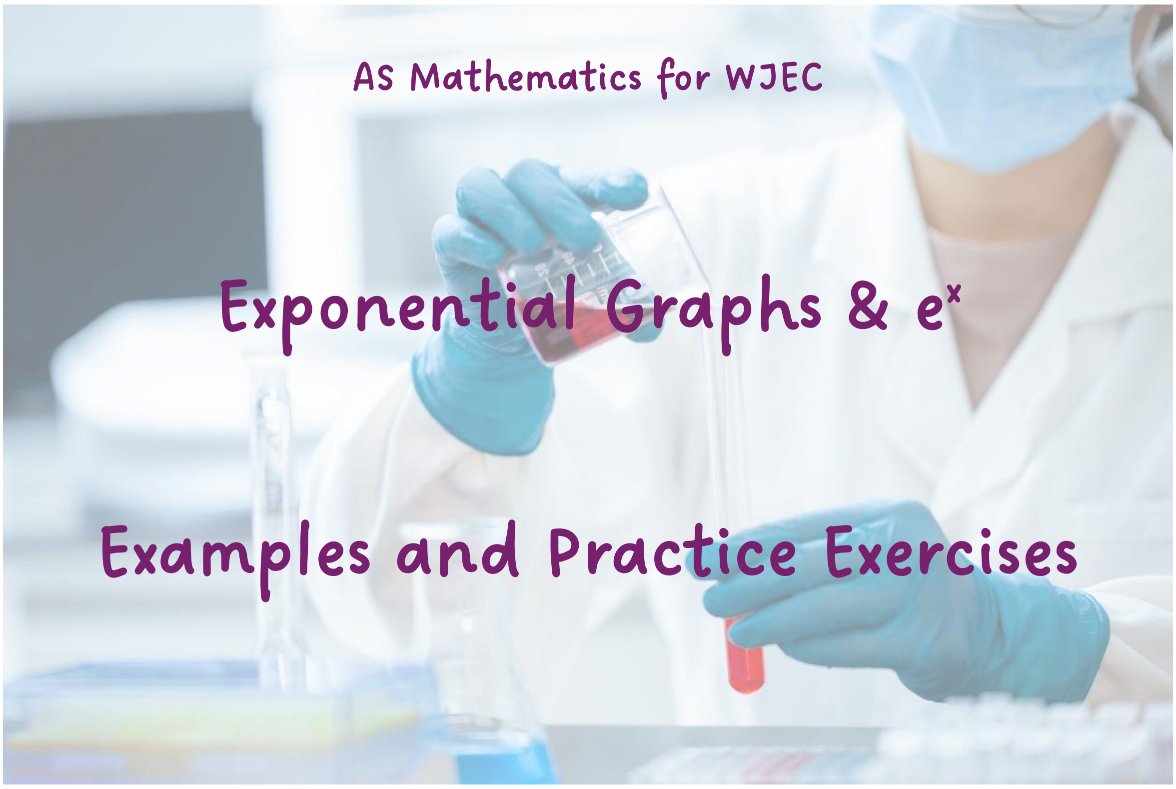
MathEVmatics

Supercharge your learning



AS Mathematics for WJEC

Exponential Graphs & e^x

Examples and Practice Exercises

Unit Learning Objectives

- Be able to sketch graphs of the forms $y = a^x$, $y = e^x$ and transformations of these graphs;
- Differentiate $y = e^{kx}$ and understand the importance of this result.
- Be able to solve simple models that use exponential functions.

Exponential growth and decay are everywhere in our world. From simple compound interest and depreciation, to lab-grown bacteria, the spread of COVID-19 (prior to immunity and vaccination), and half-lives of radioactive elements – the study of exponential behaviour is vital in many scientific, engineering and business models.

