MathEVmatics

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AS Mathematics for WJEC

The Equation of a Circle

Examples and Practice Exercises

Learning Objectives

- Understand and use the general equation of a circle, $(x-a)^2+(y-b)^2=r^2$ to find the centre and radius of a circle;
- Understand how to write a circle given in the form $x^2+y^2+2gx+2fy+c=0$ in completed square form to find the centre and radius;
- Be able to find the equation of a circle given the endpoints of its diameter;
- Be able to find the equation of a tangent to a circle, and to use the discriminant to show whether a line is a tangent or not;
- Be able to find points of intersection between circles and lines;
- Use circle and tangent properties to solve geometric problems;
- Understand and use the condition for two circles touching internally or externally.

Skill Check

1) Write x^2+10x in completed square form.

2) Find the distance between the points (2, -1) and (6, 2)

3) Find the equation of the line passing through (2, 3) and (5, 9)

4) Find the equation of the line perpendicular to y = 3x + 2 and passing through (4, 5), giving your equation in the form ax + by + c = 0 where a, b and c are integers.

Introducing the Equation of a Circle

Firstly, it would make sense to *define* a circle – as obvious as it might seem!

A circle is a set of points which are a fixed distance (the radius) away from a central point.

Let's consider a circle centred on the origin:

- **The equation of a circle, centred on the origin, with radius r is given by** $\boldsymbol{x^2 + y^2 = r^2}$**.**

However, we can extend this idea to circles that are not centred on the origin.

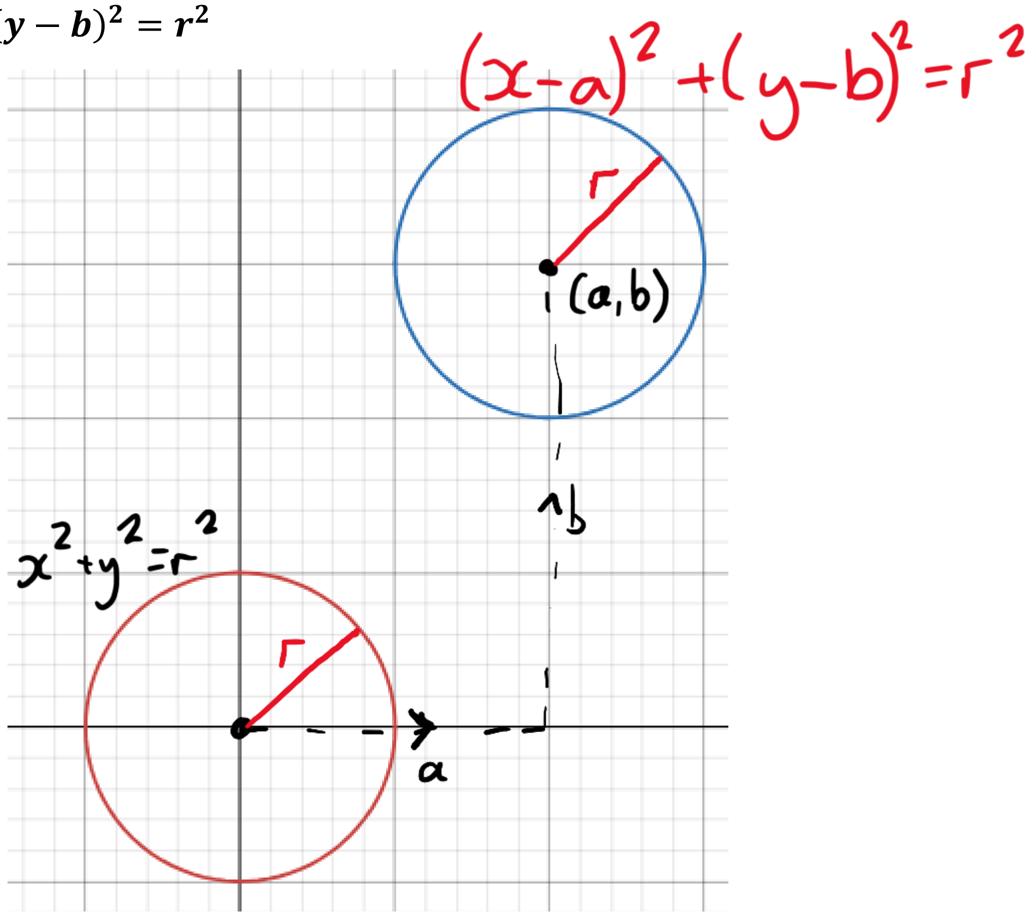
Consider our rules for translations:

$y = f(x - a)$ is a translation, a units, in the **positive** x-direction.

