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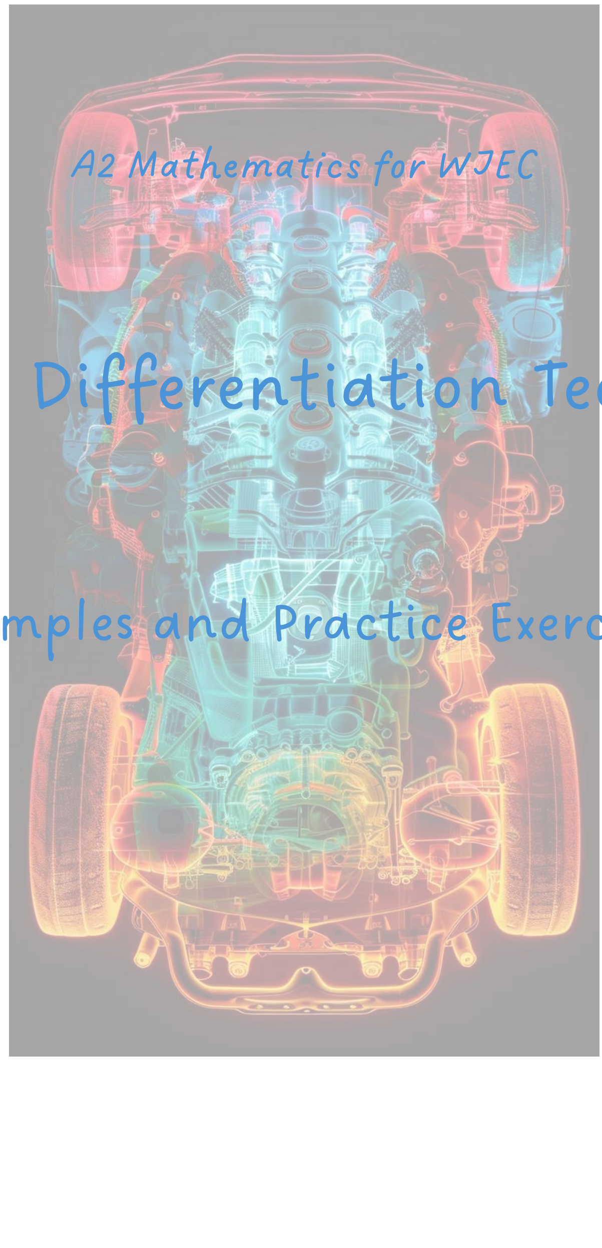

A2 Mathematics for WJEC

Unit 6 - Differentiation Techniques

Examples and Practice Exercises

Unit Learning Objectives

- ***To recall and extend the basics of differentiation, to differentiating exponential and logarithmic functions;***
- ***To be able to differentiate functions using the Chain Rule, Product Rule and Quotient Rule;***
- ***To find gradients, stationary points, tangents and normals using these new techniques;***

Prerequisite atoms:

Differentiation (AS Mathematics)

Equation of a Straight Line (AS Mathematics)

Atom Check:

1) Differentiate the following with respect to x:

a) $y = 5x^4$

b) $f(x) = 3x^2 + \frac{1}{x} - \sqrt{x}$

2) Find the gradient of the curve $y = x + \frac{3}{\sqrt{x}}$ at the point when $x = 4$.

3) Find, by differentiation, the coordinates of the stationary point on the curve with equation $y = x^2 + 3x + 2$. By considering the second derivative, show that this stationary point is a minimum value.

4) Find the equation of the normal to the curve $y = x^2 - 4$ at the point where $x = 1$.

ALSO: *It is highly advised that you are comfortable using Desmos (www.desmos.com) or other graphing software to help you during this unit!*

When you have completed the unit…

Objective	Met	Know	Mastered
I can recall and use AS mathematics differentiation techniques, and differentiate e^x and $\ln x$.			
I can understand and use the Chain Rule.			
I can understand and use the Product Rule.			
I can understand and use the Quotient Rule.			
I can apply the new techniques to problems involving stationary points, tangents and normals.			

Notes/Areas to Develop:

Differentiating Logarithmic and Exponential Terms

We briefly met the derivative of e^x at AS level. Here we will extend this learning, and in the next section will be able to fully justify the results given here.

- If $y = e^x$, then $\frac{dy}{dx} = e^x$
- If $y = ae^{kx}$, then $\frac{dy}{dx} = ake^{kx}$ (we multiply by the coefficient of the power, but the power doesn't change.
- If $y = a^{kx}$, then $\frac{dy}{dx} = a^{kx}(k \ln a)\ (a > 0)$
- $y = \ln x$ differentiates to give $\frac{dy}{dx} = \frac{1}{x}$.

