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A2 Mathematics for WJEC

Unit 9 - Further Differentiation

Examples and Practice Exercises

Unit Learning Objectives

- ***To recall and use the Chain, Product and Quotient rules;***
- ***To be able to differentiate trigonometric functions, including differentiating sin x and cos x by first principles;***
- ***Understand how to differentiate implicitly and why this is sometimes necessary.***
- ***Differentiate the inverse trigonometric functions;***
- ***Understand how to use differentiation to find points of inflection, and to consider whether functions are concave or convex.***
- ***Work with Related Rates of Change in problems.***

Prerequisite atoms:

Differentiation (AS Mathematics)

Differentiation (A2 Unit 6)

Atom Check:

Differentiate the following with respect to x:

a) $y = (5x + 3)^4$

b) $y = xe^{2x}$

c) $y = \frac{x}{3x-2}$

When you have completed the unit…

Objective	Met	Know	Mastered
I can recall and use the chain rule, product rule and quotient rule with confidence;			
I can understand differentiate sin x and cos x by first principles;			
I can differentiate trigonometric functions, and solve maxima/minima and tangent/normal problems by differentiation.			
I can differentiate a function implicity.			
I can understand and find points of inflection, concave and convex functions.			
I can solve problems involving connected rates of change.			

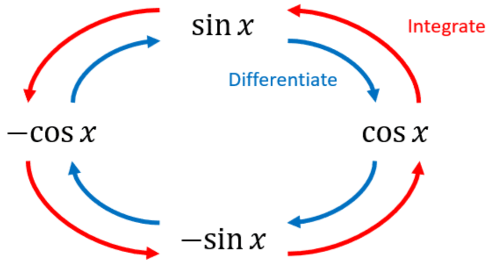
Notes/Areas to Develop:

Differentiating Trigonometric Functions

We need to be able to differentiate and integrate trigonometric functions.

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$

I sometimes will refer to the 'wheel of sine and cosine':



We also need to be able to prove the derivatives of sine and cosine by first principles.

Pro Examiner Tip: When we are differentiating or integrating trigonometric functions, we really should usually be working in radians. This is because, as you will see, we require the small angle approximations (SAAs) to differentiate by first principles.

Example 1: By first principles, show that the derivative of $\sin x$ is $\cos x$.

Let $f(x) = \sin x$

Then $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$

$$= \lim_{h \to 0}\left(\frac{\sin(x+h) - \sin x}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\sin x\left(1 - \frac{h^2}{2}\right) + \cos x(h) - \sin x}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\cancel{\sin x} - \frac{h^{\cancel{2}}}{2}\sin x + \cancel{h}\cos x - \cancel{\sin x}}{\cancel{h}}\right)$$

$$= \lim_{h \to 0}\left(-\frac{h}{2}\sin x + \cos x\right) \qquad \therefore f'(x) = \cos x$$

You will be asked to show the derivative of $\cos x$ *by first principles in TYU 1!*

We can extend the results slightly, by use of the chain rule, as follows:

- $\frac{d}{dx}(\sin kx) = k\cos kx$
- $\frac{d}{dx}(\cos kx) = -k\sin kx$

Example 2: For each of the following, find $f'(x)$:

a) $f(x) = \sin 3x - x^2$ b) $f(x) = \frac{1}{x} - \cos x$ c) $f(x) = \frac{5sin2x}{x^2}$

a) $f'(x) = 3\cos 3x - 2x$

b) $f'(x) = -\frac{1}{x^2} + \sin x$

c) $f'(x) = \frac{x^2 \cdot 10\cos 2x - 10x\sin 2x}{x^4}$

$= \frac{10(x\cos 2x - \sin 2x)}{x^3}$

Example 3: Using the chain and product/quotient rules, find the derivatives of the other four trigonometric functions. *(Note: These results are GIVEN in the formula booklet.)*

$y = \tan x$

$y = \frac{\sin x}{\cos x}$

$\frac{dy}{dx} = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$y = \sec x$

$y = (\cos x)^{-1}$

$\frac{dy}{dx} = -1(-\sin x)(\cos x)^{-2}$

$= \frac{\sin x}{\cos^2 x}$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

$= \sec x\tan x$

$y = \operatorname{cosec} x$

$y = (\sin x)^{-1}$

$\frac{dy}{dx} = -1(\cos x)(\sin x)^{-2}$

$= -\frac{\cos x}{\sin^2 x}$

$= -\cot x\operatorname{cosec} x$

$y = \cot x$

$y = \frac{\cos x}{\sin x}$

$\frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$

To summarise our differentiation results:

- $\frac{d}{dx}(\sin kx) = k\cos kx$
- $\frac{d}{dx}(\cos kx) = -k\sin kx$
- $\frac{d}{dx}(\tan x) = sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(cosec\ x) = -cosec\ x \cot x$
- $\frac{d}{dx}(\cot x) = -cosec^2 x$

The ones highlighted in green are given (though we have to be able to derive them using our other techniques), whereas the ones highlighted in yellow need to be learnt (and we have to be able to prove them via first principles).

Task 1: Using the results above, differentiate

a) $y = \tan 2x - 3\sqrt{x}$ b) $y = \sec x\, e^x$ c) $y = \ln x - 3cosec\ (3x - 10)$

a) $\frac{dy}{dx} = 2\sec^2 2x - \frac{3}{2\sqrt{x}}$

c) $\frac{dy}{dx} = \frac{1}{x} + 9cosec(3x-10)\cot(3x-10)$

b) $\frac{dy}{dx} = \sec x \tan x e^x + \sec x e^x$

$= e^x \sec x(\tan x + 1)$

Task 2:

A curve C has the equation $y = x - \sin 2x$. Find any stationary points on the curve on the interval $0 \leq x \leq \pi$.

$\frac{dy}{dx} = 1 - 2\cos 2x$

$1 - 2\cos 2x = 0 \Rightarrow \cos 2x = \frac{1}{2}$ $\quad 0 \leq 2x \leq 2\pi$

$2x = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\left(\frac{\pi}{6}, \frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{5\pi}{6} + \frac{\sqrt{3}}{2}\right)$

Task 3

Find an equation for the tangent to the curve $y = \tan 3x$ at the point where $x = \frac{\pi}{4}$.

