches and Reflections

- $y = af(x)$ is a stretch, scale factor a, in the y-direction.
- $y = -f(x)$ is a reflection in the x-axis.

- $y = f(ax)$ is a stretch, scale factor $\frac{1}{a}$, in the x-direction.
- $y = f(-x)$ is a reflection in the y-axis.

Examiner tip: Notice how anything 'inside' the function affects the x-direction, whilst anything 'outside' the function affects the y-direction. This is because, if we have a function $f(x)$, x is our 'input' and the value of $f(x)$ (our 'y') is the output. We will learn much more about this at A2 when we investigate the domain and range of functions.

Example 1: On separate diagrams, for the function $f(x) = (x-2)^2$, sketch:

a) $y = f(x)$ b) $y = f(2x)$ c) $y = 3f(x)$

d) $y = -f(x)$ e) $y = f(-x)$

Task 3: On the same set of axes, sketch the pairs of curves with equations:

a) $y = x(x-1)(x+3)$ and $y = 2[x(x-1)(x+3)]$

b) $y = x(x-1)(x+3)$ and $y = 2x(2x-1)(2x+3)$

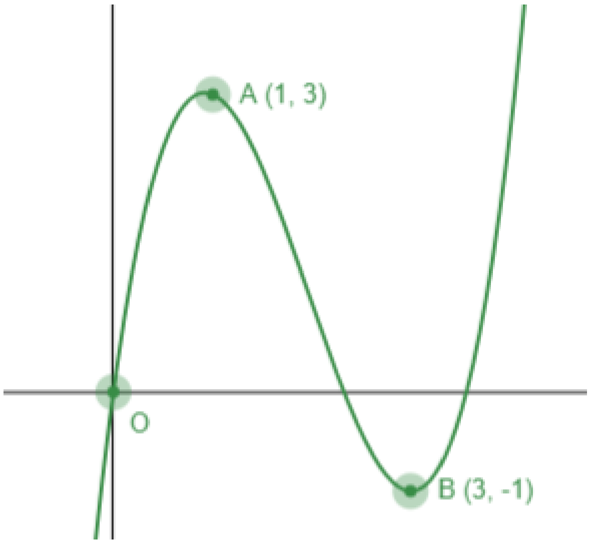
Task 4: You are given that $f(x) = x(x-2)$.

On the same diagram, sketch the graphs of $y = f(x)$, $y = -f(x)$ and $y = f(-x)$.

Now: Complete Test Your Understanding 3, Page xx.

Putting It All Together

Task 1: The following diagram shows part of the graph $y = f(x)$, with the coordinates of three points O, A and B where O is the origin.



Sketch each of the following, in each case indicating the coordinates of the images of O, A and B.

a) $y = 2f(x)$ b) $y = f(-x)$ c) $y = f(x + 1)$

Task 2:

a) Sketch the graph of $y = (x-2)^2(x-5)$.

b) The graph with equation $y = (nx-2)^2(nx-5)$ passes through the coordinate (1, 0). Find the two possible values of n.

Now: You are ready to face the Grade Enhancer™.

Test Your Understanding 1

Question 1

Sketch each of the following cubics, giving the coordinates of intersection with the axes.

a) $y = (x + 1)(x - 2)(x - 3)$

b) $y = x(x - 1)(x + 3)$

c) $y = (1 - x)(x - 2)^2$

d) $y = (2x + 1)(x - 2)(x - 5)$

e) $y = (x + 1)^3$

Question 2

Sketch each of the following quartics, giving the coordinates of intersection with the axes.

a) $y = (x + 1)(x - 1)(x - 3)(x + 3)$

b) $y = (x + 2)(x - 1)(x - 2)(x - 3)$

c) $y = x(x - 1)(x - 2)^2$

d) $y = x(2 - x)^3$

e) $y = (x + 1)^4$

Question 3

Sketch the following curves, giving the coordinates of intersection with the axes.

