Test Your Understanding 1

Question 1

Differentiate each of the following with respect to x:

a) $y = e^x$

$\frac{dy}{dx} = e^x$

b) $y = 5e^x$

$\frac{dy}{dx} = 5e^x$

c) $y = e^{5x}$

$\frac{dy}{dx} = 5e^{5x}$

d) $y = 5e^{5x}$

$\frac{dy}{dx} = 25e^{5x}$

Question 2

Differentiate each of the following with respect to x:

a) $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

b) $y = 3\ln x$

$\frac{dy}{dx} = \frac{3}{x}$

c) $y = \ln 3x$

$\frac{dy}{dx} = \frac{1}{x}$

d) $y = 3e^{2x} + 4\ln x$

$\frac{dy}{dx} = 6e^{2x} + \frac{4}{x}$

Question 3

Differentiate the following with respect to x:

a) $y = \sqrt{x} + e^{3x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + 3e^{3x}$

b) $y = \frac{3}{x} + 3\ln x$

$\frac{dy}{dx} = -\frac{3}{x^2} + \frac{3}{x}$

c) $y = 2\sqrt[3]{x} - 5e^{0.5x} - 4$

$\frac{dy}{dx} = \frac{2}{3\sqrt[3]{x^2}} - 2.5e^{0.5x}$

Question 4

Find $\frac{d^2y}{dx^2}$ if:

a) $y = x^3 + 2e^x$

$\frac{dy}{dx} = 3x^2 + 2e^x$

$\frac{d^2y}{dx^2} = 6x + 2e^x$

b) $y = \ln x - \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2\sqrt{x}}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2} + \frac{1}{4\sqrt{x^3}}$

c) $y = 5x^4 - 3e^{2x} - 4ln x$

$\frac{dy}{dx} = 20x^3 - 6e^{2x} - \frac{4}{x}$

$\frac{d^2y}{dx^2} = 60x^2 - 12e^{2x} + \frac{4}{x^2}$

Question 5

Find $f'(x)$ when

a) $f(x) = 3^x$

$\frac{dy}{dx} = 3^x ln3$

b) $f(x) = 3^{2x}$

$\frac{dy}{dx} = 3^{2x}(2ln3)$

c) $y = 2^{3x}$

$\frac{dy}{dx} = 2^{3x}(3ln2)$

Question 6

Find the value of $\frac{dy}{dx}$ at the indicated value of x:

a) $y = 3x + e^{2x}$ when $x = 0$

$\frac{dy}{dx} = 3 + 2e^{2x}$

At 0, $\frac{dy}{dx} = 5$

b) $y = \ln x - x^2$ when $x = 2$

$\frac{dy}{dx} = \frac{1}{x} - 2x$

At 2, $\frac{dy}{dx} = -\frac{7}{2}$

Question 7

Show that the curve $y = (e^x - e^{-x})^2$ has gradient 7.5 at the point where $x = \ln 2$.

$y = e^{2x} - 2 + e^{-2x}$

$\frac{dy}{dx} = 2e^{2x} - 2e^{-2x}$

$At\ x = ln2,\quad \frac{dy}{dx} = 2e^{ln4} - 2e^{-ln4}$

$= 2 \times 4 - 2 \times \frac{1}{4}$

$= 7.5$

Question 8

Find the equation of the tangent to the curve $y = e^{2x} - \ln x$ at the point where $x = 1$, giving your equation in the form $y = ax + b$, where a and b are given as exact values in terms of e.

$\frac{dy}{dx} = 2e^{2x} - \frac{1}{x} \quad => m = 2e^2 - 1$

When $x = 1$, $y = e^2$

$\therefore y - e^2 = (2e^2 - 1)(x - 1)$

$y = (2e^2 - 1)x + (1 - e^2)$

Question 9

Given that $y = 2x + ke^{2x}$, where k is a constant, show that

$$(1-2x)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y = 0$$

$\frac{dy}{dx} = 2 + 2ke^{2x}$

$\frac{d^2y}{dx^2} = 4ke^{2x}$

$\therefore (1-2x)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y$

$= (1-2x)(4ke^{2x}) + 4x(2+2ke^{2x}) - 4(2x + ke^{2x})$

$= 0$ (after convincing intermediate steps!