$\frac{dy}{dx} = 3\sec^2 3x = \frac{3}{\cos^2 3x}$

At $x = \frac{\pi}{4}$, $y = -1$, $\frac{dy}{dx} = 6$

$\therefore y + 1 = 6\left(x - \frac{\pi}{4}\right)$

$y = 6x - \left(\frac{3\pi}{2} + 1\right)$

Test Your Understanding 1

Question 1

For each of the following, find $\frac{dy}{dx}$.

a) $y = 3\sin x$

b) $y = \cos 6x$

c) $y = 5 + \tan 2x$

d) $y = \frac{1}{2}\cot 4x$

e) $y = \sin\left(2x + \frac{\pi}{6}\right)$

f) $y = 2\sec 3x$

g) $y = cosec(5x - 1)$

h) $y = sin^2 x$

i) $y = cos^3 x$

j) $y = tan^2 x$

k) $y = sec^3 x$

l) $y = 2\sec x - \tan x$

a) $\frac{dy}{dx} = 3\cos x$

b) $\frac{dy}{dx} = -6\, sin\, 6x$

c) $\frac{dy}{dx} = 2\, sec^2 2x$

d) $\frac{dy}{dx} = -2\, csc^2 4x$

e) $\frac{dy}{dx} = 2\, cos(2x + \frac{\pi}{6})$

f) $\frac{dy}{dx} = 6\, sec\, 3x\, tan\, 3x$

g) $\frac{dy}{dx} = -5\, cosec(5x - 1) cot(5x - 1)$

h) $\frac{dy}{dx} = 2\, sin\, x\, cos\, x$

i) $\frac{dy}{dx} = -3\, cos^2 x\, sin\, x$

j) $\frac{dy}{dx} = 2\, tan\, x\, sec^2 x$

k) $\frac{dy}{dx} = 3\, sec^3 x\, tan\, x$

l) $\frac{dy}{dx} = 2\, sec\, x\, tan\, x - sec^2 x$

Question 2

Use the chain rule to find $f'(x)$ when:

a) $f(x) = \ln(\sin x)$

b) $f(x) = 3e^{\cos x}$

c) $f(x) = \sqrt{cosec\ 2x}$

$f'(x) = \cot x$

$f'(x) = -3\sin x\, e^{\cos x}$

$f'(x) = -\cot 2x\sqrt{\mathrm{cosec}\, 2x}$

Question 3

In the interval $0 \le x \le 2\pi$, find the coordinates of any stationary points on the curves

a) $y = 2\sec x - \tan x$

b) $y = \sin x + \cos 2x$

$\left(\frac{\pi}{6}, \sqrt{3}\right), \left(\frac{5\pi}{6}, -\sqrt{3}\right)$

$\left(0.253, \frac{9}{8}\right), \left(\frac{\pi}{2}, 0\right), \left(2.89, \frac{9}{8}\right), \left(\frac{3\pi}{2}, -2\right)$

Question 4

Use the product or quotient rules, as appropriate, to find $\frac{dy}{dx}$ when:

a) $y = x \cos x$

$\frac{dy}{dx} = \cos x - x\sin x$

b) $y = \frac{\sin 3x}{x}$

$\frac{dy}{dx} = \frac{3x\cos 3x - \sin 3x}{x^2}$

c) $y = e^x \sec x$

$\frac{dy}{dx} = e^x \sec x\,(1 + \tan x)$

d) $y = x^2 \cot x$

$\frac{dy}{dx} = 2x \cot x - x^2 \operatorname{cosec} x$

e) $y = \cos x \cot x$

$\frac{dy}{dx} = -\cos x - \cot x \operatorname{cosec} x$

f) $y = \frac{x^3}{cosec\ 2x}$

$\frac{dy}{dx} = 3x^2 sin2x + 2x^3 cos2x$

Question 5

Find the equation of the tangent to the curve $y = \cos x$ at the point where $x = \frac{\pi}{3}$.

$$y = -\frac{\sqrt{3}}{2}x + \left(\frac{\sqrt{3} + \pi}{2\sqrt{3}}\right)$$

Question 6

Find the equation of the tangent to the curve $y = cosec\ x - 2\sin x$ at the point where $x = \frac{\pi}{6}$.

$$y = -3\sqrt{3}x + \left(\frac{2 + \sqrt{3}\pi}{2}\right)$$

Question 7

Find the value of $f'(x)$ at $x = \frac{\pi}{4}$ when $f(x) = \sin 5x \cos 3x$

$$f'(x) = 5cos5xcos3x - 3sin3xsin5x$$

$$f'\left(\frac{\pi}{4}\right) = 4$$

Question 8

Find the equation of the normal to the curve $y = \sec x$ at the point where $x = \frac{\pi}{4}$.

$$y = -\frac{1}{\sqrt{2}}x + \sqrt{2} + \frac{\pi}{4\sqrt{2}}$$

Question 9

A curve C is defined by the equation $y = \frac{\sin x + 2}{1 - sinx}$ for the domain $0 \leq x \leq 2\pi, x \neq \frac{\pi}{2}$.

a) Explain why $x = \frac{\pi}{2}$ must be excluded from the domain of C.

Since $\sin\left(\frac{\pi}{2}\right) = 1,$ which would make the denominator equal to zero.

b) Show that any turning points on the curve must satisfy the equation

$$\frac{3cosx}{(1 - \sin x)^2} = 0$$

c) Hence find the exact coordinates of the stationary point of C.

$$\left(\frac{3\pi}{2}, \frac{1}{2}\right)$$

Question 10

Given that $x = tan3y$,

a) Find $\frac{dx}{dy}$

$$\frac{dx}{dy} = 3\sec^2 3y$$

b) Hence, show that $\frac{dy}{dx} = \frac{1}{3(1+x^2)}$

$$\frac{dy}{dx} = \frac{1}{3\sec^2 3y}$$

Using the identity $1 + \tan^2 3y = \sec^2 3y$, and given $x = \tan 3y$, the result follows.

Challenge

Show that the derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1} x,$ so that $\sin y = x$

Then, $\frac{dx}{dy} = \cos y$

So $\frac{dy}{dx} = \frac{1}{\cos y}$

Since $\sin^2 A + \cos^2 A = 1,$ we can rearrange to get that $\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$

Since the graph of $y = \sin^{-1} x$ is always increasing, we take the positive root.

Hence $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

Implicit Differentiation

Generally to date, we have differentiated functions of the form $y = f(x)$ to find $\frac{dy}{dx}$.

We have seen that sometimes we have to write in the form $x = f(y)$ and then find $\frac{dx}{dy}$ first, before finding $\frac{dy}{dx}$ by taking the reciprocal.