$y = f(x) + b$ is a translation, b units, in the **positive** y-direction – however, we could rearrange this to show that $(y - b) = f(x)$ is an alternative form for this transformation.

Putting these together, that means that replacing x by $(x - a)$ and y by $(y - b)$ in our general equation, we would have a circle with radius r centred on the point (a, b).

- **The general equation of a circle with centre $(\boldsymbol{a}, \boldsymbol{b})$ and radius $\boldsymbol{r}$ is given by**
$\boldsymbol{(x - a)^2 + (y - b)^2 = r^2}$



Example 1 – Write down the centre and radius of the following circles, giving the radius as a simplified surd where required:

a) $x^2 + y^2 = 9$

b) $(x - 3)^2 + (y - 2)^2 = 25$

c) $(x + 2)^2 + (y - 1)^2 = 7^2$

d) $x^2 + (y + 3)^2 = 8$

e) $(x - 1)^2 + (y + 4)^2 = 20$

Example 2 – Write down the equation of the following circles:

a) Centre $(2, 5)$, radius 4

b) Centre $(3, -1)$, radius 8

c) Centre $(-4, 0)$, radius $\sqrt{10}$

Sometimes, to be awkward *(fact: examiners like being awkward)*, the pesky examiner will test your understanding by asking something like this:

Example 3 – Write down the centre and radius of the circle with equation

$$x^2 + y^2 + 2x - 4y - 20 = 0$$

I hope you like completing the square...

Space for additional notes:

Test Your Understanding 1

Question 1

Write down the centre and radius of the following circles:

a) $(x+1)^2 + y^2 = 4$

b) $(x+4)^2 + (y-3)^2 = 36$

c) $(x-2)^2 + (y+3)^2 = 18$

d) $x^2 + (y-13)^2 = 50$

e) $(x-2.5)^2 + (y-1.25)^2 = 121$

Question 2

Write the equation of the following circles given the centre and radius:

a) Centre $(4, 1)$, radius 9

b) Centre $(-3, 2)$, radius 7

c) Centre $(2, -4)$, radius $\sqrt{10}$

d) Centre $(0, -2)$, radius 5

e) Centre $(2m, 5m)$, radius 7m

Question 3

Verify that the point (4, 11) lies on the circle with equation $(x+1)^2 + (y+1)^2 = 169$.

Question 4

Determine whether the point (5, 6) lies inside, outside or on the circle with equation

$(x+4)^2 + (y-3)^2 = 100$.

Question 5

Find the equation of the circle with centre $(2, 3)$ which passes through $(7, 1)$.

(Hint: to know the equation of a circle we need the centre and radius...)

Question 6

Find the centre and radius of the following circles by completing the square:

a) $x^2 + y^2 - 6x + 8y = 0$

b) $x^2 + y^2 + 2x - 4y - 11 = 0$

c) $x^2 + y^2 + 4x + 2y - 4 = 0$

d) $x^2 + y^2 - 10x + 12y + 25 = 0$

e) $x^2 + y^2 + 8x + 6y - 15 = 0$

Question 7

Find the equation of a circle whose diameter has endpoints (2, -1) and (5, 3).

Question 8

Given that a circle C has equation $x^2 + y^2 - 4x + 10y = k$, find the centre of the circle and the range of possible values for k.