Question 10

Find and classify the stationary point(s) on the curves with equation:

a) $y = \ln x - 10x$

$\frac{dy}{dx} = \frac{1}{x} - 10$

At st.pt. $\frac{dy}{dx} = 0 \quad \Rightarrow x = \frac{1}{10}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

$At\ x = \frac{1}{10}, \frac{d^2y}{dx^2} = -100$

Therefore point is a maximum.

b) $y = 7 + 2x - 4\ln x$

$\frac{dy}{dx} = 2 - \frac{4}{x}$

At st.pt. $\frac{dy}{dx} = 0 \quad \Rightarrow x = 2$

$\frac{d^2y}{dx^2} = \frac{4}{x^2}$

$At\ x = 2, \frac{d^2y}{dx^2} = 1 > 0$

Therefore point is a minimum.

Question 11

Find an equation for the tangent to the curve $y = 3\ln x + \frac{2}{x}$ when $x = 1$.

$$\frac{dy}{dx} = \frac{3}{x} - \frac{2}{x^2}$$

$$When\ x = 1, \qquad y = 2, \qquad \frac{dy}{dx} = 1$$

$$\therefore y = x + 1$$

Question 12

Find an equation for the normal to the curve $y = 5 - 2e^{2x}$ at the point where $x = \ln 2$.

$\frac{dy}{dx} = -4e^{2x}$

$When\ x = ln2, \qquad y = -3, \qquad \frac{dy}{dx} = -16$

$y + 3 = \frac{1}{16}(x - ln2)$

Challenge Question

A curve has equation $y = e^{2x} - 3x$.

a) Find the exact value of x for which the tangent to the curve is parallel to the line $y = 7x$.

b) Hence, find the exact equation of the tangent to the curve at this point.

Test Your Understanding 2

Question 1

Find $\frac{dy}{dx}$ if:

a) $y = (1+3x)^5$

$\frac{dy}{dx} = 15(1+3x)^4$

b) $y = \sqrt{4x-2}$

$\frac{dy}{dx} = \frac{2}{\sqrt{4x-2}}$

c) $y = (2x-1)^{-3}$

$\frac{dy}{dx} = -\frac{6}{(2x-1)^4}$

d) $y = \frac{1}{x^2+4}$

$\frac{dy}{dx} = -\frac{2x}{(x^2+4)^2}$

e) $y = (x-3x^2)^3$

$\frac{dy}{dx} = 3(1-6x)(x-3x^2)^2$

f) $y = e^{3x-5}$

$\frac{dy}{dx} = 3e^{3x-5}$

Question 2

Find $\frac{d^2y}{dx^2}$ with respect to x when $y = (4x-3)^3$.

$\frac{dy}{dx} = 12(4x-3)^2$

$\frac{d^2y}{dx^2} = 96(4x-3)$

Question 3

Find the equation of the tangent to the curve $y = (2x-3)^3$ at the point where $x = 2$.

$\frac{dy}{dx} = 6(2x-3)^2$

$At\ (2,1), \frac{dy}{dx} = 6$

$y = 6x - 11$

Question 4

Find the equation of the normal to the curve $y = \frac{1}{2+lnx}$ at the point where $x = 1$.

$\frac{dy}{dx} = -\frac{1}{x(2+lnx)^2}$

$At\ \left(1, \frac{1}{2}\right), \frac{dy}{dx} = -\frac{1}{4}$ so $m_N = 4$

$\therefore y = 4x - \frac{7}{2}$ (OE)

Question 5

Find (you do **not** need to classify) any stationary points on the following curves:

a) $y = (4x - 1)^3$

$x = \frac{1}{4}$

$(0.25, 0)$

b) $y = e^{x^2 - x}$

$x = \frac{1}{2}$

$(0.5, e^{-0.25})$

c) $y = 2x + \frac{1}{2x}$

$x = \pm\frac{1}{2}$

$(-0.5, -2)$ and $(0.5, 2)$

Question 6

Find the rate of change of x on the curve $y = 2x - e^{3x-1}$ at the point where $x = 2$.

$$\frac{dy}{dx} = 2 - 3e^{3x-1}$$

At, $x = 2, \frac{dy}{dx} = 2 - 3e^5$

Question 7

Find the value of x for which $f'(x) = 2$ in the function $f(x) = 2\sqrt{4x - 1}$ $(x \geq \frac{1}{4})$.

$$f'(x) = \frac{4}{\sqrt{4x - 1}}$$

$$x = \frac{5}{4}$$

Question 8

A curve has equation $y = \frac{3}{(2x-1)^2}$, $x \neq \frac{1}{2}$.

Find the equation of the normal to the curve at the point with x-coordinate 3, giving your answer in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$.

$3125x - 300y - 9339 = 0$

Challenge Question

A population P of a species is modelled by the formula

$$P = ae^{kt},$$

where P is in thousands, t is in years, and a and k are constants.