These are both examples of **explicit** differentiation – differentiating in terms of a single variable.

However, what if we had a total mess, such as the equation $x^3y - y^2x^2 = 2x - xy$? Where would we even start in trying to write this as $x =$... or $y =$... ?

When we have to deal with such functions, we differentiate **implicitly**.

Once you get the hang of it, it's really not too bad!

If we differentiate 'x' terms within an expression with respect to x, we do this as normal.

However, any time we differentiate a 'y' term with respect to x, we will multiply it's derivative by dy/dx.

We will rearrange, if necessary, to find $\frac{dy}{dx}$ at the end – however this is usually in terms of ***both x and y.***

Example 1: Find $\frac{dy}{dx}$ where $x^3 + x + y^2 + 3y = 5$

Differentiate each term: $3x^2 + 1 + 2y\frac{dy}{dx} + 3\frac{dy}{dx} = 0$

$$(2y+3)\frac{dy}{dx} = -1 - 3x$$

$$\frac{dy}{dx} = \frac{-1-3x}{2y+3}$$

Task 1:

Differentiate the following implicitly, giving each answer in the form $\frac{dy}{dx}=$...

a) $x^2 + y^3 = 2$

b) $5x^3 + y^2 = x$

c) $x^2 + 4x + y^2 - 2y = 20$

d) $3x^2y = 4y$

e) $ye^x = 2e^y$

f) $\frac{x}{y+2} = x^2 - y^2$

a) $2x + 3y^2\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{2x}{3y^2}$

b) $15x^2 + 2y\frac{dy}{dx} = 1 \quad \therefore \frac{dy}{dx} = \frac{1-15x^2}{2y}$

c) $2x + 4 + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{2x+4}{2-2y}$

d) $3x^2\frac{dy}{dx} + 6xy = 4\frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{6xy}{4-3x^2}$

e) $e^x\frac{dy}{dx} + ye^x = 2e^y\frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{ye^x}{2e^y - e^x}$

f) $\frac{(y+2)(1) - x\frac{dy}{dx}}{(y+2)^2} = 2x - 2y\frac{dy}{dx}$

$y+2 - x\frac{dy}{dx} = 2x(y+2)^2 - 2y(y+2)^2\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2x(y+2)^2 - y - 2}{2y(y+2)^2 - x}$

Example 2: Given that $3xy - y^2 = 5$, use implicit differentiation to find the value of $\frac{dy}{dx}$ at the point (2, 1).

Examiner Warning: *Look out for **products** or **quotients** in implicit expressions – these are the number one cause of losing a LOT of marks in this topic!*

$$\left(3x\frac{dy}{dx}+3y\right)-2y\frac{dy}{dx}=0$$

At $(2,1)$, $3(2)\frac{dy}{dx}+3(1)-2(1)\frac{dy}{dx}=0$

$$6\frac{dy}{dx}+3-2\frac{dy}{dx}=0$$

$$4\frac{dy}{dx}=-3 \quad \text{so} \quad \frac{dy}{dx}=-\frac{3}{4}$$

(We could also rearrange to e.g. $\frac{dy}{dx}=\frac{3y}{2y-3x}$ first;

then $\frac{dy}{dx}=\frac{3(1)}{2(1)-3(2)}=-\frac{3}{4}$ also!

Example 3: Find the coordinates of the stationary point(s) on the curve defined implicitly by the equation $x^2+4y^2-6x-16y+21=0$

$$2x+8y\frac{dy}{dx}-6-16\frac{dy}{dx}=0 \qquad \frac{dy}{dx}=\frac{6-2x}{8y-16}$$

At st. pts $\frac{dy}{dx}=0 \Rightarrow 6-2x=0$

$$2x=6 \qquad x=3$$

When $x=3$, $(3)^2+4y^2-6(3)-16y+21=0$

$$4y^2-16y+12=0$$

$$y^2-4y+3=0$$

$y=1$ and 3

$\therefore (3,1)$ and $(3,3)$

Explore this curve at: https://www.desmos.com/calculator/xrrnwqvygo

Task 2: An ellipse has equation $25x^2 + 16y^2 + 200x - 160y + 400 = 0$.

Find $\frac{dy}{dx}$ in terms of x and y and hence find the coordinates of any points where there are

a) horizontal tangents $\leftarrow \frac{dy}{dx} = 0$

b) vertical tangents $\leftarrow \frac{dx}{dy} = 0$

$$\frac{dy}{dx} = -\frac{25(x+4)}{16(y-5)}$$

a) Horizontal tangents $\Rightarrow \frac{dy}{dx} = 0$ $\quad x = -4 \quad (-4, 0), (-4, 10)$

b) Vertical tangents $\Rightarrow \frac{dx}{dy} = 0$ $\quad y = 5 \quad (-8, 5), (0, 5)$

Explore this ellipse at: https://www.desmos.com/calculator/sow0luhz09

Space for notes:

Test Your Understanding 2

Question 1: Differentiate the following implicitly, giving your answer in the form $\frac{dy}{dx} =$ where possible.

a) $x^2 + y^3 = 3$

$\frac{dy}{dx} = -\frac{2x}{3y^2}$

b) $\sqrt{x} + \sqrt{y} = 1$

$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

c) $3y^2 + 4x^3 + 2x + 1 = 0$

$\frac{dy}{dx} = -\frac{6x^2+1}{3y}$

d) $sinx + cosy = x$

$\frac{dy}{dx} = \frac{\cos x - 1}{\sin y}$

e) $e^{2y} - e^{3x} = lny$

$\frac{dy}{dx} = \frac{3e^{3x}y}{2e^{2y}y-1}$

f) $\frac{2}{x^3} - \frac{5}{y^2} = x - 2$

$\frac{dy}{dx} = \frac{x^4y^3+6y^3}{10x^4}$

Question 2: Using implicit differentiation, find the gradient of the curve $3sinx + cosy = 1$ at the point P $(\frac{\pi}{6}, \frac{2\pi}{3})$

$\frac{dy}{dx} = \frac{3\cos x}{\sin y}$, at P $\frac{dy}{dx} = 3$

Question 3: By implicit differentiation, find the equation of the tangent to the circle

$$(x-3)^2 + (y+4)^2 = 25$$

at the point $(7, -1)$.

$$y = -\frac{4}{3}x + \frac{25}{3}$$

Question 4: A hyperbola has equation $4x^2 - 3y^2 = 24$. By differentiating the curve implicitly, find the equation of the normal at the point P $(3, -2)$ and hence find the coordinates of the point where the normal at P hits the curve again.