Question 9

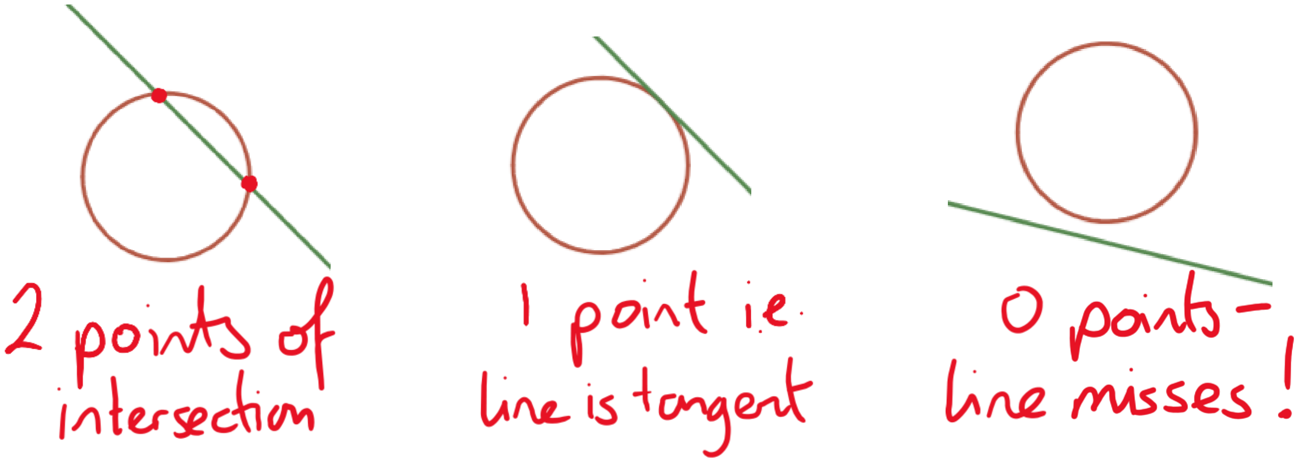
Given that the point (2, 3) lies on the circle with equation $(x - k)^2 + (y - 3k)^2 = 49$, find the possible values of k, giving each to 3 decimal places.

Challenge

A circle C has equation $(x - 2)^2 + (y - 5)^2 = r^2$. Given that the points P$(12, 5)$, Q$\left(-3, 5 + 5\sqrt{3}\right)$ and R$\left(-3, 5 - 5\sqrt{3}\right)$ lie on the circumference of the circle, find the value of r and show that PQR is equilateral.

Intersections of Lines and Circles

A straight line could intersect a circle twice, once or not at all.



Given the equation of a circle and a specific line:

- **To find the point(s) of intersection we can solve simultaneously, usually by substituting the linear expression into the equation of a circle;**
- **If we only want to know <u>how many</u> points of intersection exist, we can use the discriminant.**

Example 1

a) Show that the line x + y = 11 is a tangent to the circle with equation

$x^2 + (y - 3)^2 = 32$

b) Find the coordinates of the point of intersection.

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Example 2 – Show that the line $y = x - 10$ does not meet the circle $x^2 + y^2 - 4x = 21$

Test your Understanding 2

Question 1

Find the coordinates of the points where the circle $(x+1)^2+(y-2)^2=29$ meets the coordinate axes.

Question 2

By considering the discriminant, find how many times each of the following lines meets the circle with equation $(x-1)^2+(y-3)^2=25$

a) $x+y=3$

b) $3x+4y+10=0$

c) $4x-3y=20$

d) $2x-y=12$

Question 3

For each of the lines in question 2 which intersects the circle $(x-1)^2+(y-3)^2=25$, find the coordinates of each point of intersection.

Question 4

Show that the line $2x+y+2=0$ is tangent to the circle $(x-2)^2+(y-4)^2=20$ and find the point of intersection.

Question 5

The line $x+2y=1$ meets the circle $(x-3)^2+(y+5)^2=20$ at two points, A and B.

a) Find the coordinates of the points A and B.

b) Show that the perpendicular bisector of AB passes through the centre of the circle.

Challenge

The line $x+y=p$ meets the circle $(x+q)^2+(y-3)^2=20$ at the point $(0,7)$. State the value of p and hence find the possible values of q.

Solving Coordinate Geometry Problems

There are a number of circle properties that WJEC expect us to be able to identify and use for AS mathematics. Most of these will have been met at GCSE – I will sum these up at the end of the examples!

Example 1 – Find the equation of the tangent to the circle

$(x+2)^2 + (y-3)^2 = 20$ at the point $(2, 5)$

Pro examiner tip #1 – Always draw a diagram to help!!