At the start of measuring, the population was 20,000.

After 8 years, the population was 60,000.

a) Find the values of the constants a and k.

b) Find the population, to the nearest hundred, predicted by the model after 12 years.

c) The rate of increase of the population at 12 years.

d) Explain why the model may be unsuitable for large values of t.

Test Your Understanding 3

Question 1

Find the derivative of each of the following functions using the product rule.

a) $y = x(1+2x)^3$

$\frac{dy}{dx} = (1+2x)^3 + 6x(1+2x)^2$

b) $y = 2x^3e^x$

$\frac{dy}{dx} = 2x^3e^x + 6x^2e^x$

c) $y = \ln x\,(1+x^2)$

$\frac{dy}{dx} = 2xlnx + \frac{1+x^2}{x}$

d) $y = e^{2x}\sqrt{x}$

$\frac{dy}{dx} = 2e^{2x}\sqrt{x} + \frac{e^{2x}}{2\sqrt{x}}$

e) $y = x^2 2^x$

$\frac{dy}{dx} = 2x(2^x) + (2^x)x^2 ln2$

f) $y = x^3 \ln 2x$

$\frac{dy}{dx} = 3x^2 ln2x + x^2$

Question 2

Differentiate $y = (\ln x)^2$ using

i) the Chain Rule,

ii) the Product Rule.

Show that your answers are equal.

Both should give $\frac{dy}{dx} = \frac{2lnx}{x}$ after simplifying.

Question 3

For each of the following, find $f'(x)$ and simplify your answer as far as possible:

a) $f(x) = x(2x-1)^3$

$f'(x) = (2x-1)^3 + 6x(2x-1)^2$

$f'(x) = (2x-1)^2(8x-1)$

b) $f(x) = x\sqrt{x-1}$

$f'(x) = \sqrt{x-1} + \frac{x}{2\sqrt{x-1}}$

$f'(x) = \frac{3x-2}{2\sqrt{x-1}}$

c) $f(x) = (x+3)(x-2)^3$

$f'(x) = (x-2)^3 + 3(x+3)(x-2)^2$

$f'(x) = (x-2)^2(4x+7)$

Question 4

Find the coordinates of any stationary points on the curves with equations:

a) $y = xe^{2x}$

$\frac{dy}{dx} = e^{2x}(2x+1)$

$\left(-\frac{1}{2}, -\frac{1}{2e}\right)$

b) $y = x^2e^{3x}$

$\frac{dy}{dx} = xe^{3x}(3x+2)$

$(0,0)$ and $\left(-\frac{2}{3}, \frac{4}{9e^2}\right)$

c) $y = (1+2x)(3x-1)^2$

$\frac{dy}{dx} = 2(3x-1)^2 + (6+12x)(3x-1)$

$\left(\frac{1}{3}, 0\right)$ and $\left(-\frac{2}{9}, \frac{175}{81}\right)$

Question 5

Find the equation of the tangent to the curve $y = x^2e^{2x}$ at the point where $x = 1$.

$$y = 4e^2x - 3e^2$$

Question 6

A curve has equation $y = x^2\sqrt{x+6}$, $x > -6$.

Find the equation of the normal to the curve at the point where $x = 10$, giving your answer in the form $y = mx + c$.

$$y = -\frac{2}{185}x + \frac{18404}{37}$$

Question 7

A curve has equation $y = 8x^2(4x-1)^3$.

a) Show that

$$\frac{dy}{dx} = Ax(4x-1)^2(Bx + C)$$

Where A, B and C are integers to be determined.

$$\frac{dy}{dx} = 16x(4x-1)^3 + 96x^2(4x-1)^2$$

$$\frac{dy}{dx} = 16x(4x-1)^2(10x-1)$$

b) Hence find the coordinates of each of the stationary points on the curve. (You do not need to classify these.)

$$(0,0), \left(\frac{1}{10}, -\frac{54}{3125}\right), \left(\frac{1}{4}, 0\right)$$

Challenge

Find the derivative of $y = x^2e^x(2x+1)^3$

$$e.g. \frac{dy}{dx} = 2xe^x(2x+1)^3 + (e^x(2x+1)^3 + 6(2x+1)^2e^x)x^2$$

Test Your Understanding 4

Question 1

Find the derivative of each of the following with respect to x, simplifying your answers where possible.