Investigate this curve at: https://www.desmos.com/calculator/kccyrvtlek

$$\frac{dy}{dx} = \frac{4x}{3y}$$

At P, $\frac{dy}{dx} = -2$, so $m_N = \frac{1}{2}$

Hence equation of normal is $y = \frac{1}{2}x - \frac{7}{2}$

Other point of intersection is (-6.23, -6.62) to 3 sig. figs.

Question 5: Differentiate the following implicitly.

a) $y cos x = 1$

b) $y^3 \ln x = 1$

c) $\frac{x^2}{y^2} = x$

$\frac{dy}{dx} = y \tan x$

$\frac{dy}{dx} = -\frac{y}{3x \ln x}$

$\frac{dy}{dx} = -\frac{y^3 - 2xy}{2x^2}$

Question 6: Find the gradient of the curve $x^2 y^3 = 72$ at the point $(3, 2)$.

$$\frac{dy}{dx} = -\frac{2y}{3x}$$

$$At\ (3,2), \frac{dy}{dx} = -\frac{4}{9}$$

Question 7: The equation $x^2 - 6xy + 25y^2 = 16$ represents an ellipse, centred at the origin.

a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

b) Hence find the coordinates of all points at which a tangent is parallel to one of the coordinate axes.

Investigate this curve at: https://www.desmos.com/calculator/w3lywjhwfn

a) $\frac{dy}{dx} = \frac{3y - x}{25y - 3x}$

b) For parallel to x-axis, $\frac{dy}{dx} = 0$ which leads to $x = 3y$. Substituting this into the curve and solving leads to $(3, 1)$ and $(-3, -1)$.

For parallel to y-axis, $\frac{dx}{dy} = 0$ which leads to $x = \frac{25y}{3}$. Substituting this into the curve and solving leads to $\left(5, \frac{3}{5}\right)$ and $\left(-5, -\frac{3}{5}\right)$.

Question 8: Find the gradient of the curve with equation $\sin(xy) = \frac{1}{2}$ at the point $(1, \frac{\pi}{6})$

$\cos xy \left(y + x\frac{dy}{dx}\right) = 0$ leading to $\frac{dy}{dx} = -\frac{y}{x}$

$\therefore \frac{dy}{dx} = -\frac{\pi}{6}$

Question 9: Find the equation of the normal to the curve with equation $y^3 - 2xy + x^2 = 9$ at the point P $(4, 1)$.

$$\frac{dy}{dx} = \frac{2y - 2x}{3y^2 - 2x}$$

At P, $\frac{dy}{dx} = \frac{6}{5}$ so $m_N = -\frac{5}{6}$

Equation of normal is $y - 1 = -\frac{5}{6}(x - 4)$ or e.g. $5x + 6y - 26 = 0$

Question 10: A curve is defined implicitly by the equation $x^2 - 2xy + 4y^2 = 12$. By finding $\frac{dy}{dx}$ in terms of x and y, determine the coordinates of the point(s) where a tangent is parallel to the x-axis.

$$\frac{dy}{dx} = \frac{y - x}{4y - x}$$

Tangent parallel to x-axis when $\frac{dy}{dx} = 0$, i.e. when $y = x$

Substituting $y = x$ into the curve leads to $3y^2 = 12$, so $y = \pm 2$

$\therefore (2, 2)\ and\ (-2, -2)$

CHALLENGE QUESTION

A curve C is defined implicitly by the equation $\ln(\sec y) = x + y$. Verify that there is a vertical tangent at the point with y-coordinate $\frac{\pi}{4}$ and find the value of x at this point to 3 significant figures.

$$\frac{dy}{dx} = \frac{1}{\tan y - 1}$$

Vertical tangent when $\frac{dx}{dy} = 0$ which implies $\tan y - 1 = 0$, $\therefore y = \frac{\pi}{4}$, and $x = -0.439$

MEGA CHALLENGE QUESTION

A curve is defined implicitly by the equation $\frac{x}{y} = e^{xy}$.

a) Show that $\frac{dy}{dx} = \frac{-y^3e^{xy}+y}{x+xy^2e^{xy}}$

b) Show further that $\frac{d^2y}{dx^2} = -\frac{2y^3e^{xy}(2xy-y^4e^{2xy}-2y^2e^{xy}+3)}{x^2(y^2e^{xy}+1)^3}$

Note: This is insanity. Don't attempt this challenge.

Differentiating the Inverse Trigonometric Functions

Note: This is rarely examined, but it is in the spec and it has come up before…

We need to be able to show the following results (given in formula booklet):

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Example 1: Show that the derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$.

Hint: For all of the above, let 'y' equal the thing we want to differentiate, but we are going to find $\frac{dx}{dy}$ and then do some work!

Let $y = \sin^{-1} x \Rightarrow x = \sin y$

$$\frac{dx}{dy} = \cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $\sin^2 y + \cos^2 y = 1$

$$\therefore \cos^2 y = 1 - \sin^2 y$$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

$$= \pm\sqrt{1-x^2}$$

Since $\frac{dy}{dx} > 0$ for $y = \sin^{-1} x$, we take the positive root.

Thus, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Task 1: Show that the derivative of $\cos^{-1}x$ is $-\frac{1}{\sqrt{1-x^2}}$.

Let $y=\cos^{-1}x \quad \therefore \quad x=\cos y$

$$\frac{dx}{dy}=-\sin y$$

$$\frac{dy}{dx}=-\frac{1}{\sin y}$$

Note our $\frac{dy}{dx}$ already has a negative sign!

By the identity $\sin^2 y+\cos^2 y=1$

$$\sin y=\pm\sqrt{1-\cos^2 y}$$
$$=\pm\sqrt{1-x^2}$$

For $y=\cos^{-1}x$, $\frac{dy}{dx}<0$, so we need the positive root.