Example 2 – The circle C has equation $(x+1)^2+(y+5)^2=10$. A chord to the circle passes through the points $P(-2,-2)$ and $Q(2,-4)$.

Show that the perpendicular bisector of the line PQ passes through the centre of the circle.

Example 3 – You are given the points $P(6, 6)$, $Q(2, 3)$ and $R(4, 7)$.

a) Show that $P\hat{R}Q$ is a right angle.

b) Hence find the equation of the circle which passes through the points P, Q and R.

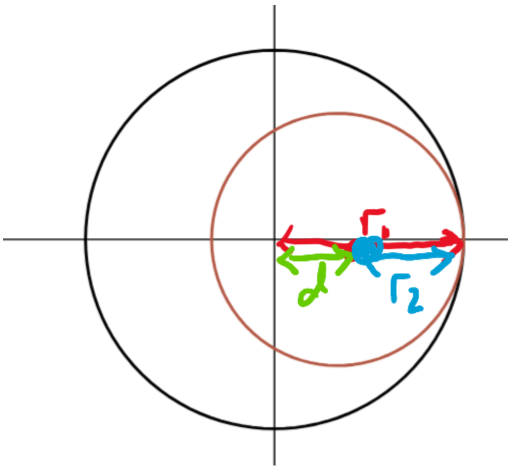
These three previous examples use properties of circles that we should have met at GCSE:

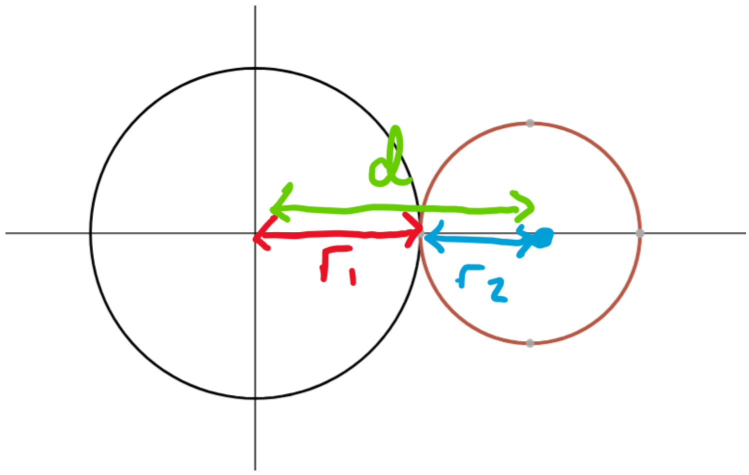
- **The tangent to a circle at a point P meets the radius at 90° at that point.**
- **The perpendicular bisector of a chord always passes through the centre of the circle.**
- **The right angle in a semi-circle is 90° (i.e. the angle opposite a diameter is always a right-angle)**

There is one final, new property that WJEC expect us to be able to use, which concerns the condition for two circles touching (tangentially) at a point.

For two circles C_1 and C_2 with radii r_1 and r_2 respectively

- **If the distance, d, between the centres circles C_1 and C_2 is equal to the difference between** r_1 and r_2, then the two circles touch internally.
- **If the distance, d, between the centres circles C_1 and C_2 is equal to the sum between** r_1 and r_2, then the two circles touch externally.





Case 1:
$d = r_1 - r_2$

Case 2:
$d = r_1 + r_2$

Example 4 – Show that the circle C_1 with equation $(x-4)^2+(y-4)^2=4$ touches the circle C_2 with equation $(x-1)^2+y^2=49$ internally.

Test Your Understanding 3

Question 1

a) Find the equation of the tangents to the circle $(x-1)^2 + (y-2)^2 = 20$ at the points:

i) A(-3, 4) ii) B(-1, 6) iii) C(-3, 0)

b) Find the coordinates of the point P where the equations at A and B intersect, and further verify that AP = BP.

Question 2

Find the equation of the tangent to the circle $(x+1)^2 + (y-2)^2 = 16$ at the point (2, 2-$\sqrt{7}$), giving your answer in the form y = mx + c where m and c are rationalised fractions with surds.

Question 3

A circle C has equation $(x+3)^2 + (y+5)^2 = 13$

a) State the radius of the circle and coordinates of the centre.