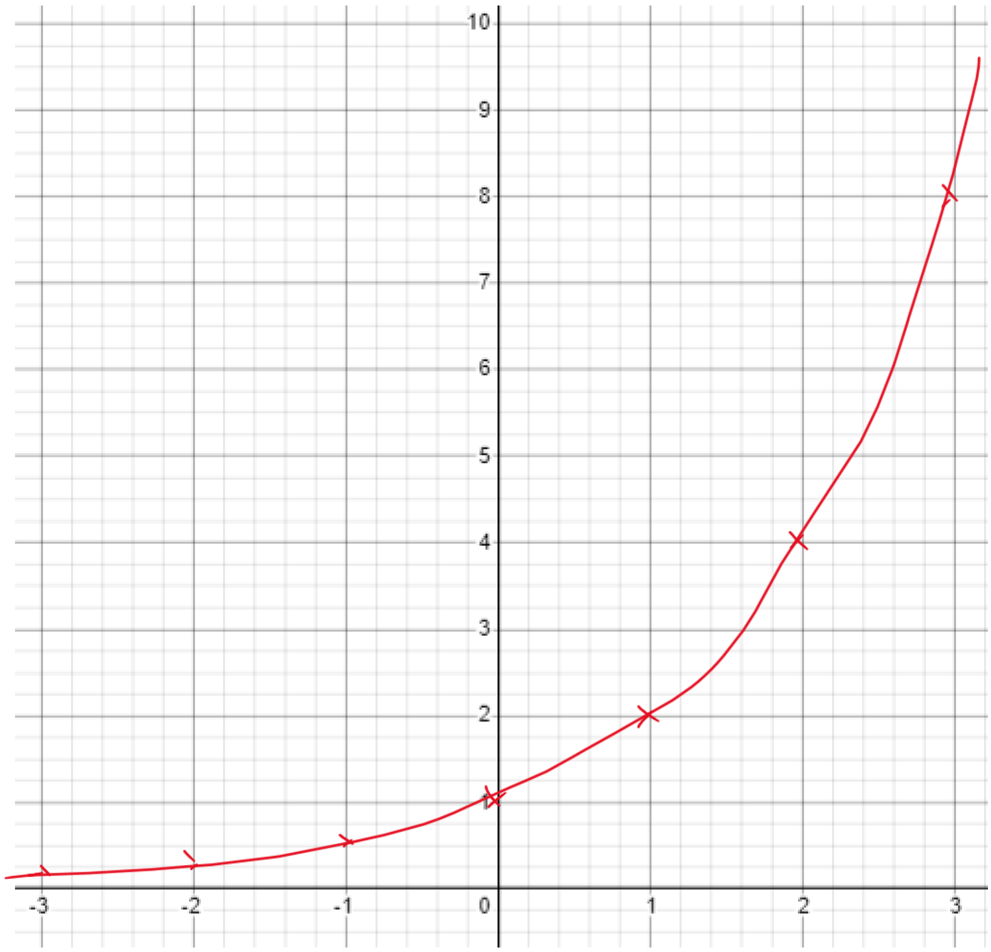
Exponential Graphs

An exponential function is a function of the form $f(x) = a^x$, where a is a constant. These graphs have a distinctive shape that you should be able to quickly recognise.

Example 1

By completing the table of values below, sketch the graph of $y = 2^x$.

x	-3	-2	-1	0	1	2	3
y	0.125	0.25	0.5	1	2	4	8

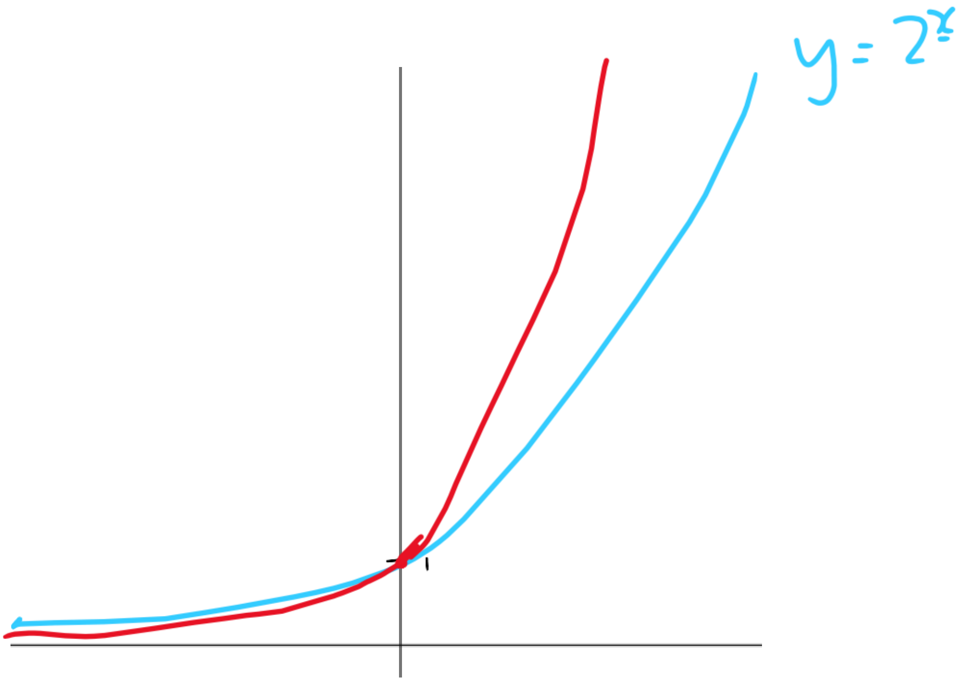


Space for key notes:

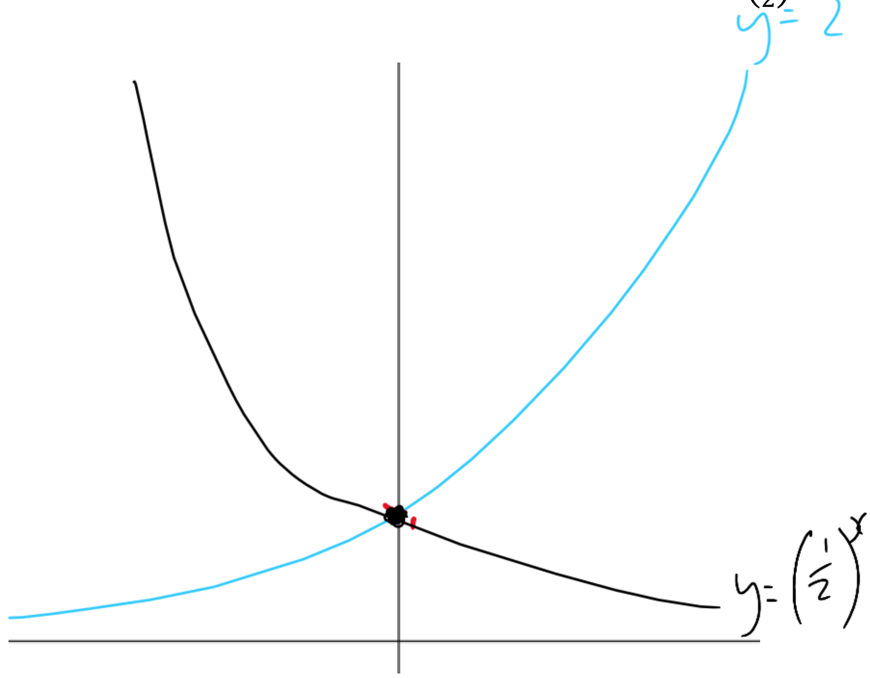
Always intercepts y-axis at 1

x-axis is an asymptote - never touches it!

Example 2: On the same set of axes, sketch $y = 3^x$ and $y = 2^x$ in different colours.



Task 1: On the same set of axes, sketch $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ in different colours.