Thus $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$

Task 2: Using the standard results, differentiate the following by the chain rule:

a) $y=\sin^{-1}(3x)$

$$\frac{dy}{dx}=\frac{3}{\sqrt{1-9x^2}}$$

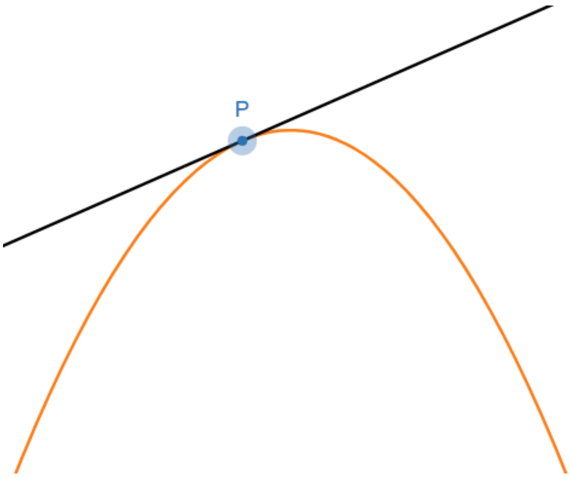
b) $y=\tan^{-1}(2x)$

$$\frac{dy}{dx}=\frac{2}{1+4x^2}$$

Convex and Concave Functions, and Points of Inflection

A curve is **concave** on an interval of values for x where the value of the second derivative is negative over that interval.

If we try to draw a tangent to a concave section of curve, it would be drawn **above** the curve.

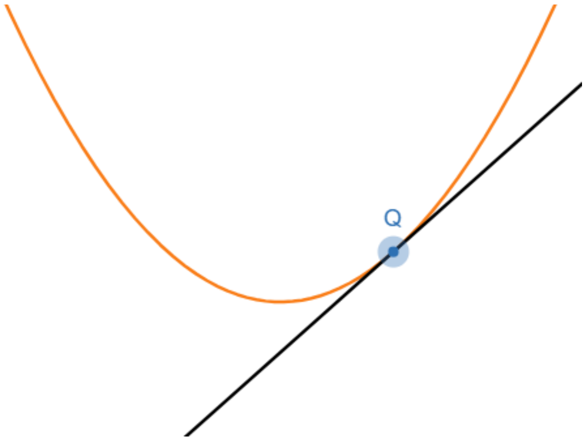


Concave section of curve – $\frac{d^2y}{dx^2} < 0$ *– rate of change of the gradient* $\frac{dy}{dx}$ *is decreasing*

Explore this at:

[*https://www.desmos.com/calculator/uvktpujoq7*](https://www.desmos.com/calculator/uvktpujoq7)

Similarly, a curve is **convex** over an interval where the value of the second derivative is positive in that interval of x-values. The tangent would always be **below** the curve:

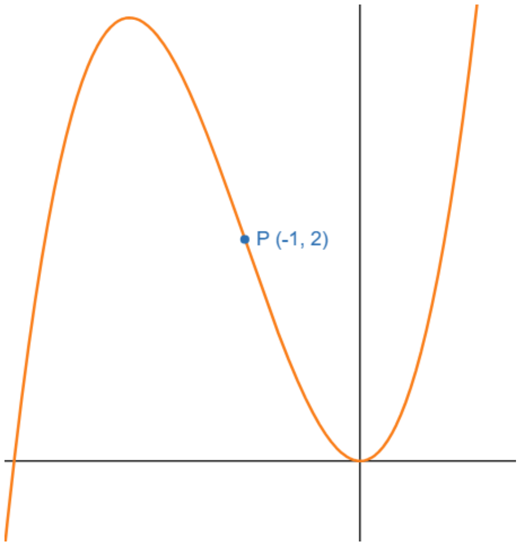


Convex section of curve – $\frac{d^2y}{dx^2} > 0$ *– rate of change of the gradient* $\frac{dy}{dx}$ *is increasing*

Explore this at:

[*https://www.desmos.com/calculator/etalmseozc*](https://www.desmos.com/calculator/etalmseozc)

The parabolas on the previous page were entirely concave or convex (as quadratic curves will always be!), but some curves will be concave in some parts and convex in others:



To the left of the point $P(-1, 2)$ the curve is concave, whereas to the right of P the curve is convex.

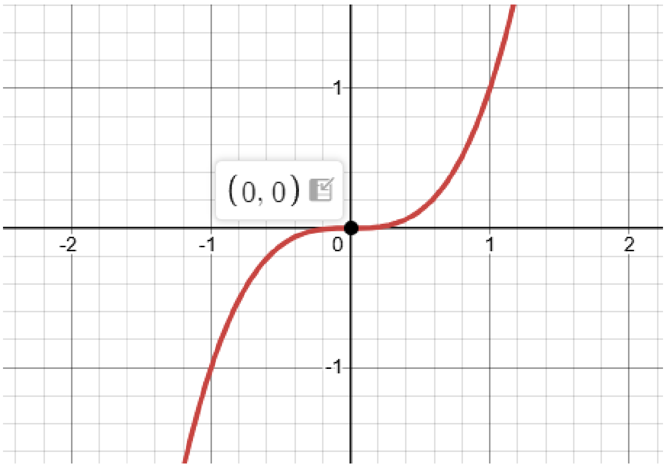
Explore this at:

https://www.desmos.com/calculator/n6gwlmxufb

The point P is called a **point of inflection** – and we have met these before!

A point of inflection occurs when the value of the second derivative equals zero **and** where there is a change in the sign of the second derivative either side of the point.

The above diagram shows a non-stationary point of inflection – that is simply a point of inflection where the gradient is non-zero. At AS mathematics we met stationary points of inflection, such as on the curve $y = x^3$:



The origin is a stationary point of inflection on $y = x^3$.

Example 1: Find the interval on which the curve C with equation $y = x^3 - 3x^2$ is concave.

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

Concave when $\frac{d^2y}{dx^2} < 0$, ie $6x - 6 < 0$

$x < 1$ or e.g $x \in (-\infty, 1)$

Example 2:

Given the curve C with equation $y = 2x^3 + 6x^2 + x$, find the coordinates of the point of inflection and determine whether it is stationary or non-stationary, giving a reason for your answer.

$$\frac{dy}{dx} = 6x^2 + 12x + 1$$

$$\frac{d^2y}{dx^2} = 12x + 12$$

At point of inflection $\frac{d^2y}{dx^2} = 0 \Rightarrow x = -1$

Note $\frac{d^2y}{dx^2} = 0$ is NOT enough for a point of inflection. There must also be a change in concavity. (ie concave → convex or vice versa!)

Test either side to ensure it is a point of inflection:

x	-2	-1	0
$\frac{d^2y}{dx^2}$	-12	0	12

Since $\frac{d^2y}{dx^2}$ changes sign, $x = -1$ is a point of inflection.