P(-5, -2) and Q(0, -3) are two points on the circle.

b) Find the midpoint of the line segment PQ

c) Find the gradient of PQ

d) Hence, show that the perpendicular bisector of PQ passes through the centre of the circle C.

Question 4

a) Show that the points P(3, 1), Q(6, 0) and R(4, 4) form a right-angled triangle.

b) Hence, find the equation of the circle which passes through P, Q and R.

Question 5

a) Show that the points J(-2, 8), K(7, 7) and L(-3, -1) form a right-angled triangle.

b) Hence, find the equation of the circle which passes through the points J, K and L.

Question 6

You are given that the points P(3, 15), Q(-13, 3) and R(-7, -5) lie on the circumference of a circle.

a) Show that PR is the diameter of the circle.

b) Find the equation of the circle and verify that the point S(8, 0) also lies on the circle.

Extended task: Verify that $S\hat{P}Q + S\hat{R}Q = 180°$

Question 7

Three circles, C_1, C_2 and C_3 are such that:

C_1: $(x-1)^2 + (y-4)^2 = 25$

C_2: $(x+1)^2 + (y-4)^2 = 9$

C_3: $(x-4)^2 + (y-4)^2 = 4$

a) Verify that C_2 and C_3 both touch C_1 internally.

b) Verify further that C_2 and C_3 touch each other externally.

Question 8

Calculate the exact shortest distance between the circle with equation $(x-6)^2 + (y-8)^2 = 36$ and the origin.

Hint: The shortest distance will always lie on the line which connects the origin to the centre of the circle.

End of Topic Assessment

These questions are all taken from A-Level papers, to test the full extent of your understanding of the content in this unit.

You should submit these questions within one week of the end of the unit.

Question 1 (OCR MEI)

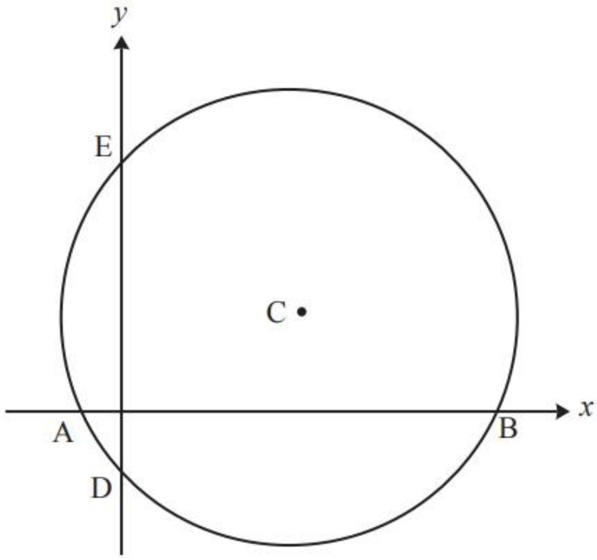


Fig. 11

Fig. 11 shows a sketch of the circle with equation $(x-10)^2 + (y-2)^2 = 125$ and centre C. The points A, B, D and E are the intersections of the circle with the axes.

(i) Write down the radius of the circle and the coordinates of C. **[2]**

(ii) Verify that B is the point (21, 0) and find the coordinates of A, D and E. **[4]**

(iii) Find the equation of the perpendicular bisector of BE and verify that this line passes through C. **[6]**

Question 2 (OCR MEI)

Fig. 10 shows a sketch of a circle with centre C(4, 2). The circle intersects the x-axis at A(1, 0) and at B.