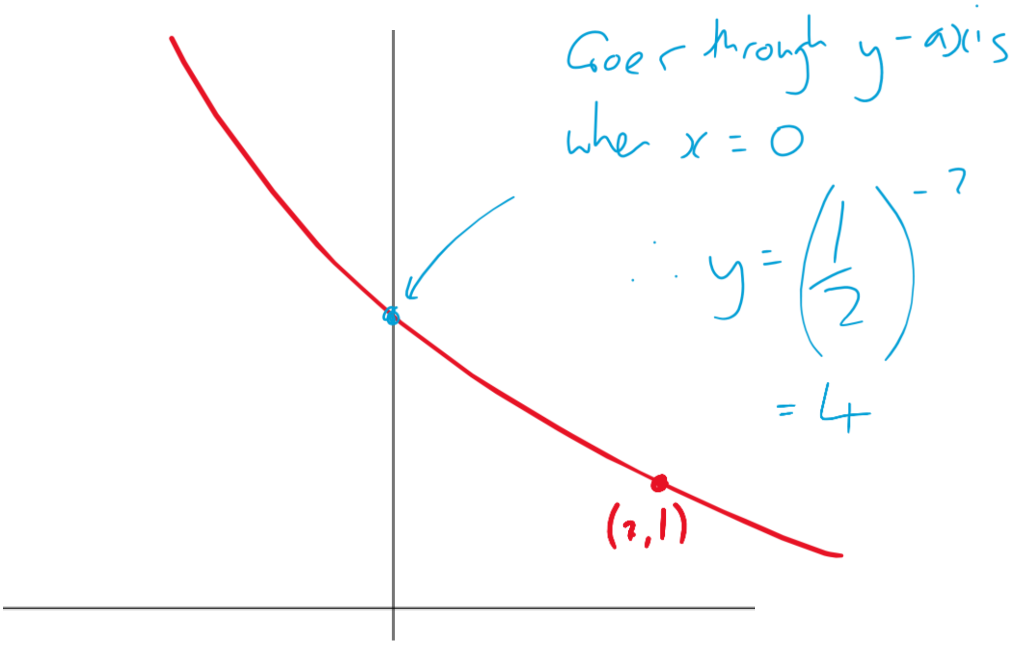


$\frac{1}{2} = 2^{-1}$

$\therefore \left(\frac{1}{2}\right)^x = 2^{-x}$

Example 3: Sketch the graph of $y = \left(\frac{1}{2}\right)^{x-2}$, showing clearly the coordinates of the y-intercept.

$f(x-2)$



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Test your Understanding 1

Question 1

On graph paper, accurately draw the graph of $y = 1.5^x$ for $-3 \leq x \leq 3$, and use your graph to estimate the solution to the equation $1.5^x = 3$.

Question 2

$f(x) = 2^x$.

For each of the following questions, sketch $y = f(x)$ and its transformation on the same set of axes, writing down the coordinates of the point where the curve crosses the y-axis, and the equation of any asymptotes.

a) $y = 3f(x)$ b) $y = f(x) - 3$ c) $y = f(2x)$

Question 3

The graph of $y = ka^x$ passes through the points (1, 6) and (3, 54). Find the values of the constants k and a.

Question 4

The graph of $y = pq^x$ passes through (-2, 0.162) and (2, 0.002). Find the values of p and q and hence find the value of y when x = 10, giving your answer in standard form rounded to 3 significant figures.

Challenge Question

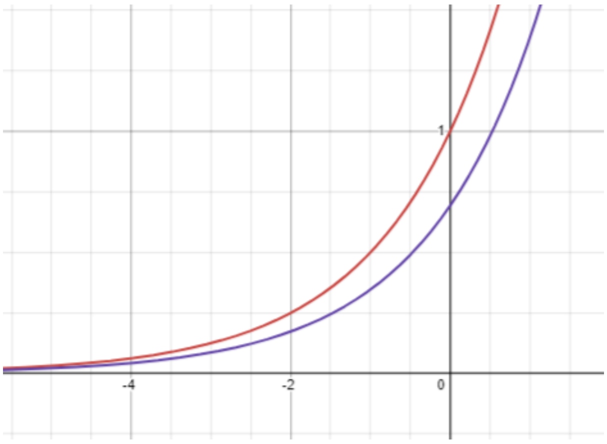
Sketch the graph of $y = 2^{x-3} - 5$, giving the coordinates of the y-intercept.

Euler's Constant and $y = e^x$

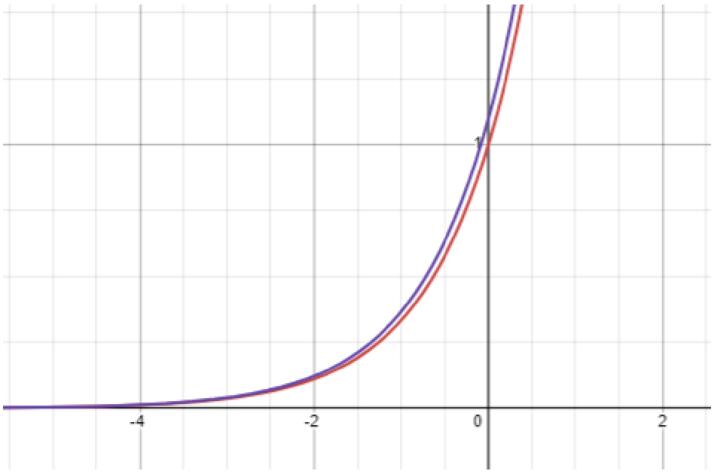
Exponential functions of the form $y = a^x$ have a very interesting property – the gradient of the function at any point on the curve is proportional to the value of the function (i.e. the y-value) at that point.

In other words, if $y = a^x$ then $\frac{dy}{dx} \propto a^x$, so $\frac{dy}{dx} = ka^x$.