a) $y = x(x + 1)(2 - x)$

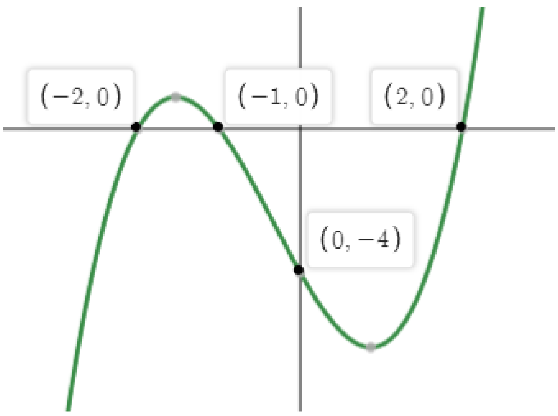
b) $y = (x^2 + 5x + 6)(x^2 - 9)$

c) $y = (x + 3)(x^2 - 1)$

d) $y = (x - 1)^2(x - 4)^2$

Question 4

The diagram below shows the graph of $y = x^3 + bx^2 + cx + d$. Find the values, of b, c and d.



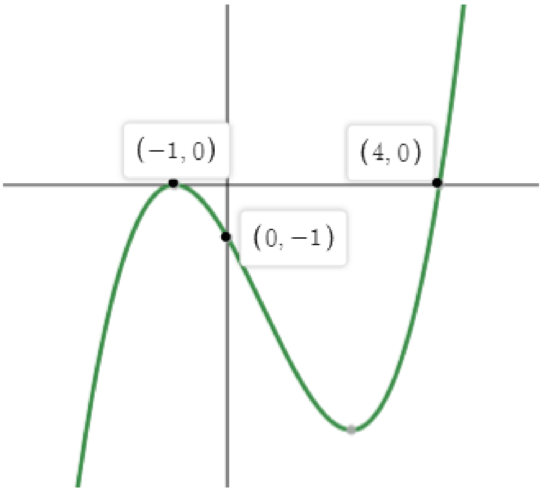
Question 5

a) Verify that $(x-5)$ is a factor of $f(x)$, where $f(x) = x^3 - x^2 - 16x - 20$.

b) Hence, sketch $y = f(x)$, showing the coordinates of all points of intersection with the coordinate axes.

Challenge 1

The diagram below shows the graph of $y = ax^3 + bx^2 + cx + d$. Find the values, of a, b, c and d.



Challenge 2

Sketch the graph of $y = (x+a)(x-a)(3x-1)$ where $a > \frac{1}{3}$, giving the coordinates of intersection with the axes in terms of a.

Test Your Understanding 2

Question 1

On the same set of axes (using a different colour for each graph), sketch each of the following pairs of graphs:

a) $y = \frac{3}{x}$ and $y = \frac{9}{x}$

b) $y = \frac{2}{x}$ and $y = -\frac{2}{x}$

c) $y = \frac{1}{x^2}$ and $y = \frac{4}{x^2}$

d) $y = -\frac{a}{x^2}$ and $y = -\frac{2a}{x^2}, a > 0$

Question 2

a) Sketch the graphs of $y = \frac{p}{x^2}, p > 0$ and $y = x^3$.

b) Giving a reason, state the number of solutions to the equation $x^3 - \frac{p}{x^2} = 0, p > 0$.

c*) *Explain what effect the condition $p < 0$ would have on the solution of this equation.*

Question 3

For each of the following pairs of curves:

i) Sketch both curves on the same diagram;

ii) State the number of points of intersection;

iii) Set up a suitable equation to solve for the x-coordinates of these points. (You do not need to solve your equation!)

a) $y = (x+1)(x-1)$ and $y = x^2(1-x)$

b) $y = \frac{2}{x}$ and $y = x^3$

c) $y = x(x-3)$ and $y = -\frac{1}{x^2}$

d) $y = x^3$ and $y = (x+1)^2(x-2)^2$

HINT: Think carefully about the behaviour of the two graphs as x grows very large!

Question 4

a) On the same set of axes, sketch $y = \frac{p}{x}$ and $y = -x(x-q)^2$, where p and q are positive constants.

b) Explain, with help of your diagram, why the equation $p + x^2(x-q)^2 = 0$ cannot have any solutions when p and q are positive constants.

Question 5

a) By sketching appropriate graphs, show that the equation $2x(4-x) = x^3 - 2x^2$ has three solutions.

b) Using an algebraic method, find the solutions to this equation, giving your answers as simplified surds where appropriate.