a) $y = \frac{3x}{x-2}$

$\frac{dy}{dx} = \frac{3(x-2)-3x}{(x-2)^2}$

$\frac{dy}{dx} = -\frac{6}{(x-2)^2}$

b) $y = \frac{e^x}{x+2}$

$\frac{dy}{dx} = \frac{(x+2)e^x - e^x}{(x+2)^2}$

$\frac{dy}{dx} = \frac{(x+1)e^x}{(x+2)^2}$

c) $y = \frac{x^2}{1+e^x}$

$\frac{dy}{dx} = \frac{2x(1+e^x)-x^2e^x}{(1+e^x)^2}$

d) $y = \frac{\ln x}{2x+1}$

$\frac{dy}{dx} = \frac{\frac{2x+1}{x} - 2\ln x}{(2x+1)^2}$

$\frac{dy}{dx} = \frac{2x+1-2x\ln x}{x(2x+1)^2}$

e) $y = \frac{\sqrt{x}+2}{(x-1)^2}$

$\frac{dy}{dx} = \frac{(x-1)^2\left(\frac{1}{2\sqrt{x}}\right) - 2(\sqrt{x}+2)(x-1)}{(x-1)^4}$

$\frac{dy}{dx} = \frac{(x-1)-4\sqrt{x}(\sqrt{x}+2)}{(x-1)^3}$

f) $y = \frac{(e^x-1)^2}{2x}$

$\frac{dy}{dx} = \frac{4xe^x(e^x-1)-2(e^x-1)^2}{4x^2}$

$\frac{dy}{dx} = \frac{(e^x-1)(2xe^x-e^x+1)}{2x^2}$

Question 2

Differentiate $y = \frac{x^2}{e^x}$ using

i) The Quotient rule,

$$\frac{dy}{dx} = \frac{e^x(2x) - x^2e^x}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{2x - x^2}{e^x}$$

ii) The Product rule.

$$\frac{dy}{dx} = x^2(-e^{-x}) + 2x(e^{-x})$$

Show that your answers are equivalent.

Question 3

Find the coordinates of any stationary points on the curve:

a) $y = \frac{x^2}{x+1}$

b) $y = \frac{e^x}{2x+1}$

c) $y = \frac{\ln x}{2x}$

(Hint: Remember to consider any restrictions on x)

a)

$\frac{dy}{dx} = \frac{2x(x+1)-x^2}{(x+1)^2}$

$\frac{dy}{dx} = \frac{x^2+2x}{(x+1)^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = 0, -2$

$(0,0)\ and\ (-2,-4)$

b)

$\frac{dy}{dx} = \frac{(2x+1)e^x - 2e^x}{(2x+1)^2}$

$\frac{dy}{dx} = \frac{(2x-1)e^x}{(2x+1)^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{1}{2}\sqrt{e}\right)$

c)

$\frac{dy}{dx} = \frac{2x\left(\frac{1}{x}\right) - 2\ln x}{4x^2}$

$\frac{dy}{dx} = \frac{2-2\ln x}{4x^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = e$

$\left(e, \frac{1}{2e}\right)$

Question 4

Find the equation of the tangent to each of the following curves at the given x-coordinate.

a) $y = \frac{2x}{x-1}$ when $x = 2$

$y = 8 - 2x$

b) $y = \frac{e^x+5}{e^x+2}$ when $x = 0$.

$y = 2 - \frac{1}{3}x$

Question 5

Find the equation of the normal to the curve $y = \frac{x}{\ln x}$ at the point where $x = e^2$, giving your answer in the form $ax + by + c = 0$ where a, b are integers and c is given as an exact real value.

$$\mathbf{4x + y - \left(\frac{e^2}{2} + 4e^2\right) = 0}$$

Question 6

You are given that $f(x) = \frac{3x}{x-3} - \frac{15x}{x^2-x-6}$

a) Show that $f(x) = \frac{3x}{x+2}$

b) Hence find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$.

$$y = \frac{2}{3}x + \frac{1}{3}$$

Question 7

A curve has equation $y = \dfrac{e^{4x}}{(x+3)^2}$, $x \neq -3$.

a) Show that $\dfrac{dy}{dx} = \dfrac{Ae^{4x}(Bx+C)}{(x+3)^3}$ where $A, B, C \in \mathbb{Z}$.

$$\frac{dy}{dx} = \frac{2e^{4x}(2x+5)}{(x+3)^3}$$

b) Find the equation of the normal to the curve at the point where $x = 0$.

$$y = \frac{1}{9} - \frac{27}{10}x$$

Challenge Question

Given the curve with equation $y = \dfrac{2\sqrt{x}-3}{x-2}$, show that any stationary points on the curve satisfy the equation $x - 3\sqrt{x} + 2 = 0$, and hence find any stationary point(s) on the curve.