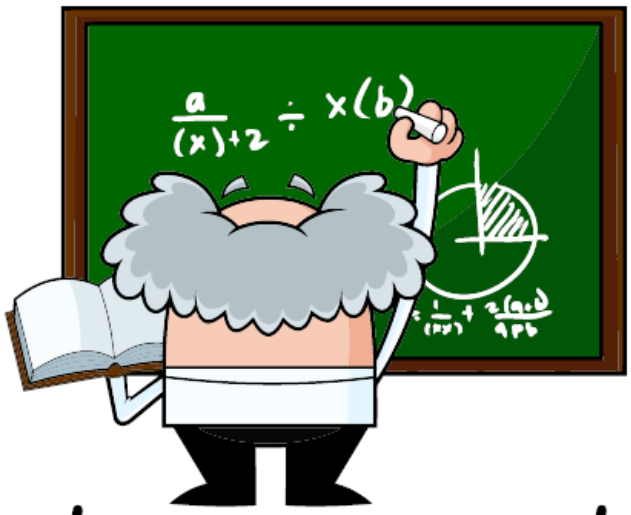

MathEVmatics

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AS Mathematics for WJEC

Unit 2: Introduction: Polynomials and Quadratics

Examples and Practice Exercises

Unit Learning Objectives

By the end of this unit, you should:

- ***Understand what is meant by a polynomial, and be able to add and subtract polynomials;***
- ***Be able to multiply two or three brackets, and to factorise a polynomial by taking out the largest factor***
- ***To be able to factorise a quadratic expression;***
- ***To be able to solve a quadratic equation by the standard techniques (factorising, or the quadratic formula);***
- ***To be able to complete the square on a quadratic expression;***
- ***To understand the information given by solving a quadratic equation and by completing the square, and be able to sketch graphs.***
- ***To understand and use the discriminant of a quadratic.***

Prior knowledge: GCSE Algebra

Now you have completed the unit…

Objective	Met	Know	Mastered
I am able to manipulate polynomials;			
I can expand two or more brackets, and factorise polynomial expressions.			
I can factorise quadratics, and solve a quadratic equation;			
I can complete the square, and use completed square form to identify the vertex and/or line of symmetry of a quadratic;			
I can sketch a quadratic graph, giving the points of intersection with the axes and the coordinates of the turning point;			
I can find the discriminant and understand the information it provides.			

Notes/Areas to Develop:

Polynomials

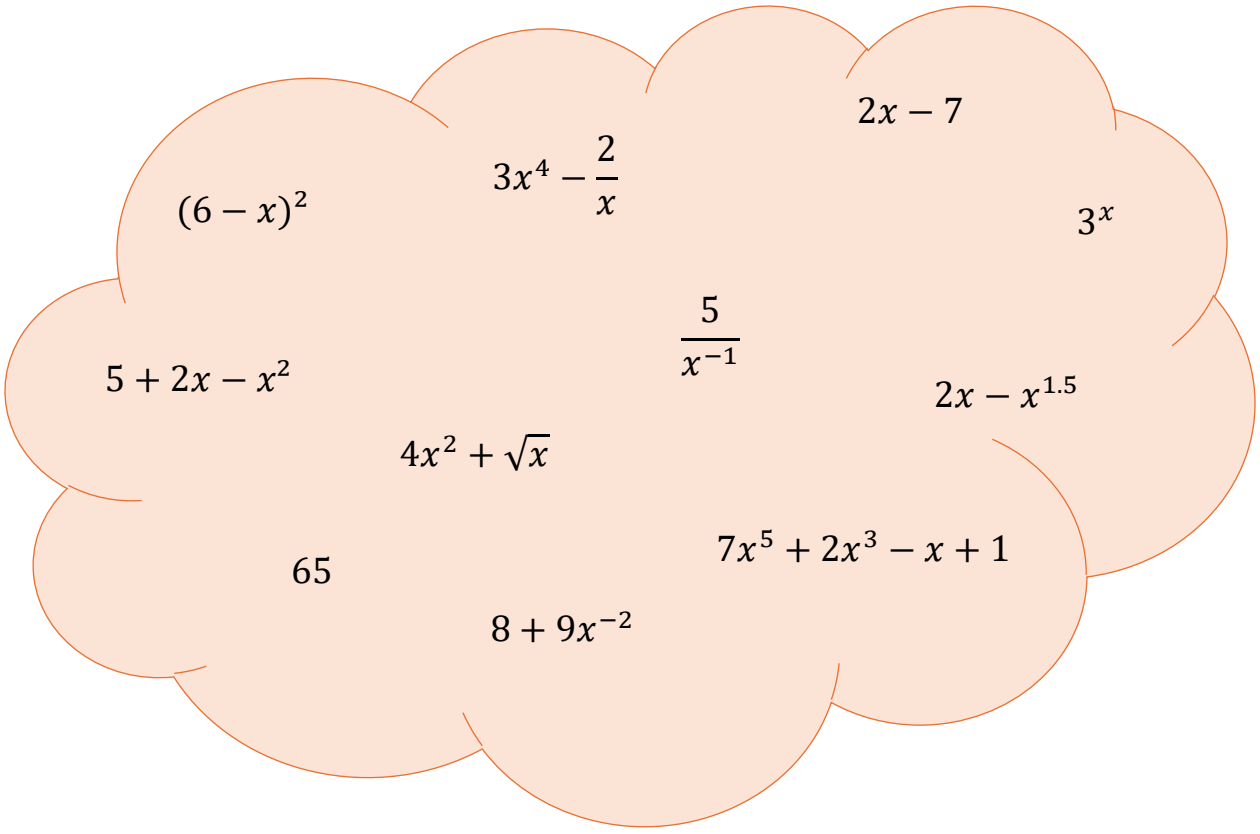
A polynomial is an algebraic expression containing one or more terms, where the powers of every term are non-negative integer values.

