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A2 Mathematics for WJEC

Unit 10A – Parametric Equations and Coordinate Geometry

Examples and Practice Exercises

Unit Learning Objectives

- ***To understand how curves can be defined parametrically;***
- ***To solve coordinate geometry problems involving parametric equations;***

Prerequisite atoms:

Coordinate Geometry, Equations and Inequalities (AS Mathematics)

Often, using parametric equations can simplify problems where trying to work in the Cartesian (x,y) system would be challenging.

Parametric equations simply treat the x- and y- component as two separate equations, allowing us to model each direction separately.

This can lead us to be able to plot and analyse all manner of weird and wonderful functions and situations which would otherwise be too complicated.

When you have completed the unit…

Objective	Met	Know	Mastered
I understand how parametric equations work, and can plot a curve defined parametrically.			
I can convert parametric equations into Cartesian form.			
I can solve coordinate geometry problems involving parametric equations.			
I can differentiate curves defined parametrically.			

Notes/Areas to Develop:

Parametric Equations

Sometimes, it is useful to describe movement of x and y directions in terms of a third parameter (often t for time, or θ for angles). This is often the case in mechanics when considering the movement of an object or the distribution of a force.

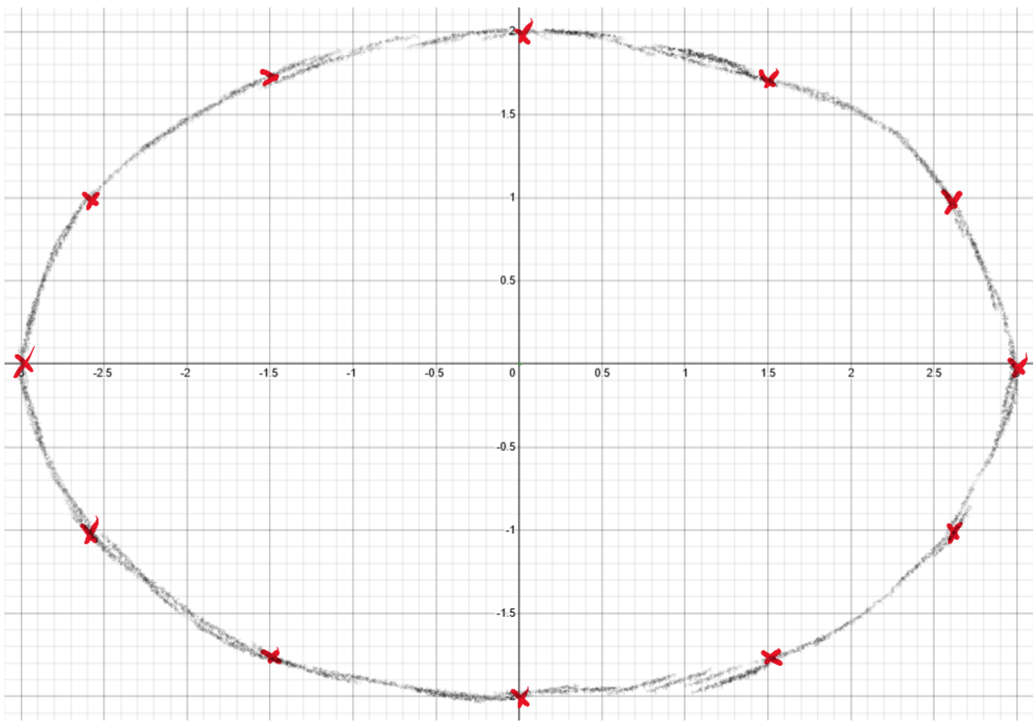
It can also be useful in pure mathematics, where to write the curve in Cartesian form (linking x and y directly) may be too complex or messy.