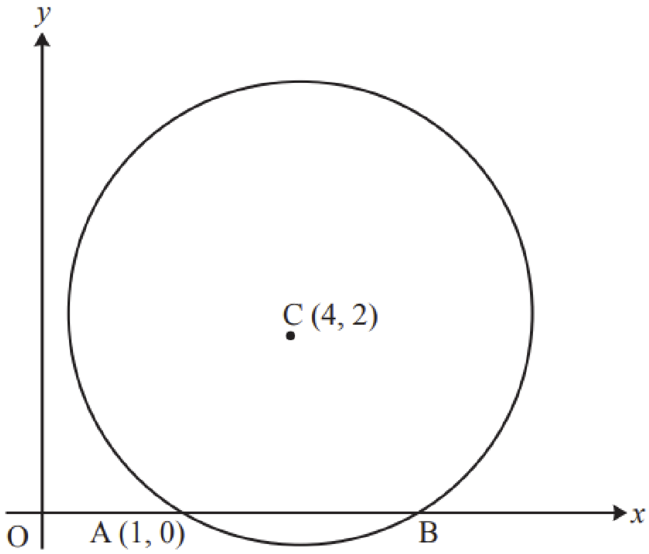


Fig. 10

(i) Write down the coordinates of B. **[1]**

(ii) Find the radius of the circle and hence write down the equation of the circle. **[4]**

(iii) AD is a diameter of the circle. Find the coordinates of D. **[2]**

(iv) Find the equation of the tangent to the circle at D. Give your answer in the form $y = ax + b$. **[4]**

Question 3 (WJEC)

The coordinates of three points *A*, *B*, *C* are (4, 6), (–3, 5) and (5, –1) respectively.

a) Show that $B\hat{A}C$ is a right angle. [3]

b) A circle passes through all three points *A*, *B*, *C*. Determine the equation of the circle. [5]

Question 4 (WJEC)

(a) The circle C_1 has centre $A(-2, 9)$ and radius 5. The circle C_2 has centre $B(10, -7)$ and radius 15.

(i) Show that C_1 and C_2 touch, justifying your answer.

(ii) Given that the circles touch at the point $P(1, 5)$, find the equation of the common tangent at P. [7]

(b) Gareth, who has been asked by his teacher to investigate the properties of another circle C_3, claims that the equation of this circle C_3 is given by

$$x^2 + y^2 + 4x - 6y + 20 = 0.$$

Show that Gareth cannot possibly be correct. [3]

Question 5 (WJEC)

The circle C_1 has centre A and equation

$$x^2 + y^2 + 6x - 20y + 59 = 0.$$

(a) (i) Find the coordinates of A and the radius of C_1.

(ii) Find the shortest distance from the origin to the circle C_1. Give your answer correct to two decimal places. [5]

(b) The line L has equation $y = 3x - 1$. The line L and the circle C_1 intersect at the points P and Q.

(i) Find the coordinates of P and Q.

(ii) The circle C_2 has centre $B(6, 7)$ and is such that PQ is the common chord of C_1 and C_2. Find the equation of C_2. [7]

Question 6 (WJEC)

The circle C has centre A and equation

$$x^2 + y^2 + 10x - 8y + 21 = 0.$$

(a) (i) Find the coordinates of A and the radius of C.

(ii) The point P has coordinates $(-2, 0)$. Determine whether P lies inside C, on C or outside C. [5]

(b) The line L has equation $y = 2x + 4$. Show that L is a tangent to the circle C and find the coordinates of the point of contact of L and C. [5]

Question 7 (OCR MEI)

A circle has equation $(x - 2)^2 + y^2 = 20$.

(i) Write down the radius of the circle and the coordinates of its centre. **[2]**

(ii) Find the points of intersection of the circle with the y-axis and sketch the circle. **[3]**

(iii) Show that, where the line $y = 2x + k$ intersects the circle,

$$5x^2 + (4k - 4)x + k^2 - 16 = 0.$$ **[3]**

(iv) Hence find the values of k for which the line $y = 2x + k$ is a tangent to the circle. **[4]**

Note: Whilst all of the individual skills within this unit are reasonably 'simple', this is often one of the hardest topics for students in terms of bringing it all together in exam questions. This booklet is the *minimum* practice that you may need on this topic, but it is possible (and highly advised) that you undertake further past paper practice on this area.

Now you have completed the unit…

Objective	Met	Know	Mastered
• *To know and use the general equation of a circle;*			
• *To be able to find the centre and radius by completing the square;*			
• *To be able to find points of intersection between lines and circles, understanding how to use the discriminant to determine the number of points of intersection.*			
• *To be able to find the equation of a tangent, and solve simple coordinate geometry problems involving tangents.*			
• *To be able to solve more difficult coordinate geometry problems involving right angled triangles and the properties of two circles touching (tangentially).*			