The order/degree of the polynomial is the highest power within the polynomial.

E.g.:

- x^5 and $3x^5 + 7x^4 - 2x^3 - 7$ are both polynomials of degree 5.
- $5x^3 - x + 4$ is a polynomial of degree 3. (We often call this a cubic.)
- 7 is a constant polynomial, i.e. a polynomial of degree 0.

Task 1: Circle/highlight the polynomials. Be prepared to explain why you have rejected any non-polynomial expressions!



We can add and subtract polynomial expressions by collecting like terms.

Example 1: Simplify $(7x^5 + 2x^3 - x + 1) - (3x^5 + 5x^4 - x^3 + 2)$

Task 2: Simplify $(3x^5 + x^4 - x^2 - x) + (3x^5 - x^4 + x^3 - 2x + 1)$

We can also multiply polynomials, using the rules of indices previously met.

Example 2: Expand and simplify $(3x^3 + x^2 - 5)(2x^5 - x + 1)$

Where we have three brackets, we simply multiply the first two together, and then multiply this answer by the third bracket:

Example 3: Expand and simplify $(x^2 + 3)(1 - x)(2 + x)$

Finally, we can factorise an expression where all the terms share a common factor:

Example 4: Factorise the following expressions:

a) $2x^5 - x^3 + 4x^2$
b) $8x^4 - 12x^3 - 20x$

Test Your Understanding 1

Question 1

Simplify the following, giving your answers in descending powers of x.

a) $(x^3 + 2x^2 + 4x + 1) + (3x^3 + x^2 + x + 2)$

b) $(4x^3 + x^2 - 3x + 5) + (2x^3 + 5x + 7)$

c) $(7x^4 + 2x^2 + 4x) - (3x^3 + x^2 - x - 2)$

d) $(x^5 + 2x^3 + 4x^2 + 1) + (3x^5 + x^4 + x + 2) - (2x^5 + 3x^4 + x^2 - 2x)$

Question 2

Fully simplify the following:

$$3(2x^3 - x^2 + 4x + 1) + 5(3x^3 + x^2 - 2) - 2(2x^3 + x - 2)$$

Question 3

Expand and simplify the following:

a) $(x^2 + x - 5)(3x - 1)$

b) $(2 - x)^3$

c) $(x^2 + x - 5)(3x^2 - 5x + 2)$

d) $(x^3 + 2x - 2)(x^2 + 5x + 2)$

Question 4

Given that

$$(x^2 + 4x + 5)(ax^2 + bx + c) \equiv 3x^4 + dx^3 - 7x^2 - 33x - 10,$$

find the values of a, b, c and d.