Challenge 1

a) On the same set of axes, sketch the curves $y = \frac{3}{x^2}$ and $y = 2x + 5$.

b) State the number of solutions to the equation $\frac{3}{x^2} = 2x + 5$

c) Verify that $x = -1$ is a solution to the equation.

d) By suitable rearrangement, find the exact x-coordinates of the point(s) of intersection.

Challenge 2

a) Show, by graphical means, that the equation $x^2 = x - 1$ has no solutions.

b) Find the range of values of k for which $x^2 + k = x - 1$ has two solutions.

Test Your Understanding 3

Question 1 – In each case give the points of intersection with the axes, and/or the equations of any asymptotes.

a) $f(x) = x^2$

Sketch i) $y = f(x)$, ii) $y = f(x+2)$ and iii) $y = f(x) + 2$

b) $f(x) = x^3$

Sketch i) $y = f(x)$, ii) $y = f(x-1)$ and iii) $y = f(x) - 1$

c) $f(x) = \frac{1}{x}$

Sketch i) $y = f(x)$, ii) $y = f(x+2)$ and iii) $y = f(x) + 2$

d) $f(x) = x^2 - 4$

Sketch i) $y = f(x)$, ii) $y = f(2x)$ and iii) $y = 2f(x)$

e) $f(x) = x(x+2)(x-2)$

Sketch i) $y = f(x)$, ii) $y = f(2x)$ and iii) $y = 2f(x)$

f) $f(x) = (x-1)^2$

Sketch i) $y = f(x)$, ii) $y = -f(x)$ and iii) $y = f(-x)$

Question 2

A curve $y = f(x)$ passes through the point $P(-2, 3)$. Find the image of P under the following transformations:

a) $y = f(2x)$ b) $y = 3f(x)$ c) $y = f(x-1)$ d) $y = f(x) + 2$

e) $y = -f(x)$ f) $y = f(-x)$

Question 3

a) Sketch the graph of $y = f(x)$, where $f(x) = x(x-3)$.

b) Determine the equation of $y = f(x-2)$ in terms of x, and hence sketch the graph of $y = f(x-2)$ including its points of intersection with the axes.

b) Determine the equation of $y = f(x) + 2$ in terms of x, and hence sketch the graph of $y = f(x) + 2$ including its points of intersection with the axes.

Question 4

a) Sketch the graph of $y = x^3 + 2x^2 - 15x$, showing the points of intersection with the x-axis.

b) Hence, sketch the graphs of

i) $y = (3x)^3 + 2(3x)^2 - 15(3x)$

ii) $y = (x+1)^3 + 2(x+1)^2 - 15(x+1)$

Question 5

a) Sketch the graph of $y = 3x - x^2$.

b) The point P $(1,0)$ lies on the curve with equation $y = 3(x+a) - (x+a)^2$. Find the possible values of a.

Question 6

a) Sketch the graph of $y = (x-2)^2(x-3)$

b) The graph of $y = (px-2)^2(px-3)$ passes through $(1,0)$. Find the two possible values of p.

Question 7

A point P $(2,5)$ lies on the curve with equation $y = f(x)$. Under a transformation, the image of P is $(-2,5)$. Give, using function notation, two possible transformations that would produce this image.

Challenge

A graph $y = f(x)$ is transformed to $y = af(bx)$.

a) Describe the effect of the combined transformation.

b) Explain why the order in which the two transformations are applied does not affect the resultant graph.

c) If, instead, $y = f(x)$ was transformed to $y = af(x) + b$, the order of applying transformations now matters – why? Which transformation should be applied first? (Use DESMOS to investigate!)

Grade Enhancer™ - Apply your knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(MathEVmatics Originals)*

