MathEVmatics

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A2 Mathematics for WJEC

Trigonometric Functions

Examples and Practice Exercises

Unit Learning Objectives

- ***To know and use the three reciprocal trigonometric functions;***
- ***To be able to sketch the reciprocal trigonometric functions and transformations of these;***
- ***To know and use the reciprocal Pythagorean identities, to solve equations and prove identities.***
- ***To understand the inverse trigonometric functions, and to solve problems using these.***

Prerequisite learning:

Trigonometric Graphs and Equations (AS Mathematics)

Transformation of Graphs (AS Mathematics)

Radian Measure (A2 Mathematics, Booklet 3)

Basic Skill Check:

1) Sketch the graph of $y = \cos x$ for $0 \leq x \leq 2\pi°$.

2) Solve the equation $5tan^2x + tanx - 4 = 0$ for $0 \leq x \leq 2\pi$

3) For a function $y = f(x)$, describe the following transformations:

(a) $y = f(x) + 3$

(b) $y = 2f(x)$

(c) $y = f(x - 5)$

(d) $y = f(2x)$

ALSO: *It is highly advised that you are comfortable using Desmos (www.desmos.com) or other graphing software to help you during this unit!*

When you have completed the unit…

Objective	Met	Know	Mastered
To know and use the three reciprocal trigonometric functions			
To be able to sketch the reciprocal trigonometric functions and transformations of these			
To know and use the reciprocal Pythagorean identities, to solve equations and prove identities			
To understand the inverse trigonometric functions, and to solve problems using these.			

Notes/Areas to Develop:

The Reciprocal Trigonometric Functions

There are three reciprocal trigonometric functions:

- secant (sec), where $\sec x = \frac{1}{\cos x}$ (not defined if $\cos x = 0$)
- cosecant (cosec), where $cosec\ x = \frac{1}{\sin x}$ (not defined if $\sin x = 0$)
- cotangent (cot), where $\cot x = \frac{1}{\tan x}$

Since $\tan x = \frac{\sin x}{\cos x}$, it also follows that $\cot x = \frac{\cos x}{\sin x}$. This implies that $\cot x$ is not defined when $\sin x = 0$, but *will* be defined at the points that $\tan x$ was not (i.e. the asymptotes).

Example 1: Evaluate the following:

a) sec 120°   b) cosec $\frac{\pi}{4}$ c*) cot $\frac{\pi}{2}$

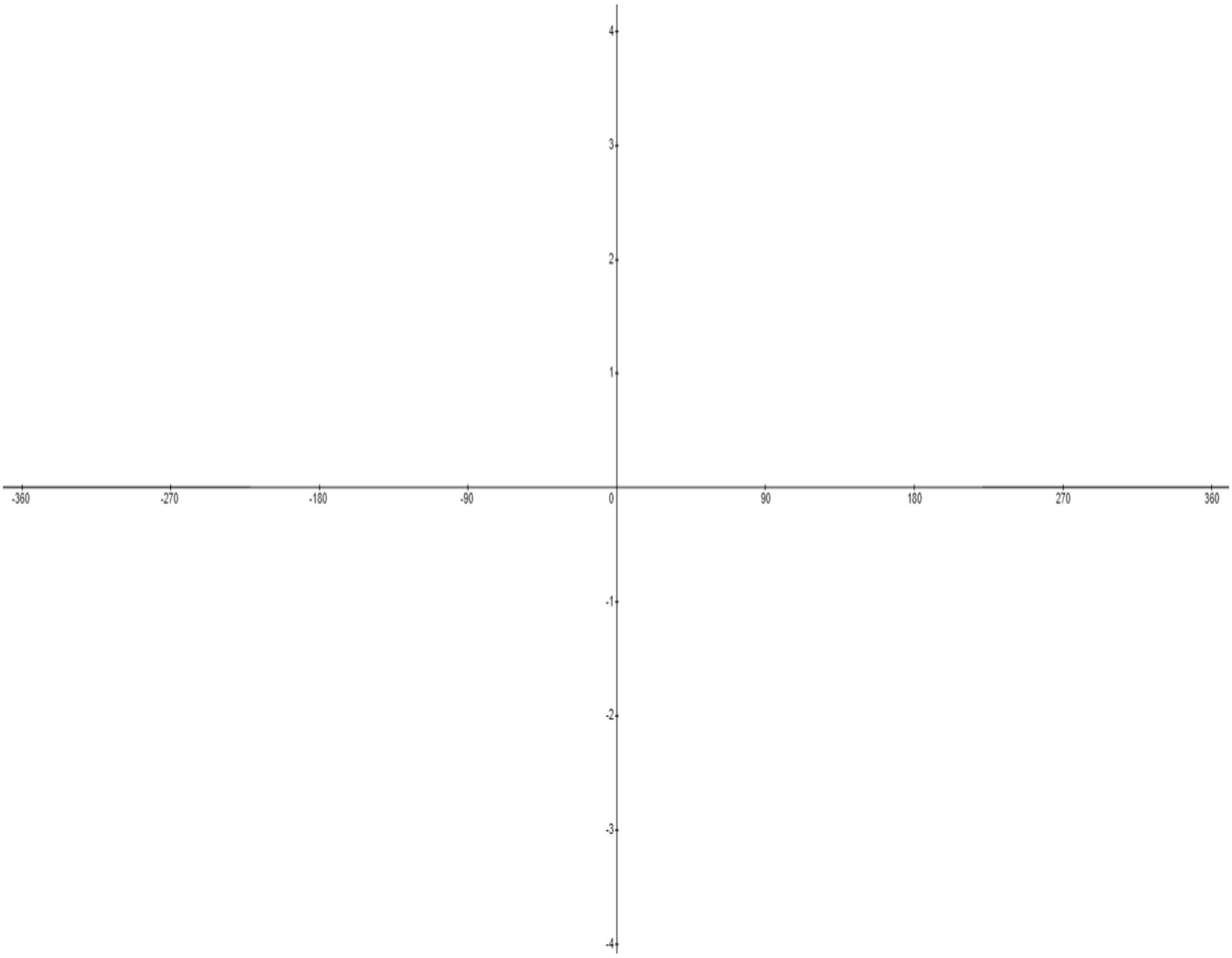
Task 1: Evaluate the following:

a) cosec 120°   b) sec $\frac{5\pi}{4}$ c) cot 80°

NOTE: In some textbooks, and on Desmos, they use 'csc' instead of 'cosec'.

As with the standard trigonometric functions, we can draw graphs of the reciprocal trigonometric functions.

Graph of $y = cosecx$



Space for notes:

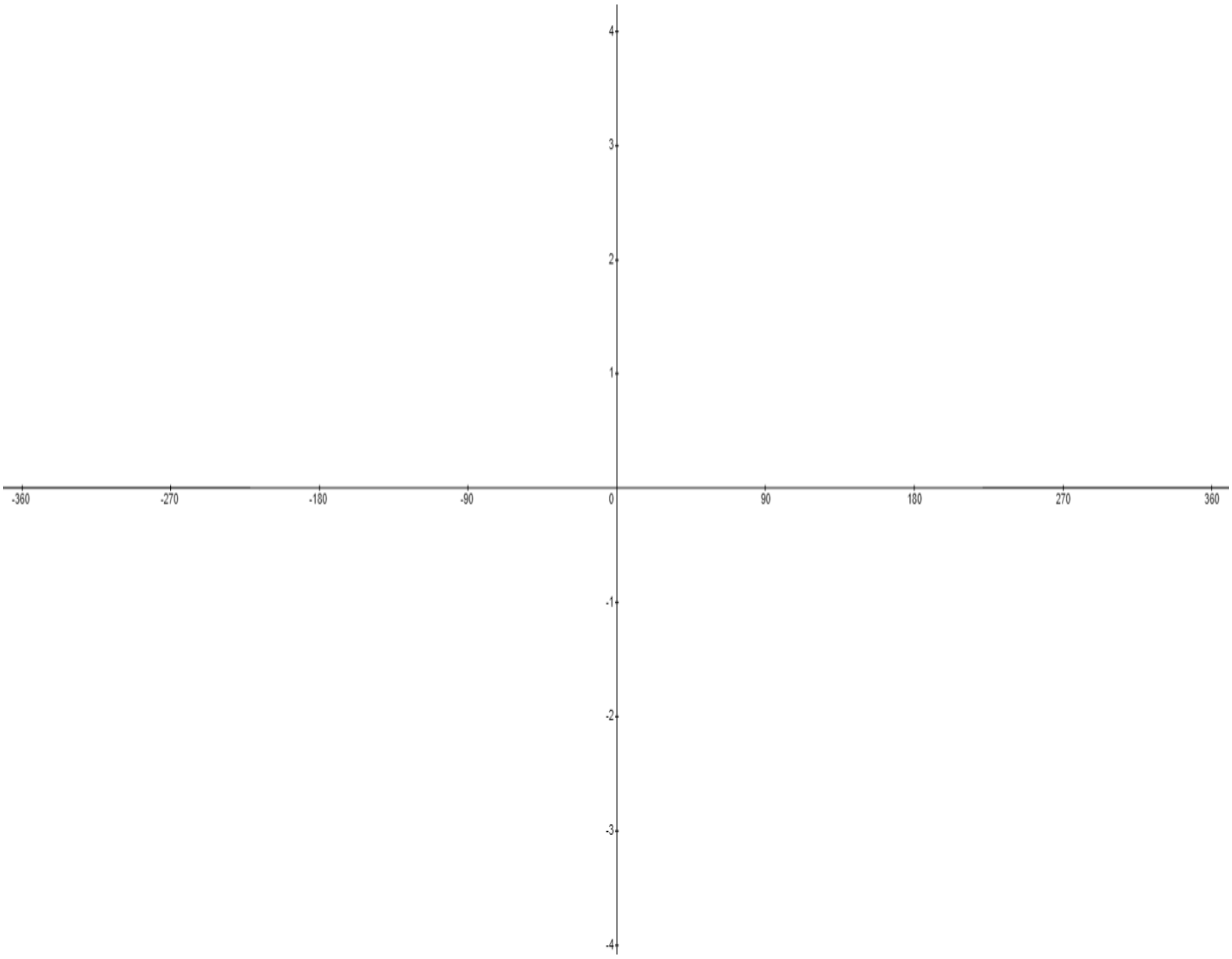
Domain:

Range:

Period:

Asymptotes:

Graph of $y = secx$



Space for notes:

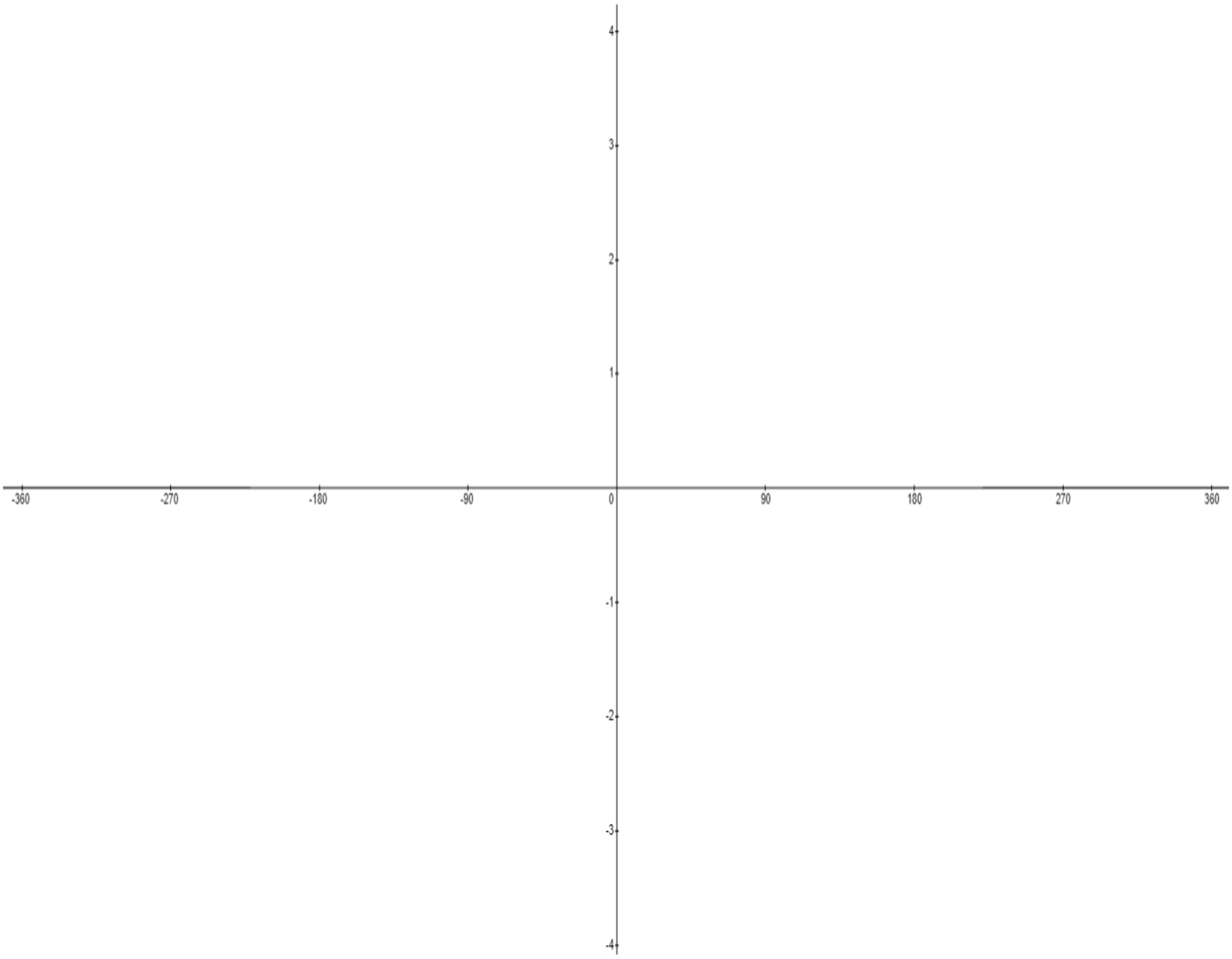
Domain:

Range:

Period:

Asymptotes:

Graph of $y = cotx$



Note that $\tan x$ *is not defined at 90°, 270° etc BECAUSE* $\cos x = 0$ *at these points. However,* $\cot x$ *__is__ defined at these points, since we are now dividing by sine instead.*

Space for notes:

Domain:

Range:

Period:

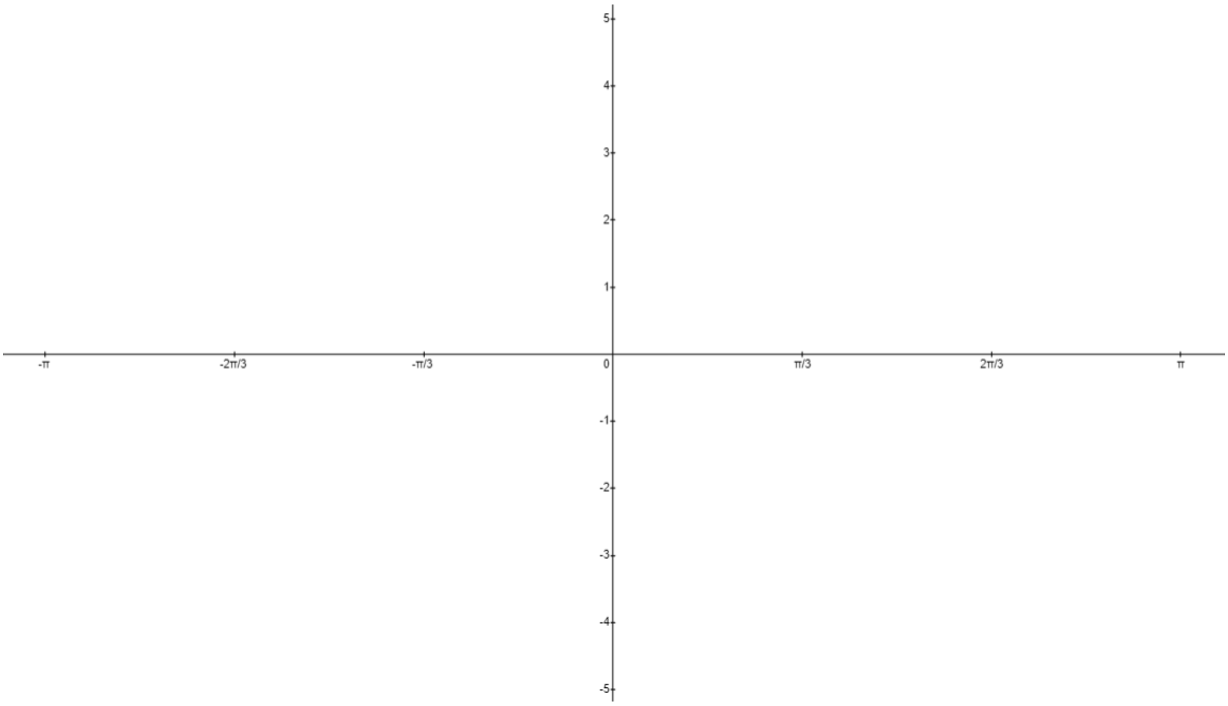
Asymptotes:

Example 2:

a) Sketch the graphs of $y = secx$ for $-\pi \leq x \leq \pi$ and $y = x + 1$ on the same graph.

b) With reference to your sketch, state the number of roots of the equation

$$secx - x - 1 = 0 \text{ for } -\pi \leq x \leq \pi$$



Task 2:

Sketch the following graphs:

a) $y = \sec 2\theta - 1$ $\quad\quad\quad\quad\quad\quad 0 \leq \theta \leq 360°$

b) $y = 2\text{cosec}\,\theta + 1$ $\quad\quad\quad\quad\quad 0 \leq \theta \leq 360°$

c) $y = 3\sec(\theta + \frac{\pi}{4})$ $\quad\quad\quad\quad\quad 0 \leq \theta \leq 2\pi°$

A note: Combined Transformations and Order of Transformation.

We have seen that multiple combinations can be combined as in the last task.

- *If there is a mix of a vertical and a horizontal transformation - like in Task 1, a) and c) – then these do not affect each other and the order does not matter (though it makes sense to follow BODMAS!).*
- *If there are two vertical transformations, as in Task 1b, then BODMAS is also followed – the stretch comes before the translation.*

However, we have seen in the past that horizontal transformations are awkward (we often talk about them doing the "opposite" to what we expect). The same is, annoyingly, true about combined horizontal transformations.

- *If we have a function $y = f(ax + b)$, we actually apply 'reverse BODMAS' – we translate by -b units and then stretch by factor $\frac{1}{a}$.*

NOTE: It is highly unclear whether WJEC could/would examine this – it is not explicitly mentioned or rejected in the specification, and there are no examination questions to date on this. **However**, it is very much a case of "better safe than sorry", and therefore I cover this for completeness and supporting those who transition to mathematics at higher levels.

Example 3:

Given the function $f(x) = x^2$, on the same axes draw $y = f(x)$ and $y = f(2x + 1)$.

Task 3:

Sketch the graph of $y = cosec(2\theta + 90)$, for $0 \leq \theta \leq 360°$.

Examiner Tip: Remember to always start off with a sketch of the parent function - here this is $y = cosec\ \theta$.

Now: Complete TEST YOUR UNDERSTANDING 1, Page 21.

Equations and Identities involving Secant, Cosecant and Cotangent

Now we have met some new trigonometric functions, it is time to look at extending our skill set with solving equations, and introduce some more identities.

Example 1:

Solve the following equations within the specified intervals.

a) $\sec\theta = \sqrt{2}$, $0 \leq \theta \leq 2\pi$ b) $3\cot\theta = -2\,, -180 \leq \theta \leq 180$

Space for sketches/notes:

Task 1: Solve the equations within the given intervals for x.

a) $cosec\ x = 2,$ $\quad -180 \leq x \leq 180$

b) $\sec 2x = \sqrt{3}$, $\quad 0 \leq x \leq 2\pi$

Example 2:

Solve the equation $2sec^2x - 5\sec x = 0$ in the interval $0 \leq x \leq 2\pi$.

Task 2: Solve the equation $2cot^2x - \cot x = 5$ in the interval $0 \leq x \leq 360$.

Pythagorean Identities

At AS-level, we met the fact that, for any angle x, the identity $sin^2x + cos^2x \equiv 1$ is true.

We can use this identity, along with the definitions of our new trigonometric identities, to form two new identities:

$$sin^2x + cos^2x \equiv 1$$

These allow us to solve further quadratic equations, as follows:

Example 1:

Solve the equation $4sec^2x - tanx = 9$ in the interval $\mathbf{0 \leq x \leq 360.}$

Now: Complete TEST YOUR UNDERSTANDING 2, Page xx.

Solving Problems and Proving Identities

We now have an expanded toolkit, which the examiner can test in multiple ways.

Example 1: Given that $\sec A = \frac{5}{3}$ and A is acute, find the value of

a) $\sin A$ b) $\cot A$

Task 1: Given that $\tan A = -\frac{7}{24}$ and that angle A is reflex, find the exact value of $\sec A$.

We will also be expected to prove identities, like at AS-level, involving our new functions.

Example 2: Prove the following identities:

a) $\sin x \cot x \sec x \equiv 1$

b) $\sin x + \cos x \equiv \sin x \cos x\,(\sec x + \operatorname{cosec} x)$

Now: Complete TEST YOUR UNDERSTANDING 3, Page xx.

The Inverse Trigonometric Functions

We have often used the inverse processes for trigonometric functions when finding angles, e.g. when we consider $tan^{-1}(2.3)$.

However, we are now going to look at the formal inverse functions – remembering that:

- we can only take the inverse of a one-to-one function,
- the graph of an inverse function is a reflection in the line y = x,
- the domain of the inverse is the range of the function (and vice-versa).

The inverse function of $\boldsymbol{sin\, x}$ is called $\boldsymbol{y = arcsin\, x}$.

To 'create' this function, we need to restrict the domain of $\sin x$ to create a one-to-one function containing the full range of values from -1 to 1.

It makes sense to use a domain for $\sin x$ from $-90 \leq x \leq 90$.

The inverse function of $cos\ x$ is called $y = arccos\ x$.

To 'create' this function, we need to restrict the domain of $\cos x$ to create a one-to-one function containing the full range of values from -1 to 1.

It makes sense to use a domain for $\cos x$ from $0 \leq x \leq 180$.

The inverse function of $tan\ x$ is called $y = arctan\ x$.

To 'create' this function, we need to restrict the domain of $\tan x$ to create a one-to-one function containing one full period. We also have to remember that $\tan x$ is not defined at the asymptotes.

It makes sense to use a domain for $\tan x$ from $-90 < x < 90$.

Putting It All Together

Task 1:

a) Using the identity $sin^2x + cos^2x \equiv 1$, prove that $tan^2x = sec^2x - 1$.

b) Solve, for the interval $0 \leq x \leq 2\pi$, the equation

$$2tan^2x + 4secx + sec^2x = 2$$

NOW – You are ready for the GRADE ENHANCER!

Test Your Understanding 1

Question 1

Evaluate the following:

a) cosec 40°

b) sec $\frac{2\pi}{5}$

c) cot $\frac{\pi}{3}$

d) sec 300°

e) cot 200°

f) cosec $\frac{2\pi}{3}$

Question 2

Find the value of $cot30sec30$.

Question 3

Show that $\sec\frac{\pi}{3} + cosec\ \frac{\pi}{3} = m + n\sqrt{3}$, where m and n are numbers to be determined.

Question 4

By sketching relevant graphs, show that the equation $4cosecx - x = 0,\ -\pi \leq x \leq \pi,$ has no solutions.

Question 5

Sketch the following graphs in the interval $0 \leq x \leq 360$.

a) $y = \sec(\frac{1}{2}x)$

b) $y = -\cot x$

c) $y = \text{cosec}(x + 30)$

d) $y = 2\text{cosec}\, x$

e) $y = \sec(x - 60)$

d) $y = \cot x + 1$

Question 6

Sketch the following combined transformations in the interval $0 \leq x \leq 360$.

a) $y = 2\text{cosec}(x - 60)$

b) $y = 3\sec x + 2$

c) $y = \text{cosec}(2x - 30)$

Question 7

Deduce the range of values of k for which $3 + 4\sec x = k$ has no solutions.

Question 8

Write down the periods of the following functions:

a) $\sec 2x$

b) $3\cot x$

c) $cosec\,\frac{1}{2}x$

Test Your Understanding 2

Question 1

Solve the following equations in the interval $-180 \leq x \leq 180$.

a) $cosec\ x = 1$ b) $\sec x = -2$ c) $\cot x = \sqrt{3}$

d) $2\sec x = 3$ e) $3\cot 2x = -1$ f) $cosec^2 x = 4$

Question 2

Solve the following equations in the interval $0 \leq x \leq 360$.

a) $2\sin x = cosec\ x$ b) $2sec^2 x - 4 = 0$ c) $2cot^2 x - cotx = 5$

d) $3\sec 2x = 4$ e) $\cot(3x + 45) = 2$ f) $2cosec^2 x - 5\ cosec\ x = 0$

Question 3

Solve the following equations in the interval $0 \leq x \leq 2\pi$.

a) $sec\ 2x = \frac{2\sqrt{3}}{3}$ b) $3cot^2 x - 8\cot x - 3 = 0$ c) $4\cos x = \cot x$

Question 4

Solve the following equations in the interval $0 \leq x \leq 360$, showing clearly any identities used.

a) $6tan^2\ x + \sec x = 9$ b) $cosec^2 x = 3\cot x - 1$ c) $cosec^2 x + \cot x = 1$

Question 5

a) Prove the identity

$$\frac{cosec^2 x - 1}{cosec^2 x} \equiv cos^2 x$$

b) Hence, find all solutions in the interval $0 \leq x \leq 2\pi$ of the equation

$$\frac{4cosec^2 x - 4}{cosec^2 x} = 1$$

Question 6 *(WJEC Summer '11)*

Find all values of θ in the range $0 \leq \theta \leq 360°$ satisfying

$$2cosec^2\theta + 3cot^2\theta + 4cosec\ \theta = 9$$

[6 marks]

Test Your Understanding 3

Question 1

Given that $\sin\theta = \frac{\sqrt{3}}{2}$, and θ is obtuse, find the exact value of:

a) $\cos\theta$ b) $\cot\theta$

Question 2

Given that $\sec\theta = \frac{3}{\sqrt{5}}$ and that θ is acute, find:

a) $\sin\theta$ b) $\tan\theta$

Question 3

Given that $\sec\theta = -\frac{4}{\sqrt{3}}$ and that θ is obtuse, find $cosec\ \theta$.

Question 4

Given that $\sec\theta = k$ and that θ is acute, find expressions for:

a) $\cos\theta$ b) $cot^2\theta$

Question 5

Prove each of the following trigonometric identities.

a) $sec^2\,x\,cosec^2x \equiv sec^2x + cosec^2x$

b) $cot^2x + cos^2x \equiv (cosec\ x + \sin x)(cosec\ x - \sin x)$

c) $sec^2x - sin^2x \equiv tan^2x + cos^2x$

d) $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} \equiv 2cosec^2x$

e) $\frac{1}{1-\cos x} - \frac{1}{1+\cos x} \equiv 2\cot x\ cosec\ x$

f) $sec^4\,x - tan^4\,x \equiv sec^2\,x + tan^2\,x$

g) $\sec x \sin x + cosec\ x \cos x \equiv \sec x\ cosec\ x$

Question 6

a) Show that, if $2\tan x - \cot x - 5\,cosec\ x = 0$, then

$$3cos^2\ x + 5\cos x - 2 = 0$$

b) Hence, for the range $0 \leq \theta \leq 2\pi$, solve the equation

$$2\tan 2\theta - \cot 2\theta = 5\ cosec\ 2\theta$$

Question 7

You are given that $a = 3\ cosec\ x$, $b = \sin x$ and $c = \cot x$.

a) Find an expression for b in terms of a.

b) Show that $c^2 = \frac{(a+3)(a-3)}{9}$.

Question 8

Given that $p = 2sec^2\ x - tan^2\ x$,

a) Write p in terms of $\sec^2 x$,

b) Write p in terms of $tan^2 x$,

c) Find $\frac{p-1}{p-2}$, giving your answer as a single trigonometric function.

Challenge Question

Simplify $sec^4 x - 2sec^2\ x\ tan^2 x + tan^4\ x$ fully.

Grade Enhancer – Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1

Sketch the following graphs, showing any points of intersection with the axes.

a) $y = 2arcsinx$

[1 mark]

b) $y = arcsinx + \frac{\pi}{2}$

[1 mark]

c) y=|arcsinx|

[1 mark]

Question 2

You are given that x satisfies the equation $\arcsin x = k$, for the interval $0 < k < \frac{\pi}{2}$.

Find expressions for $\cos k$ and $\tan k$.

[4 marks]

Question 3

Given that $f(x) = \arccos x$, sketch the graph of $y = f(2x)$, giving the domain and range.

[4 marks]

Question 4

a) Prove the identity $\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} \equiv 2\sec x.$

[4 marks]

b) Hence, or otherwise, solve $\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = \frac{4}{\sqrt{3}}$, in the range $0 \leq x \leq 4\pi$.

[4 marks]

Question 5 *(WJEC 2014)*

Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$15\operatorname{cosec}^2\theta + 2\cot\theta = 23.$$ [6]

Question 6 *(WJEC 2015)*

(a) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$7\operatorname{cosec}^2\theta - 4\cot^2\theta = 16 + 5\operatorname{cosec}\theta.$$ [6]

(b) Without carrying out any calculations, explain why there are no values of ϕ in the range $0° \leqslant \phi \leqslant 90°$ which satisfy the equation

$$4\sec\phi + 3\operatorname{cosec}\phi = 6.$$ [1]

Question 7 *(WJEC 2022)*

Solve the equation

$$6\sec^2 x - 8 = \tan x$$

for $0° \leqslant x \leqslant 360°$. [6]

Question 8 *(WJEC 2018)*

Solve the equation

$$2\tan^2\theta + 2\tan\theta - \sec^2\theta = 2,$$

for values of θ between 0° and 360°. [5]

Question 9 *(WJEC 2018)*

(a) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$3\operatorname{cosec}\theta(\operatorname{cosec}\theta - 1) = 5\cot^2\theta - 9.$$ [6]

(b) Find all values of ϕ in the range $0° \leqslant \phi \leqslant 360°$ satisfying

$$2\operatorname{cosec}\phi + 3\sec\phi = 0.$$ [3]

A TOTAL OF 52 MARKS ARE AVAILABLE.