Question 5

Expand and fully simplify:

a) $(x^2 + 3)(1 - x)(2 + x)$

b) $(3x - 1)(2 + x)(2 - x)$

c) $(3x^2 + 4)(2 - x^2)(x + 3)$

Question 6

Factorise the following expressions:

a) $3x^3 + x^2 + x$

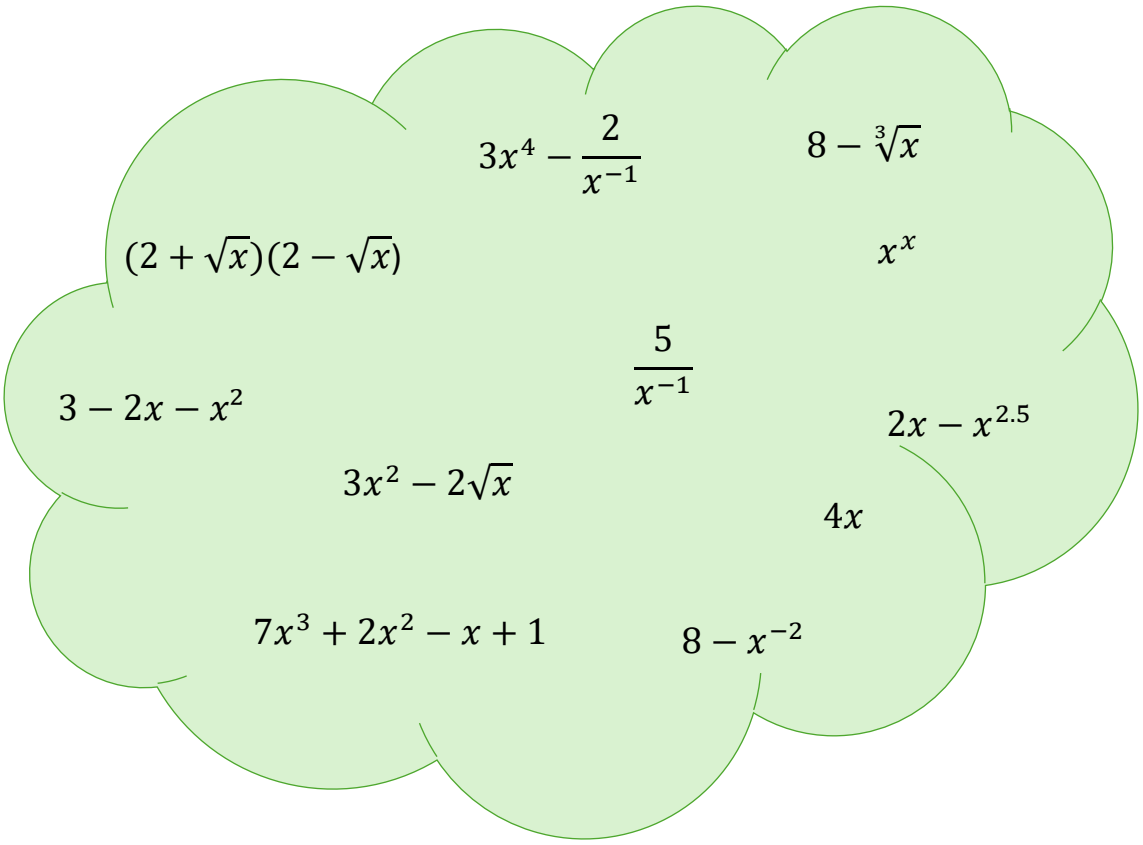
b) $8x^4 + 4x^2 - 6x$

c) $32x^5 + 16x^3 - 24x^2$

CHALLENGE QUESTION

a) Identify which of the following expressions are NOT polynomials when simplified, justifying your answer in each case.

b) For each polynomial expression, identify the order of the polynomial.



Quadratics

A **quadratic** is a polynomial expression of degree 2, e.g. $3x^2 - 5x + 2$

At GCSE you learnt how to factorise quadratic expressions, and to solve quadratic equations.

Example 1: Factorise the following quadratic expressions:

a) $x^2 - 5x - 6$

b) $3x^2 - 5x + 2$

c) $4x^2 - 9$

d) $2 - x - 6x^2$

We can use factorisation to solve some quadratic equations that are equal to zero.