Example 1: A curve is defined parametrically by the equations $x = 3\cos t\,, y = 2\sin t$, where $0 \le t \le 2\pi$.

a) By completing the table of values below and plotting the graph, show that these equations represent an ellipse.

b) Using the identity $sin^2 t + cos^2 t \equiv 1$, find the Cartesian equation of the ellipse.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$x = 3\cos t$	3	2.60	1.5	0	-1.5	-2.60	-3	-2.60	-1.5	0	1.5	2.60	3
$y = 2\sin t$	0	1	1.73	2	1.73	1	0	-1	-1.73	-2	-1.73	-1	0



b) Rearranging: $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$

Using $\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \Rightarrow 9y^2 + 4x^2 = 36$

Investigate this curve at: https://www.desmos.com/calculator/c2wl8thyhu

Where the parametric equations are non-trigonometric, we can usually find the Cartesian equation relatively simply by a combination of rearrangement and substitution.

Example 2: A curve is defined by parametric equations $x = 3 - t$, $y = 2t^2$, for $-3 \leq t \leq 3$.

a) Find the coordinates of the point where $t = 1$.

b) Find a Cartesian equation of the curve, stating the domain and range.

a) When $t=1$, $x = 3-1 = 2$ $\quad y = 2(1)^2 = 2$

$\therefore (2,2)$

b) $x = 3 - t \Rightarrow t = 3 - x$

$\therefore y = 2t^2 \Rightarrow y = 2(3-x)^2$
$= 2(x^2 - 6x + 9)$
$= 2x^2 - 12x + 18$

For domain, consider x for $-3 \leq t \leq 3$
$0 \leq x \leq 6$

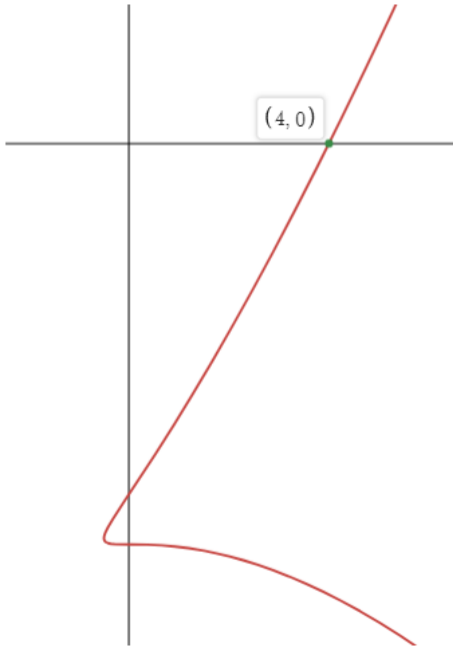
For range, consider y for $-3 \leq t \leq 3$
$0 \leq y \leq 18$

* or consider $y = 2x^2 - 12x + 18$ over the domain of x.

Key Point: To find the domain and range of a parametric curve, we can just consider the possible values of the x (domain) and y (range) parameters – it arguably makes life easier!

Example 3:

The curve shown in the image is defined parametrically by equations $x = pt^2 - t$, $y = t^3 - 8$, where p is a constant.



a) Given that the curve passes through $(4, 0)$, find the value of p.

b) Find the coordinates of the points where the curve intersects the y-axis.

a) $(\overset{x}{4}, \overset{y}{0})$ $\quad x = pt^2 - t \quad y = t^3 - 8$

$0 = t^3 - 8 \Rightarrow t = 2$

$4 = p(2)^2 - 2$

$6 = 4p \Rightarrow p = 1.5$

b) Curve intersects y-axis when $x = 0$

$\therefore 0 = 1.5t^2 - t \Rightarrow 3t^2 - 2t = 0$

$t(3t - 2) = 0$

$t = 0, \ t = \frac{2}{3}$

When $t = 0$, $y = -8$, when $t = \frac{2}{3}$, $y = -7\frac{19}{27}$

$\therefore (0, -7\frac{19}{27})$ and $(0, -8)$

Example 4

The curve C is defined by parametric equations $x = 3t$, $y = t^2$. The line with equation $x + y + 2 = 0$ meets C at the points A and B. Find the coordinates of A and B.

If solving points of intersection, the equation must be satisfied for x and y. Thus

$x + y + 2 = 0$

$\Rightarrow (3t) + (t^2) + 2 = 0$ (Substitute both parametric equations in)

$t^2 + 3t + 2 = 0$

$(t+2)(t+1) = 0$

$t = -1, \ t = -2$ Solve to find the values of t for the two coordinates.

At $t = -1$, $x = 3t$ $y = t^2$ $\therefore (-3, 1)$

At $t = -2$, $x = 3(-2)$, $y = (-2)^2$ $\therefore (-6, 4)$

Test Your Understanding 1

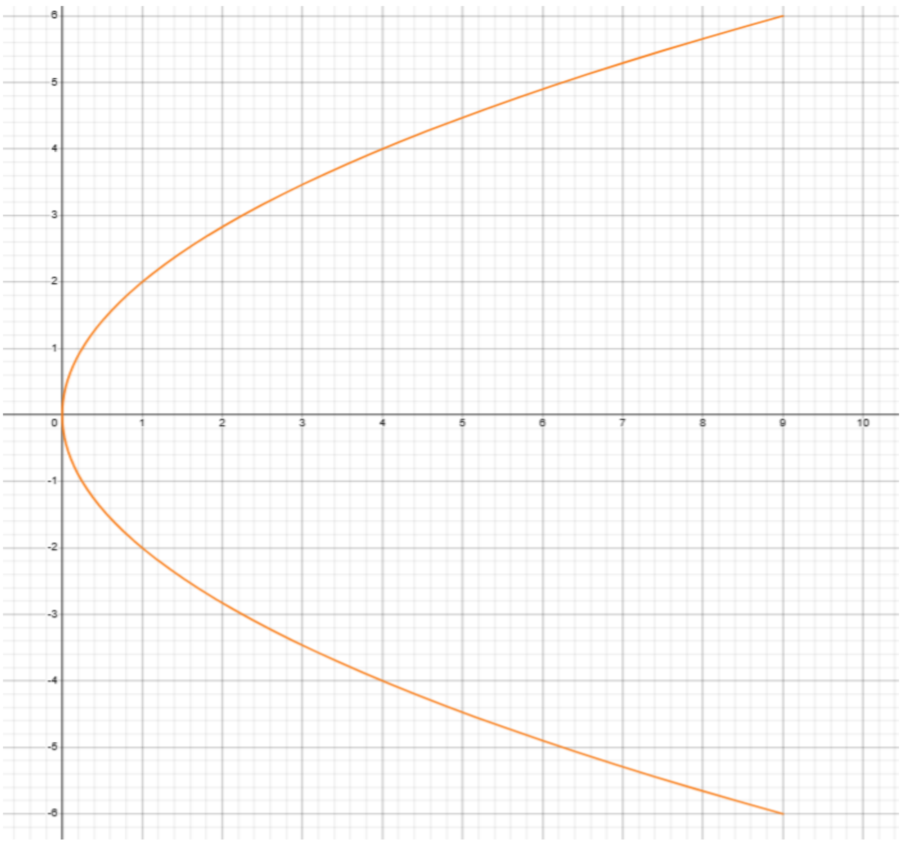
Question 1:

A curve is defined parametrically by the equations $x = t^2$, $y = 2t$, for $-3 \leq t \leq 3$.

a) By creating a table of values for t, x and y, plot the curve on graph paper.

b) Find a Cartesian equation for the curve in the form $y = f(x)$, stating the domain and range.

t	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
$x = t^2$	9	4	1	0	1	4	9
$y = 2t$	-6	-4	-2	0	2	4	6



Explore this curve and other parametric curves further at
https://www.desmos.com/calculator/kxfzx9mkcd

b) $x = t^2 \qquad y = 2t \Rightarrow t = \frac{y}{2}$

$\therefore x = \left(\frac{y}{2}\right)^2$ or $4x = y^2$ OE

Domain $0 \leq x \leq 9$, Range $-6 \leq y \leq 6$

Question 2:

a) Find the coordinates of the point on the curve $x = 3t^2$, $y = 1 - 4t^3$ where $t = -2$.

b) Find the coordinates of the **points** on the curve where $x = 3$.

a) When $t = -2$, $x = 3(-2)^2$ and $y = 1 - 4(-2)^3$

$(12, 33)$

b) $x = 3 \Rightarrow 3t^2 = 3$

$t^2 = 1 \quad \therefore t = \pm 1$

$t = 1 \Rightarrow y = 1 - 4(1)^3 = -3 \quad \therefore (3, -3)$

$t = -1 \Rightarrow y = 1 - 4(-1)^3 = 5 \quad \therefore (3, 5)$

Question 3

Determine Cartesian equations for the following curves defined parametrically.

a) $x = \cos t\,, y = 2\sin t$

b) $x = 3\cos t\,, y = sin^2 t$

c) $x = 2t - 3, y = 5 - 3t$

d) $x = t,\ \ y = 1 + \frac{1}{t^2}$

a) $\cos t = x \quad \sin t = \frac{y}{2} \quad \therefore x^2 + \left(\frac{y}{2}\right)^2 = 1$

or $4x^2 + y^2 = 4$

b) $\cos t = \frac{x}{3} \quad \therefore y + \left(\frac{x}{3}\right)^2 = 1$

$y = 1 - \frac{x^2}{9}$

c) $x = 2t - 3 \Rightarrow t = \frac{x+3}{2} \quad \therefore y = 5 - 3\left(\frac{x+3}{2}\right)$

d) $y = 1 + \frac{1}{x^2}$ by substitution.

Question 4

The curve defined by parametric equations $x = 5 + t$, $y = 3 - t$ meets the x-axis at A and the y-axis at B. Find the coordinates of A and B.

At A, $y=0$ $\therefore 3-t=0$ $t=3$

$x=5+3$ $\therefore (8,0)$

At B, $x=0$ $\therefore 5+t=0$ $t=-5$

$y=3-(-5)$ $\therefore (0,8)$

Question 5

Find the coordinates of the points where the curve defined by the parametric equations $x = t^2 - 1, y = \frac{1}{t} - 1$ meets the y-axis.

Meets y-axis when $x=0$ $\therefore t^2-1=0$

$t=\pm 1$

At $t=1$, $y=\frac{1}{1}-1=0$ $\therefore (0,0)$

At $t=-1$, $y=\frac{1}{-1}-1=-2$ $\therefore (0,-2)$

Question 6

A curve is defined parametrically as $x = \frac{t-1}{t+1}$, $y = 2t^2$, $t \neq -1$.

Find the coordinates of any points of intersection with the x- or y- axes.

Meets y-axis when $x=0$ $\therefore \frac{t-1}{t+1}=0$ $\Rightarrow t=1$

$\Rightarrow (0,2)$

Meets x-axis when $y=0$ $\therefore 2t^2=0$ $\Rightarrow t=0$

$\Rightarrow (-1,0)$

Question 7

A line L_1 is defined parametrically by the equations $x = 3t + 2, y = 1 - t$. The line L_2 has Cartesian equation $y = 2 - x$. Find the point of intersection of L_1 and L_2.

For points of intersection, sub eqⁿ's for x and y into L_2.

$$y = 2 - x$$

$$(1 - t) = 2 - (3t + 2)$$

$$1 - t = 2 - 3t - 2$$

$$2t = -1 \quad t = -\tfrac{1}{2} \quad \Rightarrow \left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

Question 8

Find the coordinates of the points of intersection of the line $y = 6 - 3x$ and the curve with parametric equations $x = t^2$, $y = 3t$.

$$y = 6 - 3x$$

$$3t = 6 - 3(t^2)$$

$$3t^2 + 3t - 6 = 0$$

$$t^2 + t - 2 = 0$$

$$(t + 2)(t - 1) = 0$$

$$t = -2, \ t = 1$$

At $t = 1$, $x = (1)^2$, $y = 3(1)$

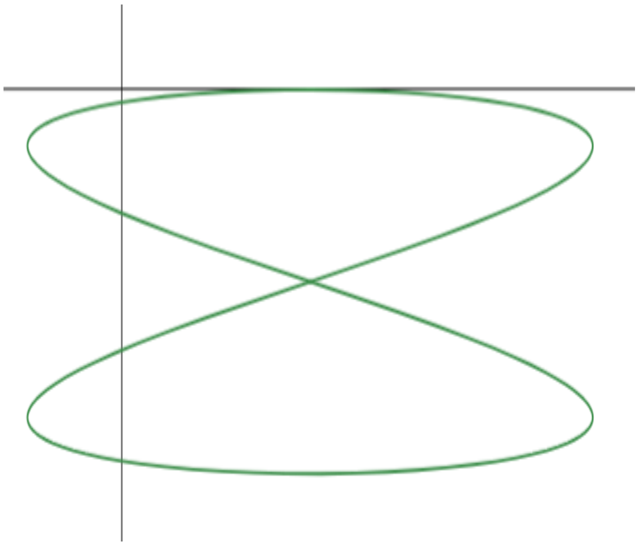
$(1, 3)$

At $t = -2$, $x = (-2)^2$, $y = 3(-2)$

$(4, -6)$

Question 9

The curve shown in the diagram on the right is defined parametrically by the equations $x = 2 + 3\sin 2\theta$, $y = \cos\theta - 1$ over the interval $0 \le \theta < 2\pi$.



a) Show that the curve meets the x-axis at the point $(2, 0)$.

b) Find the coordinates of the points where the curve meets the y-axis.

a) On x-axis, $y = 0$

$\therefore \cos\theta - 1 = 0$

$\cos\theta = 1 \qquad \theta = 0$

$x = 2 + 3\sin(2 \cdot 0)$
$= 2 + 3(0) = 2 \quad \therefore (2, 0)$ as req^d.

b) Meets y-axis when $x = 0$

$2 + 3\sin 2\theta = 0$

$\sin 2\theta = -\frac{2}{3}$

Remember: Since $0 \le \theta < 2\pi$
Use $0 \le 2\theta < 4\pi$
Then $\div 2$ for θ

$2\theta = \cancel{-0.73}, \pi + 0.73, 2\pi - 0.73, 3\pi + 0.73, 4\pi - 0.73$
$= 3.87, 5.55, 10.15, 11.84$

$\theta = 1.94, 2.78, 5.08, 5.92$

$\theta = 1.94 \Rightarrow y = \cos(1.94) - 1$
$= -1.36 \qquad \therefore (0, -1.36)$

$\theta = 2.78 \Rightarrow y = \cos(2.78) - 1$
$= -1.94 \qquad$ and $(0, -1.94)$

$\theta = 5.08 \Rightarrow y = \cos(5.08) - 1$
$= -0.64 \qquad$ and $(0, -0.64)$

$\theta = 5.92 \Rightarrow y = \cos(5.92) - 1$
$= -0.065 \qquad$ and $(0, -0.07)$

Question 10

Given that the line with equation $y = 2x - k$ does not intersect the curve defined by parametric equations $x = 1 - t, y = 3t^2 + 1$, find the range of possible values for k.

Sub into $y = 2x - k$:

$$3t^2 + 1 = 2(1-t) - k$$

$$3t^2 + 2t + (k-1) = 0$$

For no solutions, $b^2 - 4ac < 0$ $\therefore (2)^2 - 4(3)(k-1) < 0$

$$4 - 12k + 12 < 0$$

$$k > 4/3$$

Question 11

Two curves are defined parametrically as follows:

$$C_1 : x = 2t, y = t^2$$

$$C_2 : x = t, y = 3t$$

Find the coordinates of the points where C_1 and C_2 intersect.

Hint: For questions where two parametric equations meet, it is usually best to convert to Cartesian equations first!

$C_1 : t = x/2 \Rightarrow y = \frac{x^2}{4}$

$C_2 : x = t \Rightarrow y = 3x$

$\therefore \frac{x^2}{4} = 3x$

$$x^2 - 12x = 0$$

$x(x-12) = 0$ $\quad x = 0, y = 0 \quad (0, 0)$

$x = 12, y = 36 \quad (12, 36)$