Example 1: Show that $y = \ln(kx)$ differentiates to give $\frac{dy}{dx} = \frac{1}{x}$ for any $k > 0$.

Example 2: Show that, if $y = a^x$, the derivative is given by $\frac{dy}{dx} = a^x \ln a$, $a > 0$

Example 3: For each of the following, find $f'(x)$:

a) $f(x) = 3^x - e^{4x}$

b) $f(x) = \ln(x^4) - \frac{1}{x}$

c) $f(x) = \frac{5e^{4x}-3}{e^x}$

Task 1: Find the derivative for each of the following:

a) $y = 3e^{3x} - 2^x$

b) $y = \ln(2x)$

c) $y = \frac{4e^{3x}+5}{2e^{2x}}$

Now: *Complete Test Your Understanding 1, Pg 13.*

The Chain Rule

So far, we have looked at differentiating 'simple' functions where we can differentiate individual terms.

However, most mathematical functions are not so straightforward. For example, what if we were asked to differentiate $y = (2x - 3)^7$? (This is sometimes called a function of a function.)

Only a sadist would want to expand everything out… but what if we then also needed to *simplify* the derivative? There must be a better way! Enter the **Chain Rule.**

Example 1: Differentiate $y = (2x - 3)^7$

It is **really helpful** if we can do this quickly (and, in a departure from norms, with a minimum of workings) in the exam:

$$y = (2x - 3)^7$$

$$\frac{dy}{dx} =$$

Example 2: Differentiate $y = \sqrt{2x^2 + 3}$

Long(er) method:

Short(er) method:

Task 1: Differentiate the following using the chain rule:

a) $(5x - 3)^6$

b) $\frac{1}{(x^2-3)^3}$

c) e^{x^2}

NOW: *Complete Test Your Understanding 2, Page 15.*

The Product Rule

Next, we can consider finding the derivative of two functions being multiplied together.

If we have two functions of x, say u and v, then:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

*This **looks** scary, but it isn't!*

Example 1: Find $\frac{dy}{dx}$ where $y = x^3e^{2x}$

Task 1: Find $\frac{dy}{dx}$ where $y = x^4\sqrt{3x+1}$

Examiners love to test understanding of other skills within this topic area, particularly understanding of roots of equations and exponential graphs.

Example 2: Find the coordinates of any stationary points on the curve with equation

$$y = xe^{2x}$$

Space for additional notes:

NOW: *Complete Test Your Understanding 3, Page 17.*

The Quotient Rule

Our final 'basic' technique is concerned with where one function of x is being divided by another.

If we have two functions of x, u **and** v, **so that** $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Note here that the order is vital as subtraction is not commutative (i.e. the order matters in subtraction!)

It's also worth noting that this is really just a particular case of the product rule for $y = uv^{-1}$.

Example 1: Find $\frac{dy}{dx}$ when:

a) $y = \frac{x}{3x-2}$

b)) $y = \frac{\ln x}{x^3}$

Task 1: Find the equation of the tangent to the curve $y = \frac{x^2}{3x-2}$ at the point where $x = 1$.

NOW: *Complete Test Your Understanding 4, Page 19.*

A useful trick to know:

Sometimes, a function cannot be explicitly written as $y = \ldots$ in order to differentiate it. However, a special case of the chain rule gives us the result that

$$\frac{dy}{dx} = 1 \div \frac{dx}{dy}$$

This can allow us to find derivatives of some awkward functions, though unusually (at least for now) the derivatives will appear to be in terms of the 'wrong' variable!

Example 1: Given that $y^2 + y = x$, find $\frac{dy}{dx}$.

Task 1: Given that $y^3 + 4y = x$, find the value of $\frac{dy}{dx}$ at the point (5, 1).

Now: You are ready for the Grade Enhancer™, Page 21.

Space for additional notes:

Test Your Understanding 1

Question 1

Differentiate each of the following with respect to x:

a) $y = e^x$ b) $y = 5e^x$ c) $y = e^{5x}$ d) $y = 5e^{5x}$

Question 2

Differentiate each of the following with respect to x:

a) $y = \ln x$ b) $y = 3\ln x$ c) $y = \ln 3x$ d) $y = 3e^{2x} + 4\ln x$

Question 3

Differentiate the following with respect to x:

a) $y = \sqrt{x} + e^{3x}$ b) $y = \frac{3}{x} + 3\ln x$ c) $y = 2\sqrt[3]{x} - 5e^{0.5x} - 4$

Question 4

Find $\frac{d^2y}{dx^2}$ if:

a) $y = x^3 + 2e^x$ b) $\mathrm{y} = \ln x - \sqrt{x}$ c) $y = 5x^4 - 3e^{2x} - 4lnx$

Question 5

Find $f'(x)$ when

a) $f(x) = 3^x$ b) $f(x) = 3^{2x}$ c) $y = 2^{3x}$

Question 6

Find the value of $\frac{dy}{dx}$ at the indicated value of x:

a) $y = 3x + e^{2x}$ when $x = 0$ b) $y = \ln x - x^2$ when $x = 2$

Question 7

Show that the curve $y = (e^x - e^{-x})^2$ has gradient 7.5 at the point where $x = \ln 2$.