$$f(x) = x^3 - 4x^2 - 3x + 18$$

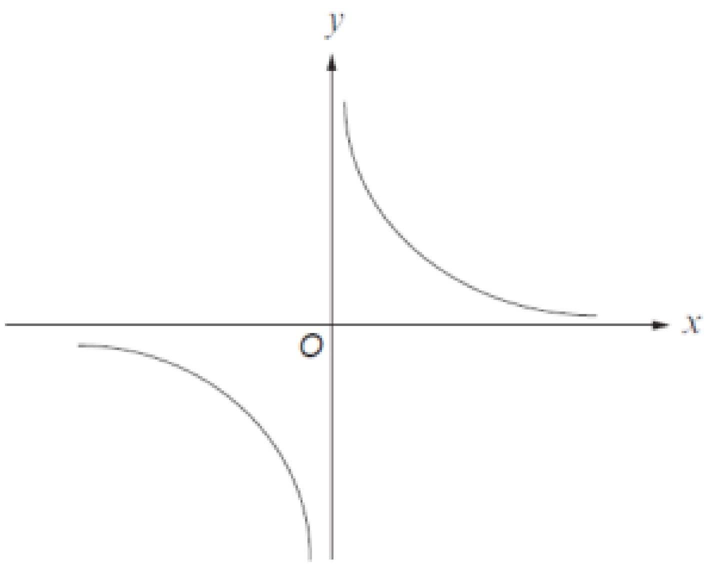
a) Show that $(x+2)$ is a factor of $f(x)$. [2]

b) Hence, sketch the graph of $y = f(x)$, showing the points of intersection with the axes. [5]

c) The graph of $y = f(x+a)$ has a root at (1, 0). Find the two possible values of a. [2]

Question 2 *(WJEC 2018)*

The diagram below shows a sketch of $y = f(x)$.



a) Sketch the graph of $y = 4 + f(x)$, clearly indicating any asymptotes. [2]

b) Sketch the graph of $y = f(x-3)$, clearly indicating any asymptotes. [2]

Question 3 *(WJEC January 2014)*

Figure 1 shows a sketch of the graph of $y = f(x)$. The graph has a maximum point at (2, 6) and intersects the x-axis at the points (–4, 0) and (8, 0).

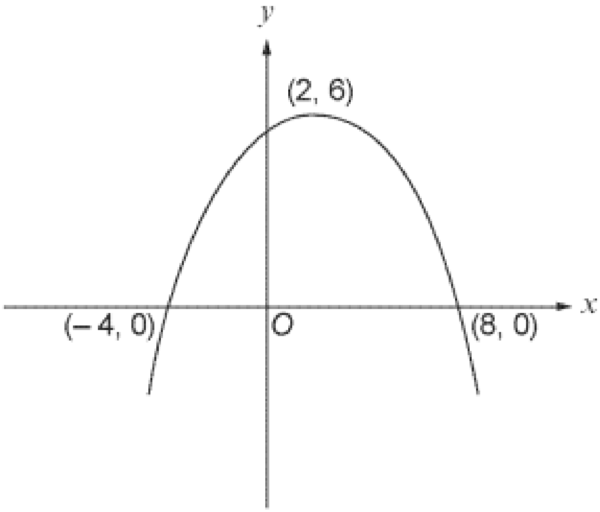


Figure 1

(a) Sketch the graph of $y = f(x - 3)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) **Figure 2** shows a sketch of the graph having **one** of the following equations with an appropriate value of p, q or r.

$y = f(x) + p$, where p is a constant
$y = f(qx)$, where q is a constant
$y = rf(x)$, where r is a constant

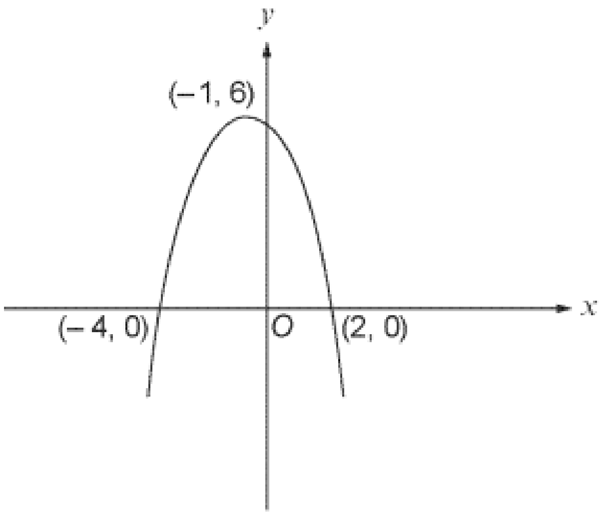
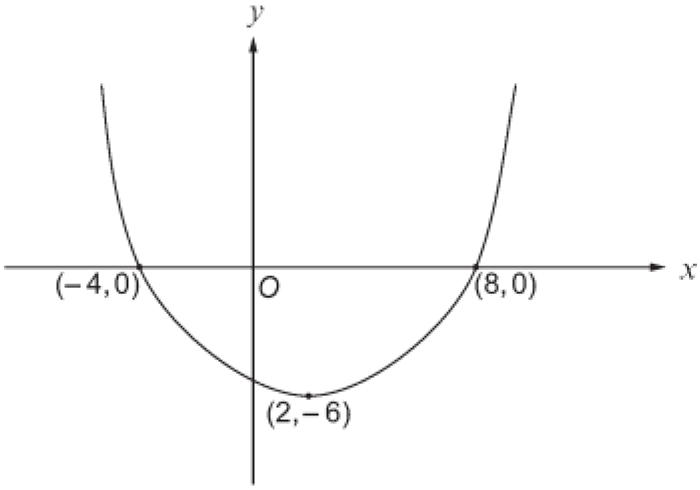


Figure 2

Write down the equation of the graph sketched in **Figure 2**, together with the value of the corresponding constant. [2]

Question 4 *(WJEC 2017)*

The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-4, 0)$ and $(8, 0)$ and has a minimum point at $(2, -6)$.



(a) Sketch the graph of $y = -\frac{1}{2}f(x)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) Siân is asked by her teacher to draw the graph of $y = f(ax)$ for various non-zero values of the constant a. Write down two facts about the stationary point on Siân's graph which will always be true whatever her choice of a. [2]

Question 5 *(WJEC 2016)*

Figure 1 shows a sketch of the graph of $y = f(x)$. The graph has a minimum point at $(1, -3)$ and intersects the x-axis at the points $(-4, 0)$ and $(6, 0)$.

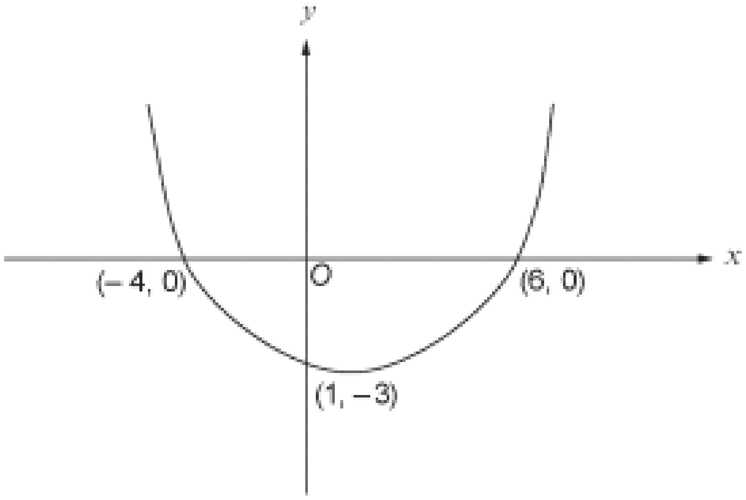


Figure 1

(a) Sketch the graph of $y = -3f(x)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) Figure 2 shows a sketch of the graph of $y = g(x)$, where

$g(x) = f(x) + p$, where p is a constant,

or $g(x) = f(qx)$, where q is a constant,

or $g(x) = rf(x)$, where r is a constant,

or $g(x) = f(x + s)$, where s is a constant.

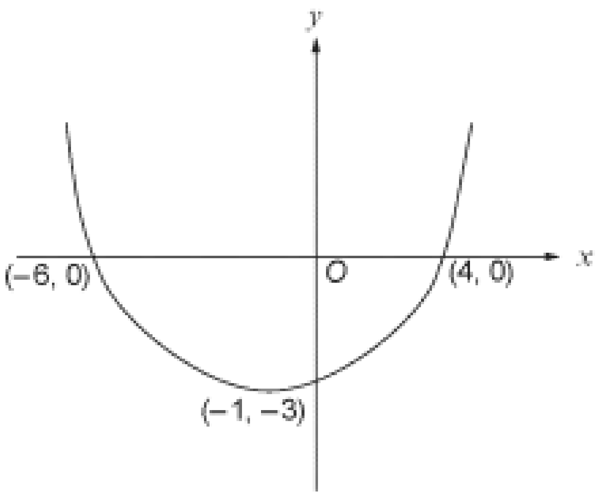
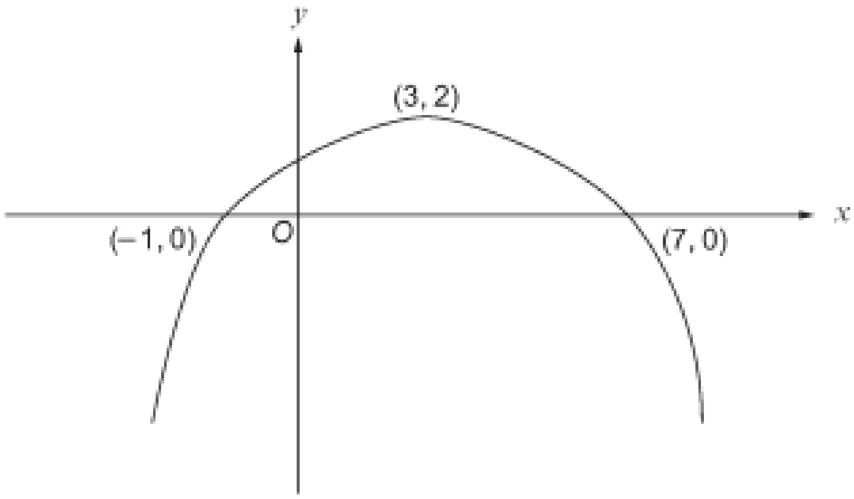


Figure 2

The function g can in fact be any one of **two** of the above functions. In each of these two cases, write down the expression for $g(x)$, including the value of the corresponding constant. [2]

Question 6 *(WJEC Summer 2014)*

The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-1, 0)$ and $(7, 0)$ and has a maximum point at $(3, 2)$.



(a) Sketch the following graphs, using a separate set of axes for each graph. In each case, you should indicate the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis.

(i) $y = f(x + 4)$

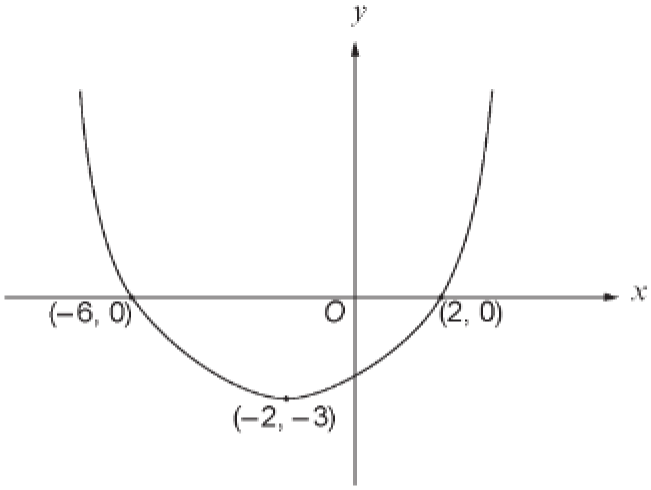
(ii) $y = -2f(x)$ [6]

(b) Hence write down one root of the equation

$$f(x + 4) = -2f(x) + 4.$$ [1]

Question 7 *(WJEC 2015)*

The diagram shows a sketch of the graph of $y = f(x)$. The graph passes through the points $(-6, 0)$ and $(2, 0)$ and has a minimum point at $(-2, -3)$.



(a) Sketch the graph of $y = f(\frac{1}{2}x)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the x-axis. [3]

(b) Angharad is asked by her teacher to draw the graph of $y = af(x)$ for various non-zero values of the constant a. One of Angharad's graphs passes through the origin O. Explain why this cannot possibly be correct. [1]

Question 8 (*WJEC 2023*)

The function f is defined by $f(x) = \frac{8}{x^2}$.

a) Sketch the graph of $y = f(x)$. [2]

b) On a separate set of axes, sketch the graph of $y = f(x-2)$. Indicate the vertical asymptote and the point where the curve crosses the y-axis. [3]

c) Sketch the graphs of $y = \frac{8}{x}$ and $y = \frac{8}{(x-2)^2}$ on the same set of axes.

Hence state the number of roots of the equation $\frac{8}{(x-2)^2} = \frac{8}{x}$. [2]

TOTAL MARKS AVAILABLE: 46 MARKS.