In this figure, the red curve is $y = 2^x$ and the purple curve is the gradient function – notice how it is the same shape but grows slightly slower.



In this next figure, the red curve is $y = 3^x$ and the purple curve is its gradient function. Notice the same shape again, except this time it is growing slightly more quickly than the original function.



But this suggests an interesting thing: in the first figure the purple curve was "below" the red one, but in the second figure the purple curve is slightly above the red one. So, wasn't there a point where they crossed over? I.e. there must have been some value between a=2 and a=3 for which $y = a^x$ was identical to $\frac{dy}{dx}$...

Euler's constant is represented by e, where $e \approx 2.71828$. Like π, e is an irrational number (a never-ending, never-repeating decimal which cannot be written as an exact fraction involving integers) and an incredibly important mathematical constant.

- If $y = e^x$ then $\frac{dy}{dx} = e^x$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

Notice that, when we differentiate an exponential function, the e^{kx} 'bit' doesn't change – we just bring down the number multiplying the exponent.

Examiner comment – in A2 Mathematics we will learn the chain rule to understand ***why*** *this occurs.*

Example 1

Differentiate the following with respect to x

a) $y = e^{3x}$ b) $y = e^{-0.1x}$ c) $y = 2e^{4x}$

Example 2

Sketch the graphs of the following equations, giving the coordinates of any points where the graphs cross the y-axis, and the equation of the vertical asymptote.

Examiner tip: When sketching transformations of a function, always start off by sketching f(x) and then thinking about what transformation has been applied.

a) $y = e^{3x}$

b) $y = 6e^{-x}$

c) $y = 2 + 3e^{2x}$

Test your Understanding 2

Question 1

Use your calculator to evaluate e^x for the following x-values, giving the value to 4 decimal places.

a) x = 2 b) x = 5 c) x = -1 d) x = 0.3

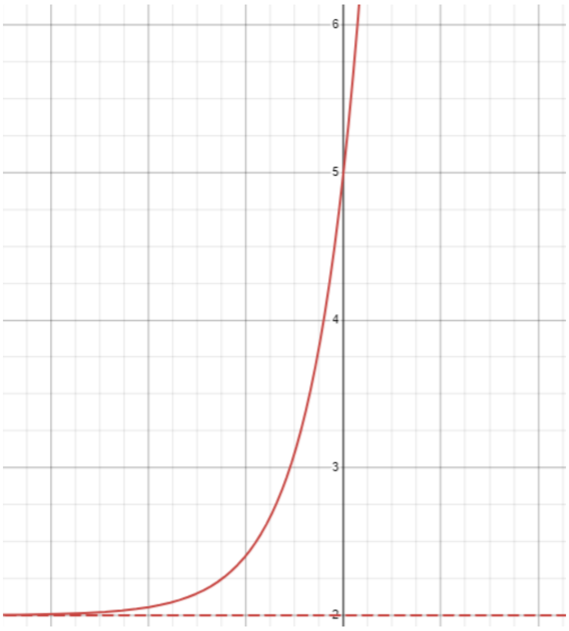
Question 2

Sketch the graph of $y = e^{x+2}$ and give the exact coordinates of the y-intercept.

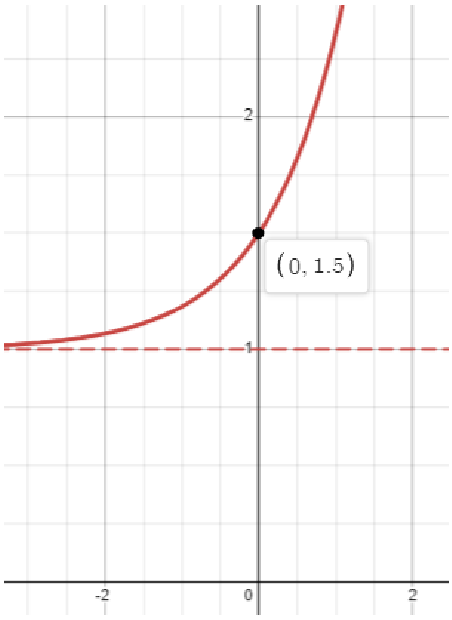
Question 3

For each of the following sketches, find the equation in the form $y = Ae^x + B$ where A and B are constants to be determined.

a)



b)



Question 4

Rewrite $f(x) = e^{2x+3}$ in the form $y = Ae^{kx}$ stating the value of k and giving the value of A to 4 significant figures.