Question 8

Find the equation of the tangent to the curve $y = e^{2x} - \ln x$, giving your equation in the form $y = ax + b$, where a and b are given as exact values in terms of e.

Question 9

Given that $y = 2x + ke^{2x}$, where k is a constant, show that

$$(1-2x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

Question 10

Find and classify the stationary point(s) on the curves with equation:

a) $y = \ln x - 10x$

b) $y = 7 + 2x - 4\ln x$

Question 11

Find an equation for the tangent to the curve $y = 3\ln x + \frac{2}{x}$ when $x = 1$.

Question 12

Find an equation for the normal to the curve $y = 5 - 2e^{2x}$ at the point where $x = \ln 2$.

Challenge Question

A curve has equation $y = e^{2x} - 3x$.

a) Find the exact value of x for which the tangent to the curve is parallel to the line $y = 7x$.

b) Hence, find the exact equation of the tangent to the curve at this point.

Test Your Understanding 2

Question 1

Find $\frac{dy}{dx}$ if:

a) $y = (1 + 3x)^5$

b) $y = \sqrt{4x - 2}$

c) $y = (2x - 1)^{-3}$

d) $y = \frac{1}{x^2+4}$

e) $y = (x - 3x^2)^3$

f) $y = e^{3x-5}$

Question 2

Find $\frac{d^2y}{dx^2}$ with respect to x when $y = (4x - 3)^3$.

Question 3

Find the equation of the tangent to the curve $y = (2x - 3)^3$ at the point where $x = 2$.

Question 4

Find the equation of the normal to the curve $y = \frac{1}{2+lnx}$ at the point where $x = 1$.

Question 5

Find (you do **not** need to classify) any stationary points on the following curves:

a) $y = (4x - 1)^3$

b) $y = e^{x^2-x}$

c) $y = 2x + \frac{1}{2x}$

Question 6

Find the rate of change of x on the curve $y = 2x - e^{3x-1}$ at the point where $x = 2$.

Question 7

Find the value of x for which $f'(x) = 2$ in the function $f(x) = 2\sqrt{4x - 1}$ $(x \geq \frac{1}{4})$.

Question 8

A curve has equation $y = \frac{3}{(2x-1)^2}$, $x \neq \frac{1}{2}$.

Find the equation of the normal to the curve at the point with x-coordinate 3, giving your answer in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$.

Challenge Question

A population P of a species is modelled by the formula

$$P = ae^{kt},$$

where P is in thousands, t is in years, and a and k are constants.

At the start of measuring, the population was 20,000.

After 8 years, the population was 60,000.

a) Find the values of the constants a and k.

b) Find the population, to the nearest hundred, predicted by the model after 12 years.

c) The rate of increase of the population at 12 years.

d) Explain why the model may be unsuitable for large values of t.

Test Your Understanding 3

Question 1

Find the derivative of each of the following functions using the product rule.

a) $y = x(1 + 2x)^3$

b) $y = 2x^3 e^x$

c) $y = \ln x\,(1 + x^2)$

d) $y = e^{2x}\sqrt{x}$

e) $y = x^2 2^x$

f) $y = x^3 \ln 2x$

Question 2

Differentiate $y = (\ln x)^2$ using

i) the Chain Rule,

ii) the Product Rule.

Show that your answers are equal.

Question 3

For each of the following, find $f'(x)$ and simplify your answer as far as possible:

a) $f(x) = x(2x - 1)^3$

b) $f(x) = x\sqrt{x - 1}$

c) $f(x) = (x + 3)(x - 2)^3$

Question 4

Find the coordinates of any stationary points on the curves with equations:

a) $y = xe^{2x}$

b) $y = x^2 e^{3x}$

c) $y = (1 + 2x)(3x - 1)^2$

Question 5

Find the equation of the tangent to the curve $y = x^2 e^{2x}$ at the point where $x = 1$.

Question 6

A curve has equation $y = x^2\sqrt{x + 6}$, $x > -6$.

Find the equation of the normal to the curve at the point where $x = 10$, giving your answer in the form $y = mx + c$.

Question 7

A curve has equation $y = 8x^2(4x-1)^3$.

a) Show that

$$\frac{dy}{dx} = Ax(4x-1)^2(Bx+C)$$

Where A, B and C are integers to be determined.

b) Hence find the coordinates of each of the stationary points on the curve. (You do not need to classify these.)

Challenge

Find the derivative of $y = x^2e^x(2x+1)^3$

Test Your Understanding 4

Question 1

Find the derivative of each of the following with respect to x, simplifying your answers where possible.

a) $y = \frac{3x}{x-2}$ b) $y = \frac{e^x}{x+2}$ c) $y = \frac{x^2}{1+e^x}$

d) $y = \frac{\ln x}{2x+1}$ e) $y = \frac{\sqrt{x}+2}{(x-1)^2}$ f) $y = \frac{(e^x-1)^2}{2x}$

Question 2

Differentiate $y = \frac{x^2}{e^x}$ using

i) The Quotient rule,

ii) The Product rule.

Show that your answers are equivalent.

Question 3

Find the coordinates of any stationary points on the curve:

a) $y = \frac{x^2}{x+1}$ b) $y = \frac{e^x}{2x+1}$ c) $y = \frac{\ln x}{2x}$

(Hint: Remember to consider any restrictions on x)

Question 4

Find the equation of the tangent to each of the following curves at the given x-coordinate.

a) $y = \frac{2x}{x-1}$ when $x = 2$ b) $y = \frac{e^x+5}{e^x+2}$ when $x = 0$.

Question 5

Find the equation of the normal to the curve $y = \frac{x}{\ln x}$ at the point where $x = e^2$, giving your answer in the form $ax + by + c = 0$ where a, b are integers and c is given as an exact real value.

Question 6

You are given that $f(x) = \frac{3x}{x-3} - \frac{15x}{x^2-x-6}$

a) Show that $f(x) = \frac{3x}{x+2}$

b) Hence find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$.

Question 7

A curve has equation $y = \frac{e^{4x}}{(x+3)^2}$, $x \neq -3$.

a) Show that $\frac{dy}{dx} = \frac{Ae^{4x}(Bx+C)}{(x+3)^3}$ where $A, B, C \in \mathbb{Z}$.

b) Find the equation of the normal to the curve at the point where $x = 0$.

Challenge Question

Given the curve with equation $y = \frac{2\sqrt{x}-3}{x-2}$, show that any stationary points on the curve satisfy the equation $x - 3\sqrt{x} + 2 = 0$, and hence find any stationary point(s) on the curve.

Grade Enhancer™ - Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(MathEVmatics Originals)*

You are given that $y = \ln(px + q)$.

By writing x in terms of y, show that $\frac{dy}{dx} = \frac{p}{px+q}$. [5]

Question 2 *(MathEVmatics Originals)*

A curve C has equation $x = y^3 - 2y^2$.

a) Verify that the point (9, 3) lies on the curve. [1]

b) Find an equation for the tangent to the curve at the point (9, 3). [5]

Question 3 *(WJEC 2014)*

Differentiate each of the following with respect to x, simplifying your answer wherever possible.

(i) $\dfrac{1}{\sqrt[4]{9 - 4x^5}}$ (ii) $\dfrac{3 + 2x^3}{7 - x^3}$ [5]

Question 4

Differentiate each of the following with respect to x, simplifying your answer wherever possible.

(i) $\ln(4x^2 - 3x - 5)$ [2]

(ii) $e^{\sqrt{x}}$ [2]

Question 5 *(WJEC 2014, partial)*

Differentiate each of the following with respect to x, simplifying your answer wherever possible.

(a) $(5x^3 - x)^{10}$ [2]

(b) $x^4 \ln(2x)$ [3]

(c) $\frac{e^{4x}}{(2x+3)^6}$ [4]

Question 6 *(WJEC 2019)*

Differentiate each of the following functions with respect to x.

i) $x^5 \ln x$ [3]

ii) $\frac{e^{3x}}{x^3 - 1}$ [3]

Question 7 *(MathEVmatics Originals)*

A scientist is monitoring the growth of algae on a pond. They model the area A m² from the start of monitoring by the equation

$$A = 20 - \frac{3e^{0.2t}}{(t+1)^2}, \ t > 0,$$

where t is the time in days since monitoring began.

a) State the initial area of algae on the pond. [1]

b) Show that

$$\frac{dA}{dt} = \frac{e^{0.2t}(At - B)}{(t+1)^3},$$

stating the values of A and B. [4]

c) Hence, find the time when the rate of change of algae reaches zero. By considering the gradient either side of this value, show that this value is a maximum. [4]

d) Explain why this model is unsuitable for large values of t. [1]

Total Mark Available is 45.