Example 2: Solve the following quadratic equations:

a) $x^2 - 3x - 4 = 0$

b) $x^2 - 2x + 1 = 0$

c) $12x^2 + 8x - 15 = 0$

Many quadratic expressions cannot be factorised; a quadratic equation which cannot be solved by factorising can instead be solved by the quadratic formula.

Given a quadratic equation of the form

$$ax^2 + bx + c = 0$$

then the solutions of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Task 3: Solve the following quadratics by the quadratic formula, leaving your answers in surd form where appropriate:

a) $2x^2 - 3x - 4 = 0$

b) $30x^2 - 11x - 30 = 0$

Think: What do your solutions to question b) tell you about this quadratic?

Ensure: You can use your calculator's equation solver to check your answers.

Understanding solutions graphically

At GCSE you learnt to sketch quadratic graphs. The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$ correspond to the point(s) of intersection that the curve $y = ax^2 + bx + c$ has with the x-axis.

We also learnt that the value of 'c' gave us the y-intercept of the curve.

Finally, we should remember that a quadratic graph is a U-shaped parabola (if the coefficient of the x^2 term is positive), or an $\cap$-shaped parabola (if the x^2 coefficient is negative).

Task 2:

Sketch the graph of $y = 2x^2 - 5x - 3$, showing clearly the points of intersection with the coordinate axes.

Test Your Understanding 2

Question 1: Factorise each of the following quadratic expressions:

HINT: Look for opportunities to use the Difference of Two Squares (DOTS).

a) $m^2 - 7m + 12$ b) $x^2 - 12x - 28$ c) $t^2 + 3t - 28$

d) $b^2 - b - 6$ e) $k^2 - 49$ f) $m^2 - 5m$

g) $3y^2 - 8y + 4$ h) $3n^2 - 4n - 4$ i) $8x^2 - 15x - 2$

j) $25 - a^2$ k) $4m^2 - 81$ l) $(2x + 1)^2 - (x - 3)^2$

Question 2: Solve each of the following quadratics by factorisation:

a) $x^2 - 5x - 14 = 0$ b) $2x^2 - 5x - 2 = 0$ c) $6x^2 - 5x - 6 = 0$

Question 3: Solve each of the following quadratics by use of the formula, leaving your answers in surd form:

a) $x^2 + 3x - 5 = 0$ b) $2x^2 + 7x - 4 = 0$ c) $8 - 3x - x^2 = 0$

Question 4: Solve each of the following quadratics using an appropriate method. Give answers to 3 significant figures where appropriate.

a) $x^2 - 2x - 35 = 0$ b) $x^2 - 6x + 9 = 0$ c) $5 + 4x - 6x^2 = 0$

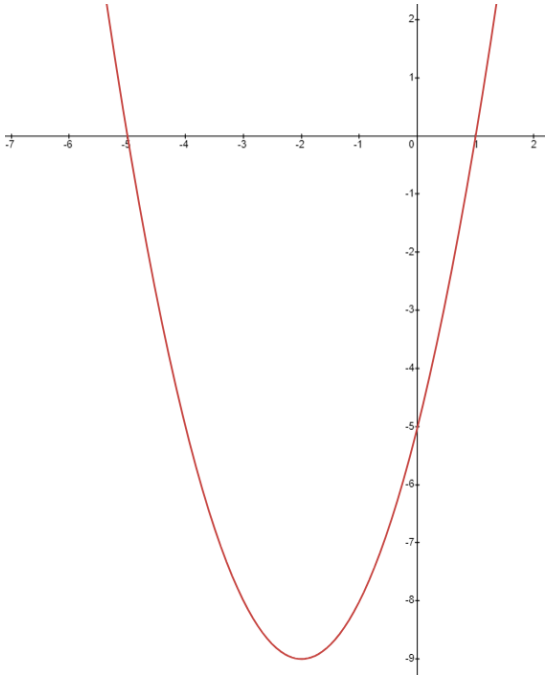
d) $9x^2 - 64 = 0$ e) $7x^2 - 2x - 3 = 0$ f) $24x^2 + 18x - 1 = 0$