Question 5

Differentiate the following with respect to x:

a) $y = e^{5x}$

b) $y = e^{-x}$

c) $y = 4e^{3x}$

d) $y = e^{3x} - e^{-2x}$

e) $y = e^{x}(e^{x} - 2)$

f) $y = 0.018e^{-0.025x}$

Question 6

Find, to 3 significant figures, the gradient of the curve of $y = f(x)$ when

a) $f(x) = e^{2x}$ at x = 5

b) $f(x) = e^{-5x}$ at x = 0.01

Question 7

A function f is defined by $f(x) = e^{0.1x}$.

Find the equation of the tangent to $y = f(x)$ at the point $(10, e)$, stating the exact gradient.

Modelling with Exponential Functions

As mentioned into the introduction to the chapter, exponential functions are incredibly useful for modelling a range of real-life situations.

Example 1

The population, P, of a species of cicada within a forest can be modelled by the equation $P = 150000e^{-0.04t}$, where t is the time in months since the population was first calculated.

a) Interpret the meaning of the constant 150000 in the model.

b) Find, to 3 significant figures, the population of cicadas after one year as given by the model.

c) Find the *rate of change* of population over time, and interpret the significance of the sign.

d) Sketch the graph of P against t.

e) Given that, after 5 years, the population of cicadas reaches approximately 70,000, evaluate the accuracy of the model as t gets larger.

Example 2

a) Two variables x and y are such that the rate of change of y with respect to x is proportional to y. State a possible relationship between x and y.

b) The concentration, Y units, of a certain drug in a patient's body decreases exponentially with respect to time. At time t hours the concentration can be modelled by $Y = 5e^{-0.3466t}$.

Find i) The concentration of the drug in the patient's body after 3 hours.

ii) The rate of change of concentration over time at 5 hours.

Test your Understanding 3

Question 1

The value, £V, of a car is modelled by the formula $V = 32000e^{-0.07t}$, where t is its age in years from new.

a) State its value when new.

b) Find its value (to the nearest £) after 5 years.

c) Sketch the graph of V against t.

d) Given that the car is worth £1250 after 10 years, evaluate the accuracy of the model as t gets larger.

Question 2

The population of a country is modelled using the formula $P = 9.6 + 8e^{0.02t}$ where P is the population (in millions) and t is the time in years after 1st Jan 2020.

a) State the population on 1st Jan 2020.

b) Calculate the population, as given by the model, on 1st July 2025.

c) Sketch the graph of P against t.

d) Justifying your answer, explain whether the model is suitable to predict the population of the country in 2320.

Question 3

The number of rabbits, R, in a population after m months is modelled by the formula $R = 8e^{0.25m}$.

a) Use this model to estimate the number of rabbits in the population after

i) 1 month   ii) 1 year

b) Interpret the meaning of the constant 8 in this model.

c) Find the rate of change of population after 6 months.

d) It is known that this model is not valid for suitably large values of t. Give one reason why the model may become invalid at larger values of t.

Question 4

The approximate pressure, p, in bars can be modelled by the formula $p = e^{-0.000128h}$ where h is the height above sea level in metres.

The five tallest mountains in the world, and their peak height in feet, is summarised in the following table:

Mountain	**Height (feet)**
Everest	29032
K2	28251
Kangchenjunga	28169
Lhotse	27940
Makalu	27838

Given that there are 12 inches in a foot, and one inch is equal to 2.54 centimetres, calculate the pressure in bars at the peak of each mountain.

Question 5

Martin purchases a rally car for £45000. He wishes to model the depreciation of the value of the car, £C, after t years. His friend suggests two plausible models:

Model 1: $C = 45000e^{-0.26t}$

Model 2: $C = 43000e^{-0.28t} + 2000$

a) Use both models to predict the value of the rally car after:

i) One year ii) Five years

b) Interpreting the meaning of the constant 2000 in model 2, suggest why the second model may be more realistic for large values of t.

Now you have completed the unit…

Objective	Met	Know	Mastered
Be able to sketch graphs of the forms $y = a^x$, $y = e^x$ *and transformations of these graphs;*			
Differentiate $y = e^{kx}$ *and understand the importance of this result.*			
Be able to solve simple models that use exponential functions.			

Notes/Areas to Develop: