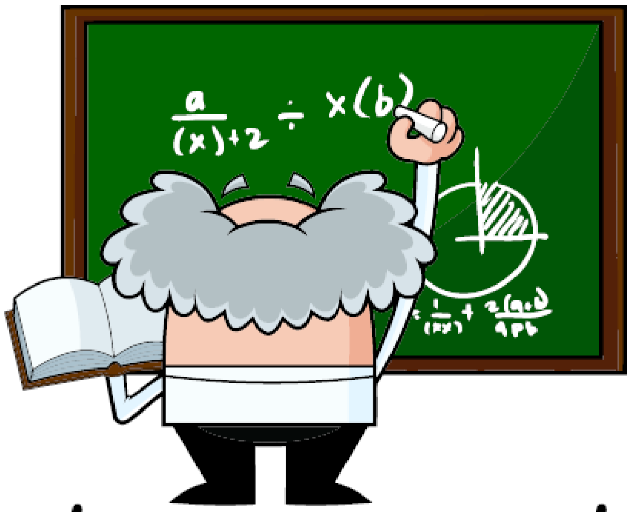

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A2 Mathematics for WJEC

Unit 1 – Partial Fractions

Examples and Practice Exercises

Unit Learning Objectives

- ***To recall techniques for working with algebraic fractions;***
- ***To be able to split a fraction into two or more partial fractions;***
- ***To understand how to modify the technique in cases where there are repeated factors.***

Prerequisite atoms:

The Four Operations with Fractions (KS3 Mathematics)

Basic Algebra (GCSE Mathematics)

Atom Check: Basics of Algebraic Fractions (*See section 1*).

When you have completed the unit…

Objective	Met	Know	Mastered
I can add, subtract, multiply and divide algebraic fractions.			
I understand what is meant by the term 'Partial Fractions' and can split a fraction into two or three Partial Fractions.			
I can extend the technique to working with repeated factors (extension: and improper fractions).			

Notes/Areas to Develop:

Atom Recap – Algebraic Fractions

"'Cause tonight is the night when two become one" – The Spice Girls

We met the techniques for the four operations with algebraic fractions, and their further simplification, at GCSE. This section serves as a recap of these techniques.

Example 1: Simplify the following fully:

a) $\frac{3}{x}+\frac{2}{x-1}$ b) $\frac{4q}{q^2+3q}-\frac{3}{4q+12}$ c) $\frac{4x^2}{3x-6}\times\frac{8x-16}{2x}$ d) $\frac{10x-10}{5x+15}\div\frac{4-3x-x^2}{x^2+7x+12}$

a) $\frac{3}{x}+\frac{2}{x-1}$

$=\frac{3(x-1)}{x(x-1)}+\frac{2x}{x(x-1)}$ Write both fractions over a common denominator.

$=\frac{3(x-1)+2x}{x(x-1)}$ Add (remember we only add the numerators!).

$=\frac{5x-3}{x(x-1)}$ Simplify. (No need to expand denominator!)

b) $\frac{4q}{q^2+3q}-\frac{3}{4q+12}$

$=\frac{4q}{q(q+3)}-\frac{3}{4(q+3)}$ Factorise the denominators – look for any shared factors.

$=\frac{4}{q+3}-\frac{3}{4(q+3)}$ Cancel any common factors within a fraction.

$=\frac{16}{4(q+3)}-\frac{3}{4(q+3)}=\frac{13}{4(q+3)}$ Write both fractions over a common denominator, subtract and simplify.

c) $\frac{4x^2}{3x-6}\times\frac{8x-16}{2x}$

$=\frac{4x^2}{3(x-2)}\times\frac{8(x-2)}{2x}$ Factorise each numerator/denominators – look for any shared factors.

$=\frac{32x^2(x-2)}{6x(x-2)}=\frac{16x}{3}$ Multiply the fractions and cancel any common factors.

d) $\frac{10x-10}{5x+15}\div\frac{4-3x-x^2}{x^2+7x+12}$

$=\frac{10x-10}{5x+15}\times\frac{x^2+7x+12}{4-3x-x^2}$ "Kiss 'n' flip"

$=\frac{10(x-1)}{5(x+3)}\times\frac{(x+3)(x+4)}{(x+4)(1-x)}$ Factorise each numerator/denominators – look for any shared factors.

$=\frac{10(x-1)(x+3)(x+4)}{5(x+3)(x+4)(1-x)}=-2$ Multiply the fractions and cancel any common factors.

Task 1: For each of the following, simplify fully.

a) $\frac{3}{x} + \frac{1}{x+1}$ b) $\frac{4q}{q^2+4q} - \frac{3}{3q+12}$ c) $\frac{2x^3}{3x+9} \times \frac{8x+24}{2x}$ d) $\frac{x-1}{3x+9} \div \frac{4-3x-x^2}{x^2+7x+12}$

Task 2: Simplify the fraction $\frac{x(x^2-4)}{x^3-5x^2+6x}$ fully

Now:

- *If fully confident, move onto the next section.*
- *If not fully confident, ask for additional practice questions.*

Introducing Partial Fractions

We have seen how to combine two fractions by addition/subtraction.

However, as A-level mathematicians this should beg the question – given one single fraction, is there some way to 'split' it into two (or more) constituent parts?

Spoiler alert: Yes.

Example 1: Show that $\frac{3x+1}{(x+2)(x-3)}$ can be written in the form $\frac{A}{x+2}+\frac{B}{x-3}$.

$$\frac{3x+1}{(x+2)(x-3)} \equiv \frac{A}{x+2}+\frac{B}{x-3}$$

Multiply through by LHS denominator and cancel common factors

$$\therefore\ 3x+1 \equiv A(x-3)+B(x+2)$$

Sub in $x=3$

$$3(3)+1 = B(3+2) \Rightarrow B=2$$

Sub in $x=-2$

$$3(-2)+1 = A(-2-3) \Rightarrow A=1$$

In turn, substitute values of x to make each bracket zero (eliminating A or B) to solve for A and B.

$$\therefore\ \frac{3x+1}{(x+2)(x-3)} \equiv \frac{1}{x+2}+\frac{2}{x-3}$$

State your answer clearly.

Task 1: Express the following in partial fractions:

a) $\frac{17-x}{(x-2)(x+3)}$

b) $\frac{7x-13}{x^2-5x+4}$

We can extend this method to any cubic denominator that we can express as a product of three linear factors!

Example 2: Express $\frac{x^2+9x+26}{(x+3)(x-1)(x+2)}$ in the form $\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{x+2}$.

$$\frac{x^2+9x+26}{(x+3)(x-1)(x+2)} \equiv \frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{x+2}$$

$$\therefore x^2+9x+26 \equiv A(x-1)(x+2)+B(x+3)(x+2)+C(x+3)(x-1)$$

Sub in $x=-3$

$$(-3)^2+9(-3)+26 = A(-3-1)(-3+2) \Rightarrow A=2$$

Sub in $x=1$

$$(1)^2+9(1)+26 = B(1+3)(1+2) \Rightarrow B=3$$

Sub in $x=-2$

$$(-2)^2+9(-2)+26 = C(-2+3)(-2-1) \Rightarrow C=-4$$

$$\therefore \frac{x^2+9x+26}{(x+3)(x-1)(x+2)} \equiv \frac{2}{x+3}+\frac{3}{x-1}-\frac{4}{x+2}$$

NOW: *Complete Test Your Understanding 1, Page 10.*

Further Partial Fractions:

Repeated Factors

Sometimes we may have to deal with repeated factors as per the following example.

Example: Express $\frac{2x+3}{(x+3)^2}$ using partial fractions.

$$\frac{2x+3}{(x+3)^2} \equiv \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

$$\therefore\ 2x+3 \equiv A(x+3) + B$$

If we were adding two fractions and had a result with a denominator of $(x+3)^2$, our fractions could have either had a denominator of $x+3$ or $(x+3)^2$.

Multiply through by LHS denominator and cancel common factors

$Sub\ in\ x = -3$

$2(-3) + 3 = B \ \Rightarrow B = -3$

$Sub\ in\ (e.g.)\ x = 0$

$2(0) + 3 = A(0+3) + (-3) \ \Rightarrow A = 2$

In turn, substitute the value of x to make the bracket zero (eliminating A) to solve for B.

Then, choose another value of x, as well as substituting in our value of B, to find A.

$$\therefore \frac{2x+3}{(x+3)^2} \equiv \frac{2}{x+3} - \frac{3}{(x+3)^2}$$

Key Points:

- If our expression contains a factor in the denominator of the form $(x+a)^2$ then we will need two fractions to account for the two possible fractions that could be added with that denominator, i.e. $\frac{1}{x+a}$ and $\frac{1}{(x+a)^2}$.
- When we reach the 'substituting values for x' stage, we will be one value 'short' – however we can substitute any value for x at this stage to find the remaining numerator.

Task: Express the following using partial fractions.

a) $\frac{3x-7}{(x-2)^2}$

b) $\frac{19-13x}{(x+1)(x-3)^2}$

NOW: *Complete Test Your Understanding 2, Page 11.*

Extension(ish) – Improper Algebraic Fractions

An **IMPROPER** algebraic fraction is defined as one where the numerator is at least equal in degree (highest power) to the denominator – e.g. if the highest-powered term on top and bottom of the fraction, when expanded, were both x^2.

According to the specification, this should be beyond the scope of the course. However, I:

- *Hate surprises;*
- *Love knowing more than I need to, as it makes the stuff I need to know seem simpler;*
- *Never trust specifications.*

Since we need no new learning to tackle this problem, and it provides a helpful revision on a skill learnt at AS-level… let's take a look.

Example 1: Show that $\frac{2x^2+x-45}{x^2-9} \equiv A + \frac{B}{x+3} + \frac{C}{x-3}$, where A, B and C are integers to be found.

$$\frac{2x^2+x-45}{x^2-9} \equiv A + \frac{B}{x+3} + \frac{C}{x-3}$$

$$\therefore\ 2x^2 + x - 45 \equiv A(x+3)(x-3) + B(x-3) + C(x+3)$$

Sub in $x = 3$

$$2(3)^2 + (3) - 45 = C(3+3) \quad \Rightarrow C = -4$$

Sub in $x = -3$

$$2(-3)^2 + (-3) - 45 = B(-3-3) \quad \Rightarrow B = 5$$

Sub in e.g. $x = 0$ *with* $B = 5,\ C = -4$

$$2(0)^2 + (0) - 45 = A(0+3)(0-3) + 5(0-3) - 4(0+3) \qquad \Rightarrow A = 2$$

$$\therefore\ \frac{2x^2+x-45}{x^2-9} \equiv 2 + \frac{5}{x+3} - \frac{4}{x-3}$$

Interested in more practice on this? See **Beyond the Boundaries™**, Page 12.

Otherwise: You are ready for the Grade Enhancer™, Page 13.

Test Your Understanding 1

Question 1

Show that $\frac{2}{(x+1)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+3}$

Question 2

Express the following in partial fractions:

a) $\frac{x+1}{(x-5)(x-3)}$

b) $\frac{15}{(x+2)(3-x)}$

c) $\frac{x-4}{x(x-2)}$

Question 3

Show that $\frac{8x+14}{(x-2)(x+3)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{x+1}$

Question 4

Express the following in partial fractions:

a) $\frac{7-5x}{(x-1)(x+1)(x-2)}$

b) $\frac{16x-2}{(x+1)(x-2)(x+3)}$

c) $\frac{4x-48}{x(x^2-16)}$

Question 5

Express the following in partial fractions:

a) $\frac{1-3x}{(3x+4)(2x+1)}$

b) $\frac{12x+12}{(x+3)(x-1)(x+5)}$

c) $\frac{2x-10}{8x^2+10x-3}$

d) $\frac{4x-1}{x^2+x-2}$

e*) $\frac{2x^2+4}{x(x-1)(x-4)}$

Challenge Question

Express in partial fractions

$$\frac{2x-10}{x^3+6x^2+11x+6}$$

Test Your Understanding 2

Question 1

Show that $\frac{5x-4}{(x-1)^2} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2}$

Question 2

Express the following as partial fractions:

a) $\frac{3x+8}{(x+3)^2}$ b) $\frac{2x-5}{(x-3)^2}$ c) $\frac{x+4}{(x+1)^2}$

Question 3

You are given that

$$\frac{-25}{(x+3)(x-2)^2} \equiv \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

where A, B and C are integers. Find the values of A, B and C.

Question 4

Express the following as partial fractions:

a) $\frac{3}{x(x+1)^2}$ b) $\frac{3x-2}{(x-1)(x-2)^2}$ c) $\frac{9}{(x-2)(x+1)^2}$

Question 5

Express the following in partial fractions, simplifying your coefficients where possible.

$$\frac{9}{x^2(x-2)}$$

Challenge Question

Express in partial fractions

$$\frac{3x^3-20x^2+47x-38}{(x-1)^2(x-3)^2}$$

Beyond The Boundaries™

This section contains questions at/beyond the specification or at the very top end of the A grade range. These questions are optional!*

BtB1.

a) Find the quotient and remainder when $3x^2 - 4x - 13$ is divided by $x^2 - x - 2$.

b) Hence, show that

$$\frac{3x^2 - 4x - 13}{x^2 - x - 2} \equiv A + \frac{B}{x+p} + \frac{C}{x-q}$$

BtB2.

Show that

$$\frac{4x^3 - 15x^2 - 32x + 14}{(x+2)(x-5)} \equiv Ax + B + \frac{C}{x+2} + \frac{D}{x-5}$$

BtB3.

a) Show that

$$\frac{2x^3 - 2x^2 - 2}{x^2 - 1} \equiv Ax + B + \frac{C}{x+1} + \frac{D}{x-1}$$

b) Hence, or otherwise, solve

$$\frac{2x^3 - 2x^2 - 2}{x^2 - 1} = \frac{2x - 4}{(x+1)(x-1)}$$

Grade Enhancer™ - Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Note: Many of these questions are (no pun intended, honestly) 'partial', as this topic is often co-examined as part of another topic.

Question 1 *(WJEC 2015)*

Express

$$\frac{2}{n(n+2)}$$

in partial fractions. [3]

Question 2 *(WJEC 2016)*

The function f is defined by

$$f(x)=\frac{17+4x-x^2}{(2x-1)(x-3)^2}.$$

Express $f(x)$ in terms of partial fractions. [4]

Question 3 *(WJEC 2017)*

The function f is defined by

$$f(x)=\frac{24x^2+31x+9}{(x+1)(2x+1)(3x+1)}.$$

Express $f(x)$ in partial fractions. [4]

Question 4 *(WJEC 2018)*

Show that

$$\frac{3x}{(x-1)(x-4)^2}\equiv\frac{A}{(x-1)}+\frac{B}{(x-4)}+\frac{C}{(x-4)^2},$$

where A, B and C are constants to be found. [3]

Question 5 *(WJEC 2019)*

Express $\dfrac{9}{(x-1)(x+2)^2}$ in terms of partial fractions. [4]

Question 6 *(WJEC 2014)*

(a) Express $\dfrac{5x^2+7x+17}{(x+1)^2(x-4)}$ in terms of partial fractions. [4]

(b) **Use your answer to part *(a)*** to express $\dfrac{5x^2+9x+9}{(x+1)^2(x-4)}$ in terms of partial fractions. [2]

Question 7 *(WJEC 2015)*

Given that $f(x)=\dfrac{2x^2+5x+25}{(x+3)^2(x-1)}$ express $f(x)$ in terms of partial fractions [4]

Question 8 *(WJEC 2017)*

(a) Express $\dfrac{8x^2+7x-25}{(x-1)^2(x+4)}$ in terms of partial fractions. [4]

(b) **Use your result to part *(a)*** to express $\dfrac{9x^2+5x-24}{(x-1)^2(x+4)}$ in terms of partial fractions. [3]

Total Mark Available is 35.