Question 5: Showing your method, deduce the points of intersection with the axes of the following quadratic graphs, and hence sketch the following graphs showing clearly the coordinates of any intersections with the axes:

a) $y = x^2 - 5x + 4$ b) $y = x^2 - 4x + 4$ c) $y = 6x^2 - x - 2$

Challenge Question: Write, in the form $y = ax^2 + bx + c$, the equations of the curves shown in the following graphs.

a)



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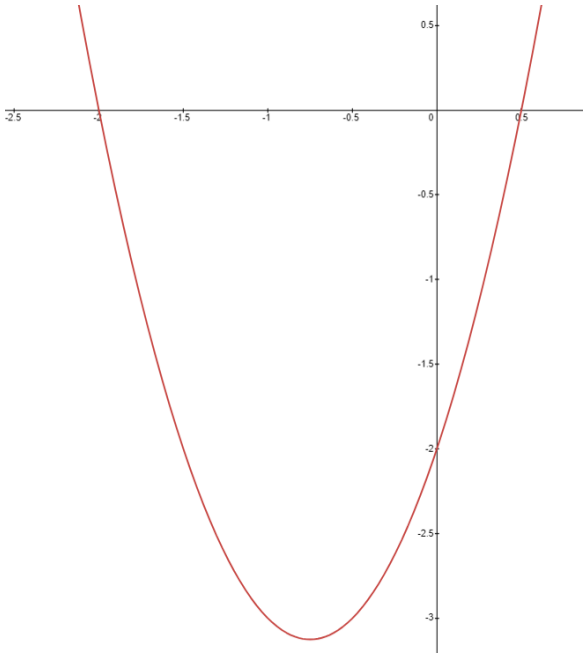

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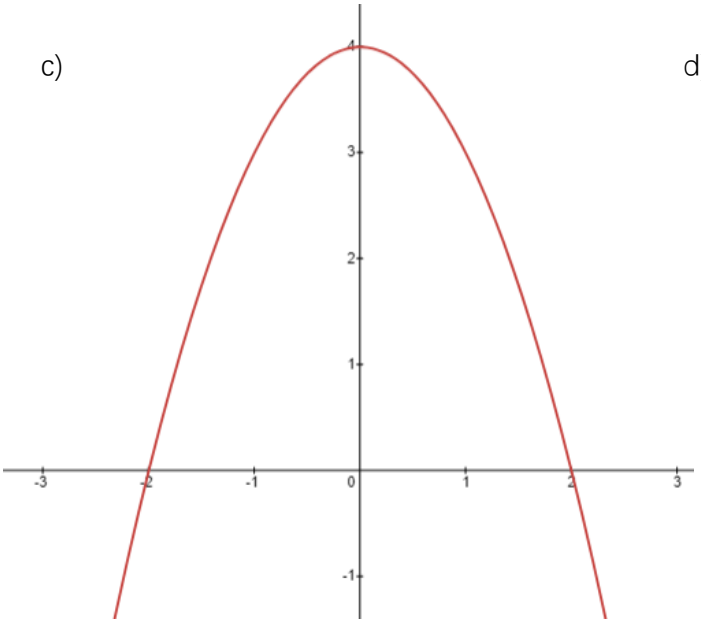
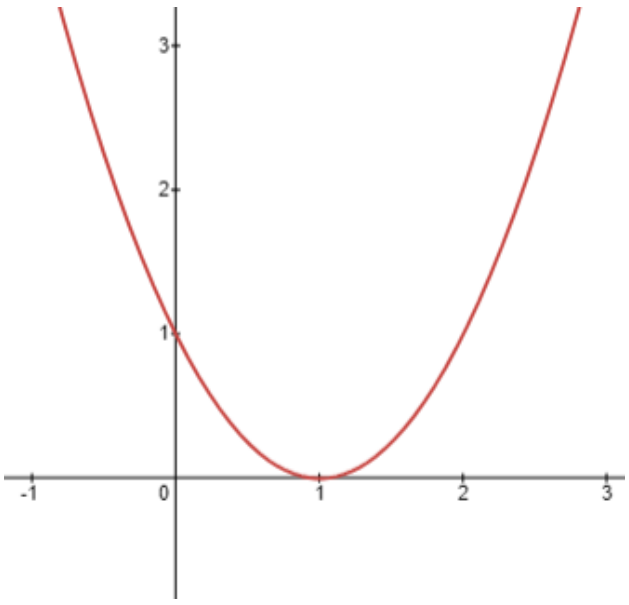


d)



Completing the Square

One of the most important skills with quadratic expressions is a technique known as 'completing the square'.

This is not in WJEC's GCSE Mathematics course, but you will have met this if you studied an English GCSE board, or the WJEC Level 2 Additional Mathematics qualification.

Example 1: Complete the square on the following expressions:

a) $x^2 + 6x + 10$ b) $x^2 - 4x - 2$ c) $x^2 + 3x + 1$

d) $x^2 - 12x$

In simple terms, to complete the square on $x^2 + bx + c$,

- write $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$
- simplify.

However, at A-level, understand is everything, and you should ensure you understand both the *how* and **why**.

It can be trickier to complete the square when the coefficient of x^2 is not 1.

Example 2:

Write the following expressions in the form $a(x+b)^2+c$, where a, b and c are rational:

a) $2x^2+8x+3$ b) $3x^2+9x-1$ c) $3+4x-2x^2$

This covers the basic mechanics of how to complete the square.

Next, we need to understand **why** we would choose to complete the square.

Example 3:

i) Write $x^2 + 2x - 8$ in completed square form.

ii) Using the answer above, sketch the graph of $y = x^2 + 2x - 8$, giving:

a) the coordinates of intersection with the x-axis,

b) the coordinates of the vertex (turning point) of the graph,

c) the equation of the line of symmetry of the graph,

d) the minimum value of y and the value of x for which it occurs.

To put it simply: completing the square is incredibly powerful!

Task 1:

i) Write $2x^2 + 3x - 1$ in completed square form.

ii) Using the answer above, sketch the graph of $y = 2x^2 + 3x - 1$, giving:

a) the coordinates of intersection with the x-axis,

b) the coordinates of the vertex (turning point) of the graph,

c) the equation of the line of symmetry of the graph,

d) the minimum value of y and the value of x for which it occurs.

The Discriminant

When we consider a quadratic equation of the form $ax^2 + bx + c = 0$, it is possible that we get:

- two distinct, real roots (solutions);
- one repeated, real root;
- no real roots.

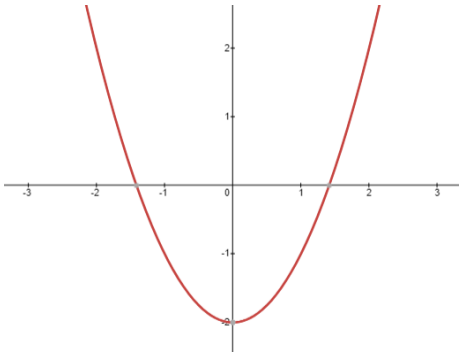
Sometimes we are only interested in finding how many roots a quadratic has (or have to solve problems based on a quadratic having a specified number of roots).

We can use the **discriminant** to identify the number of roots a quadratic equation $ax^2 + bx + c = 0$ has.

The discriminant is given by $b^2 - 4ac$ – you may recognise this as the 'bit' inside the square root in the quadratic equation.

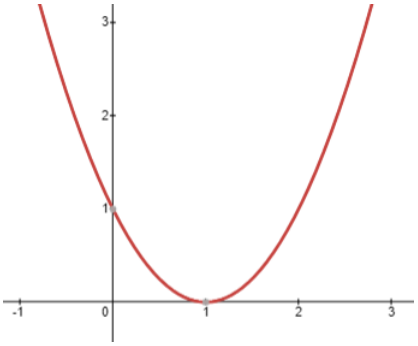
The number of roots corresponds to the following graphical situations:

Case 1: Two distinct real roots.



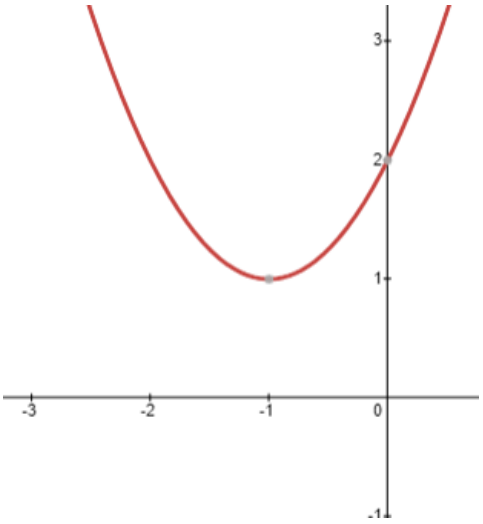
$b^2 - 4ac > 0$

Case 2: One real, repeated root.



Here, $b^2 - 4ac = 0$, and the x-axis is tangent to the curve.

Case 3: No real roots.



Here, $b^2 - 4ac < 0$ and the curve does not intersect the x-axis.

In AS mathematics you will be expected to use this knowledge in many different ways to solve problems.

Example 1: Calculate the value of the discriminant in the following expressions.

(a) $2x^2 - 3x + 1$ (b) $x^2 - x + 8$ (c) $4x^2 - 4x + 1$

Task 1: Find the value(s) of k for which $y = x^2 + kx + 9$ has a repeated real root.

Task 2: Given that the equation $x^2 + 5x + k = 0$ has two distinct real roots, find the range of possible values for k.

Task 3: You are given that $f(x) = x^2 + (k+4)x + 3k$.

(a) Find the discriminant of $f(x)$ in terms of k.

(b) By writing the discriminant in the form $(k+p)^2 + q$, explain why $f(x) = 0$ always has two real distinct roots for any real value of k.

Space for additional notes:

Grade Enhancer – Apply your Knowledge!

These 'Grade Enhancer' questions are designed in examination style, to test your understanding of the content learnt.

You should complete this task and submit full solutions within one week of the end of unit.

Question 1 *(WJEC 2019)*

Find all the values of k for which the equation $x^2 + 2kx + 9k = -4x$ has two distinct real roots. [7]

Question 2 *(WJEC 2016)*

Find the range of values of k for which the quadratic equation

$$9x^2 + 8x - 2k = 0$$

has **two distinct real roots**. [4]

Question 3 *(WJEC 2014)*

Show that $x^2 + 1{\cdot}6x - 24{\cdot}36$ may be expressed in the form $(x + p)^2 - 25$, where p is a constant whose value is to be found.
Hence solve the quadratic equation $x^2 + 1{\cdot}6x - 24{\cdot}36 = 0$. [5]

Question 4 *(WJEC 2015)*

(a) Express $4x^2 - 24x - 189$ in the form $a(x + b)^2 + c$, where the values of the constants a, b and c are to be found. [3]

(b) **Using your answer to part *(a)***, solve the equation

$$4x^2 - 24x - 189 = 0.$$ [3]

Question 5 *(WJEC 2017)*

(a) Express $-2x^2 - 20x + 35$ in the form $a(x + b)^2 + c$, where the values of the constants a, b and c are to be found. [3]

(b) **Without carrying out any further calculation**, write down the stationary value of $y = -2x^2 - 20x + 35$ and state whether this stationary value is a maximum or a minimum. [2]

Question 6 *(WJEC 2015)*

(a) Find the range of values of k for which the quadratic equation

$$kx^2 + (2k - 5)x + (k - 6) = 0$$

has **no real roots**. [4]

(b) Without carrying out any further calculation, write down the value of k for which the quadratic equation

$$kx^2 + (2k - 5)x + (k - 6) = 0$$

has **two equal roots**. [1]

Question 7 *(WJEC 2014)*

Given that the quadratic equation

$$(k - 1)x^2 + 2kx + (7k - 4) = 0$$

has no real roots, show that

$$6k^2 - 11k + 4 > 0.$$

Find the range of values of k satisfying this inequality. [7]

Question 8 *(WJEC 2014)*

Given that the quadratic equation

$$(2k - 3)x^2 + 8x + (2k + 3) = 0$$

has no real roots, show that k satisfies an inequality of the form

$$m - nk^2 < 0,$$

where m, n are integers whose values are to be found.

Hence find the range of values of k such that the quadratic equation

$$(2k - 3)x^2 + 8x + (2k + 3) = 0$$

has no real roots. [6]

Question 9 *(WJEC 2014)*

(a) Express $4x^2 - 8x + 11$ in the form $a(x + b)^2 + c$, where a, b and c are constants whose values are to be found. [3]

(b) **Use your answer to part *(a)*** to find the greatest value of $\dfrac{1}{4x^2 - 8x + 29}$. [2]