At stationary point $\frac{dy}{dx} = 0$, but at $x = -1$ $\frac{dy}{dx} = -5$

$\therefore$ A non-stationary point of inflection.

Task 1: You are given that $f(x) = x^3 + 2x^2 - 8x + 5$. By finding $f''(x)$, find the range of values for which $f(x)$ is convex.

$f'(x) = 3x^2 + 4x - 8$

$f''(x) = 6x + 4$

For convex, $f''(x) > 0$

$6x + 4 > 0$

$x > -\frac{2}{3}$

Task 2: A curve has equation $y = x^4 + 4x^3 - 18x^2 + 1$. Verify that the curve has two points of inflection at $x = -3$ and $x = 1$.

$\frac{dy}{dx} = 4x^3 + 12x^2 - 36x$

$\frac{d^2y}{dx^2} = 12x^2 + 24x - 36$

Setting $\frac{d^2y}{dx^2} = 0$, $x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3 \quad x = 1$

Testing for change in concavity:

x	-4	-3	-2
$\frac{d^2y}{dx^2}$	60	0	-36

x	0	1	2
$\frac{d^2y}{dx^2}$	-36	0	60

Thus both are points of inflection.

Test Your Understanding 3

Question 1

Find $\frac{dy}{dx}$ where:

a) $y = \sin^{-1}(3x - 2)$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-(3x-2)^2}}$$

b) $y = \cos^{-1}(4x)$

$$\frac{dy}{dx} = -\frac{4}{\sqrt{1-16x^2}}$$

c) $y = \tan^{-1}(x^2)$

$$\frac{dy}{dx} = \frac{2x}{1+x^4}$$

Question 2

For each of the following functions, find the interval on which the function is convex.

a) $y = x^3$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{d^2y}{dx^2} = 6x$$

Convex when $x > 0$

b) $y = x^3 - 5x^2 + x$

$$\frac{dy}{dx} = 3x^2 - 10x + 1$$

$$\frac{d^2y}{dx^2} = 6x - 10$$

Convex when $x > \frac{5}{3}$

c) $y = x^2 - x^3$

$$\frac{dy}{dx} = 2x - 3x^2$$

$$\frac{d^2y}{dx^2} = 2 - 6x$$

Convex when $x < \frac{1}{3}$

Question 3

Show that $y = e^{0.2x} - x$ is convex for all values of x.

$$\frac{dy}{dx} = 0.2e^{0.2x} - 1$$

$$\frac{d^2y}{dx^2} = 0.04e^{0.2x}$$

Since $e^{0.2x} > 0 \ \forall x \in \mathbb{R}$, $\therefore$ function is always convex.

Question 4

Find the coordinates of the point of inflection on the curve with equation $y = x^3 - x^2 - x$.

$$\frac{dy}{dx} = 3x^2 - 2x - 1$$

$$\frac{d^2y}{dx^2} = 6x - 2$$

If $\frac{d^2y}{dx^2} = 0$, $x = \frac{1}{3}$

Testing:

x	0	$\frac{1}{3}$	1
$\frac{d^2y}{dx^2}$	-2	0	4

$\therefore \left(\frac{1}{3}, -\frac{11}{27}\right)$

Question 5

Find the exact intervals on which the curve $f(x) = 2 + x^2 - x^4$ is concave.

$f'(x) = 2x - 4x^3$

$f''(x) = 2 - 12x^2$

$2 - 12x^2 < 0 \quad \therefore x^2 > \frac{1}{6}$

Concave for $x < -\frac{1}{\sqrt{6}}$ and $x > \frac{1}{\sqrt{6}}$

Question 6

Show that (1) $y = \ln x, x > 0$ and (2) $y = \sqrt{x}, x > 0$ are both always concave across their domain.

(1) $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$. For $x > 0$, $-\frac{1}{x^2} < 0 \quad \therefore$ concave

(2) $\frac{d^2y}{dx^2} = -\frac{1}{4\sqrt{x^3}}$ For $x > 0$, $\sqrt{x^3} > 0$ and thus $-\frac{1}{4\sqrt{x^3}} < 0$

$\therefore$ Concave

Question 7

Show that, if $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$.

$x = \tan y$

$\frac{dx}{dy} = \sec^2 y$

$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y}$

But $\sec^2 y = 1 + \tan^2 y = 1 + x^2$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$

Question 8

A curve C is defined by the equation $y = x^2 e^x$. Show that the curve has two points of inflection at $x = -2 \pm \sqrt{2}$, and two stationary points at $x = -2$ and $x = 0$, classifying these. Hence, make a rough sketch of the graph of $y = x^2 e^x$.

$y' = 2xe^x + x^2 e^x = (x^2 + 2x)e^x$

$y'' = (x^2 + 2x)e^x + (2x + 2)e^x$

$= (x^2 + 4x + 2)e^x$

Note:

y' is a convenient (lazy) notation for $\frac{dy}{dx}$, and y'' is $\frac{d^2y}{dx^2}$!

When $y'' = 0$, $e^x \neq 0$ $\therefore$ $x^2 + 4x + 2 = 0$ $\Rightarrow x = -2 + \sqrt{2}$, $x = -2 - \sqrt{2}$

Testing either side we see a change of sign in y'' $\therefore$ inflection points.

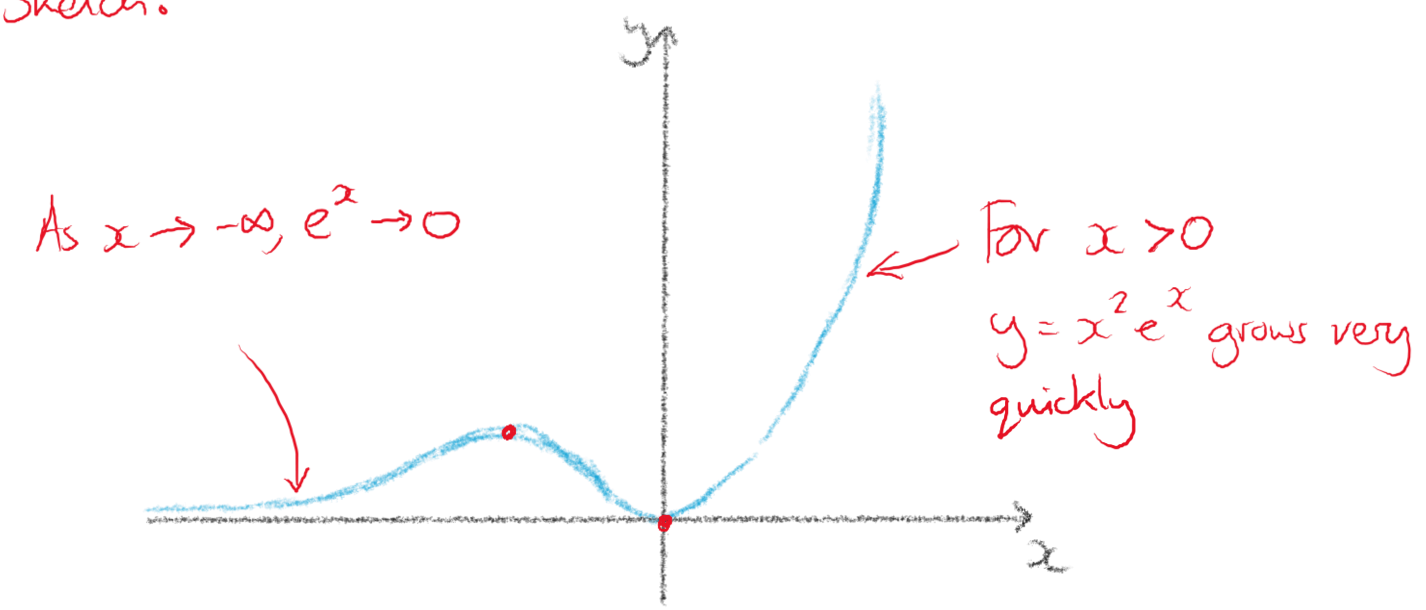
Similarly $y' = 0 \Rightarrow x^2 + 2x = 0$ (since $e^x \neq 0$)

$\therefore$ $x(x+2) = 0$

$x = 0$ and $x = -2$

Min   Max

Sketch:



Related Rates of Change

Differentiation, at it's heart, is about **rates of change**. The operator $\frac{dy}{dx}$ is the rate of change of y with respect to x; similarly, $\frac{dr}{dt}$ may be the rate of change of the radius of an object over time, and so on.

However, often there are multiple variables related to each other. Imagine, for example, a spherical balloon being inflated. As the radius of the balloon increases, both the surface area and the volume are also increasing.

This final section of Unit 9 looks at how we can connect together rates of change to solve problems. This is really just a further extension of the chain rule!!

$\frac{dV}{dt} = 2$

Volume

Example 1: A spherical balloon is being inflated at a constant rate of $2cm^3$ per second. At the instant at which the radius is $3cm$, find how fast the radius is increasing. $\frac{dr}{dt}$

Pro Examiner Tip: These type of questions generally involve needing one or more formulae for shapes such as spheres, cones, cylinders etc. Make sure you know the formulae for volume and surface area!

$V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$ $\qquad \frac{dr}{dV} = 1 \div \frac{dV}{dr}$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} \quad \therefore \frac{dr}{dt} = \frac{1}{4\pi r^2} \times 2 = \frac{1}{2\pi r^2}$

When $r = 3$, $\frac{dr}{dt} = \frac{1}{18\pi}$ cm/sec

Example 2:

a) Write down formulae for the volume V and surface area A of a hemisphere of radius r.

b) A hemispherical object is inflated such that the rate of increase of volume is $4cm^3/sec$. Find the rate of increase of the surface area.

$\frac{dV}{dt} = 4$

a) $V = \frac{2}{3}\pi r^3 \qquad A = 3\pi r^2$

b) $\frac{dV}{dr} = 2\pi r^2 \qquad \frac{dA}{dr} = 6\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt} = 6\pi r \times \frac{1}{2\pi r^2} \times 4 = \frac{12}{r} cm^2 s^{-1}$

Task 1: Given a cube with length x cm, surface area $A\ cm^2$ and volume $V\ cm^3$, and given further that the surface area of the cube is expanding at a constant rate of $3\ cm^2/sec$, find the rate of change of volume in cm^3/sec.

Remember: Start off by finding formulae for the volume and surface area!

$V = x^3 \qquad A = 6x^2$

$\frac{dV}{dx} = 3x^2 \qquad \frac{dA}{dx} = 12x \qquad \frac{dA}{dt} = 3$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$$

$$= 3x^2 \times \frac{1}{12x} \times 3 = \frac{3x}{4}\ cm^3 s^{-1}$$

Math EY matics

Test Your Understanding 4

Question 1

Given that $y = xe^x$ and that $\frac{dx}{dt} = 3$, find the exact value of $\frac{dy}{dt}$ at the point where $x = 3$.

$$\frac{dy}{dx} = xe^x + e^x$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\therefore \frac{dy}{dx} = 3e^x(x+1)$$

At $x = 3, \frac{dy}{dx} = 12e^3$

Question 2

A curve is modelled by $r = 1 + 2\sin\theta$. Given that $\frac{d\theta}{dt} = 2$, find $\frac{dr}{dt}$ at the instant where $\theta = \frac{\pi}{3}$.

$$\frac{dr}{d\theta} = 2cos\theta$$

$$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt}$$

$$\therefore \frac{dr}{dt} = 4cos\theta$$

When $\theta = \frac{\pi}{3}, \frac{dr}{dt} = 2$

Question 3

A water tank is in the shape of a hemisphere where $V = \frac{2}{3}\pi r^3$ and is being filled at a rate of $3cm^3/sec$. Find the rate at which the radius is changing at the instant where $r = 2$.

$$\frac{dv}{dr} = 2\pi r^2$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{3}{2\pi r^2}$$

$$When\ r = 2, \frac{dr}{dt} = \frac{3}{8\pi} cm\ s^{-1}$$

Question 4

The radius of a circle is increasing at a constant rate of $0.2cm$ per second.

a) Show that $\frac{dC}{dt} = 0.4\pi\ cm$ per second, where C is the circumference of the circle.

b) Find the rate at which the area is increasing when the radius is $8cm$.

c) Find the radius of the circle when $\frac{dA}{dt} = 20cm^2$ per second.

$$A = \pi r^2 \text{ and } C = 2\pi r$$

$$\frac{dA}{dr} = 2\pi r \text{ and } \frac{dC}{dr} = 2\pi$$

a) $\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$

$\frac{dC}{dt} = 2\pi \times 0.2 = 0.4\pi\ cm\ s^{-1}$

b) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dA}{dt} = \frac{16}{5}\pi\ cm^2\ s^{-1}$

c) $\frac{dA}{dt} = 0.4\pi r$

$When \frac{dA}{dt} = 20, r = \frac{50}{\pi}\ cm$

Question 5

The volume of a sphere is increasing uniformly at a constant rate of $3cm^3/sec$.

Find: a) the value of $\frac{dr}{dt}$ when $r = 4cm$,

b) the rate at which the surface area of the sphere is increasing at this instant.

$$V = \frac{4}{3}\pi r^3 \ and\ A = 4\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2 \ and\ \frac{dA}{dr} = 8\pi r$$

a) $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\frac{dr}{dt} = \frac{3}{64\pi} cm\ s^{-1}$

b) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dA}{dt} = 1.5\ cm^2\ s^{-1}$

Question 6

The radius of a circular oil spill floating on water is changing at a rate of 3m per hour.

Find: a) the rate at which the circumference is changing at the instant when the radius is 20m. (What do you notice about your answer?)

b) the radius at the instant that the rate of change of area of the oil spill reaches $222\pi\ m^2/second$.

$$\boldsymbol{A = \pi r^2} \text{ and } \boldsymbol{C = 2\pi r}$$

$$\frac{dA}{dr} = 2\pi r \text{ and } \frac{dC}{dr} = 2\pi$$

a) $\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$

$\frac{dC}{dt} = 6\pi$ which is constant (independent of r)

b) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dA}{dt} = 6\pi r$; when $\frac{dA}{dt} = 222\pi, r = 37m$

CHALLENGE QUESTIONS

Challenge 1:

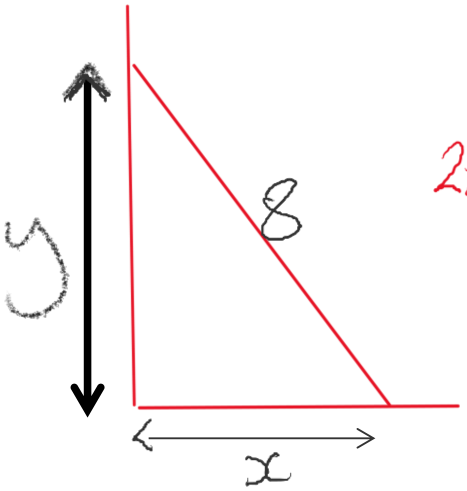
A cylindrical tin of radius 5cm is being filled with liquid at a rate of $10cm^3$ *per second. Find the rate at which the level of the liquid rises.*

$$\frac{dh}{dt} = \frac{2}{5\pi}\ cm\ s^{-1}$$

Challenge 2:

A ladder of length 8 metres leans against a wall with its feet resting on the ground perpendicular to the wall. It slips (in such a way that the ends of the ladder retain contact with the wall and ground respectively), and the end against the wall falls at a rate of 2 metres per second. Find the speed of the lower end of the ladder when it is 6 metres from the wall. (Hint: We may need implicit differentiation here!)

A sketch helps here:



$x^2+y^2=8^2$

We want $\frac{dx}{dt}$ when $x=6$

$\frac{dy}{dt}=-2$

$2x+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{x}{y}$

$\frac{dx}{dt}=\frac{dx}{dy}\times\frac{dy}{dt}$

$=-\frac{y}{x}\times -2$

At $x=6, y=\sqrt{28}$,

$\frac{dx}{dt}=\frac{\sqrt{28}}{3}$ m/s

Challenge 3:

A chemical compound is shaped into a cube. This cube is then uniformly compacted such that it retains a cubic shape. The rate of change of surface area of the cube is equal to $-0.5\text{cm}^2\text{s}^{-1}$. *Find the rate of change of the volume at the instant where the cube's length has halved from its initial size.*

$V=x^3$ $\qquad A=6x^2$ $\qquad \frac{dA}{dt}=-0.5$

$\frac{dV}{dx}=3x^2$ $\qquad \frac{dA}{dx}=12x$

$\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dA}\times\frac{dA}{dt}$

$=3x^2\times\frac{1}{12x}\times -0.5 = -\frac{1}{8}x$

Let start length be X.

When $x=\frac{1}{2}X$

$\frac{dV}{dt}=-\frac{1}{16}X$ cm³/s

Now: You are ready for the Grade Enhancer™.

Grade Enhancer™ - Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(WJEC 2016)*

Differentiate each of the following with respect to x, simplifying your answer wherever possible.

(a) $\ln(\cos x)$ [3]

(b) $\tan^{-1}(\frac{x}{3})$ [3]

(c) $e^{6x}(3x-2)^4$ [4]

Question 2 *(WJEC 2017)*

Given that

$$x^4 - 3x^2y + 2y^3 - 4x = 7,$$

find an expression for $\frac{dy}{dx}$ in terms of x and y. [4]

Question 3 *(WJEC 2019)*

a) Differentiate the following functions with respect to x, simplifying your answer wherever possible.

i) $e^{3\tan x}$,

ii) $\frac{\sin 2x}{x^2}$. [5]

b) A function is defined implicitly by

$$3x^2y + y^2 - 5x = 5.$$

Find the equation of the normal at the point (1, 2). [6]

Question 4 *(WJEC 2015)*

The curve C has equation

$$x^4 + 3x^2y - 2y^2 = 34.$$

(a) Show that $\frac{dy}{dx} = \frac{4x^3 + 6xy}{4y - 3x^2}$. [3]

(b) Find the coordinates of each of the points on C where the tangent is parallel to the y-axis. [4]

Question 5 *(WJEC 2023)*

A curve C has equation $f(x) = 5x^3 + 2x^2 - 3x$.

a) Find the x-coordinate of the point of inflection. State, with a reason, whether the point of inflection is stationary or non-stationary. [5]

b) Determine the range of values of x for which C is concave. [2]

Question 6 *(WJEC s2014)*

(i) Sketch the graph of $y = \sin^{-1}x$ for values of x satisfying $-1 \leqslant x \leqslant 1$.

(ii) By first rewriting $y = \sin^{-1}x$ as $x = \sin y$, find an expression for $\frac{dy}{dx}$ in terms of x. You should justify any choice of sign that you make. [6]

Question 7 *(WJEC 2015)*

The function f is defined on the domain $(0, \pi)$ by

$$f(x) = 2^x \sin x.$$

Obtain an expression for $f'(x)$.

[4]

Total Mark Available is 49.