# UNIT 10.2 – PARAMETRIC DIFFERENTIATION

# STARTER – REVIEW OF PRIOR LEARNING

A curve $C$ is defined parametrically by $x = \sqrt{t-5}$, $y = 7 - 2t$

a) Find the coordinates where the curve meets the $x-$ and $y-$ axes.

b) Find a Cartesian equation for $C$ in the form $y = \ldots$

# LEARNING OBJECTIVES: PARAMETRIC DIFFERENTIATION

- To understand how to differentiate parametrically using the chain rule;
- To solve problems involving gradients, tangents, normals and stationary points on curves parametrically.

# INTRODUCTION

- Since we have now looked at parametric curves, it makes sense to consider how we would find their gradient, and to solve other problems linked to our prior learning, such as those involving tangents, normals and stationary points.
- Again, the chain rule is our friend!

# INTRODUCTION

- If we have a curve $C$ defined parametrically, that means I have two equations for $x$ and $y$ given in terms of $t$.
- This means that I can (hopefully) easily find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
- Then, by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

**Some people learn the equivalent result:**

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Either one is fine - they're the same!

# PARAMETRIC DIFFERENTIATION

For example, if given a curve $C$ defined parametrically such that $x = t^2 + \text{t}$, $y = 3 - 2t$

$$\frac{dx}{dt} = 2t + 1 \qquad \frac{dy}{dt} = -2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = -2 \times \frac{1}{2t + 1}$$

$$\frac{dy}{dx} = -\frac{2}{2t + 1}$$

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# QUICKFIRE DERIVATIVES - WHITEBOARDS

Find $\frac{dy}{dx}$ for each of:

| | |
|---|---|
| $x = 2t, y = t^3$ | $\frac{dy}{dx} = \frac{3t^2}{2}$ |
| $x = \sin t\,, y = t^2$ | $\frac{dy}{dx} = \frac{2t}{\cos t}$ or $\frac{dy}{dx} = 2t\, sec\ t$ |
| $x = \tan t\,, y = \ln t$ | $\frac{dy}{dx} = \frac{1}{t\sec^2 t}$ |
| $x = e^t, y = t - t^3$ | $\frac{dy}{dx} = \frac{1-3t^2}{e^t}$ |
| $x = (t^2-3)^3, y = ln\ t$ | $\frac{dy}{dx} = \frac{1}{6t^2(t^2-3)^2}$ |
| $x = cos\ 3t, y = t - sint$ | $\frac{dy}{dx} = -\frac{1-\cos t}{3\sin 3t}$ |

**Differentiation Results:**

| | |
|---|---|
| $\tan x$ | $\sec^2 x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\cot x$ | $-\mathrm{cosec}^2 x$ |
| $\mathrm{cosec}\, x$ | $-\mathrm{cosec}\, x \cot x$ |
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |

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# EXAMPLE 1

A curve $C$ is defined parametrically by $x = t^3 - t$, $y = 2t - t^2$.

Find the equation of the tangent to $C$ at the point $P$ where $t = 3$.

$$\frac{dx}{dt} = 3t^2 - 1 \qquad\qquad \frac{dy}{dt} = 2 - 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2 - 2t}{3t^2 - 1}$$

When $t = 3$, $\dfrac{dy}{dx} = \dfrac{2 - 2(3)}{3(3)^2 - 1} = -\dfrac{2}{13}$ and $x = 24$, $y = -3$

$\therefore$ equation of tangent is: $y + 3 = -\dfrac{2}{13}(x - 24)$

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## TASK 1

A curve is defined by parametric equations $x = 2\cos\theta, y = 3\sin\theta$. Find the equation of the normal to the curve at the point where $\theta = \frac{\pi}{6}$.

$$\frac{dx}{d\theta} = -2\sin\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{3\cos\theta}{2\sin\theta}$$

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{3\left(\frac{\sqrt{3}}{2}\right)}{2\left(\frac{1}{2}\right)} = -\frac{3\sqrt{3}}{2}$

so $m_N = \frac{2\sqrt{3}}{9}$ and $x = \sqrt{3}$, $y = \frac{3}{2}$

$\therefore$ equation of normal is: $y - \frac{3}{2} = \frac{2\sqrt{3}}{9}(x - \sqrt{3})$

## TASK 2

Find the coordinates of the stationary points on the curve given by the parametric equations

$$x = 2 - t, \quad y = 2t^2 - t^3.$$

*You do not need to determine the nature of these points.*

$$\frac{dx}{dt} = -1 \qquad \frac{dy}{dt} = 4t - 3t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = 3t^2 - 4t$$

At stationary points, $\frac{dy}{dx} = 0$

$$3t^2 - 4t = 0 \Rightarrow t = 0, t = \frac{4}{3}$$

When $t = 0$, $(2,0)$

When $t = \frac{4}{3}$, $\left(\frac{2}{3}, \frac{32}{27}\right)$

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# TEST YOUR UNDERSTANDING 2

Complete TYU 2 from your pack.

You can now complete the PPQ booklet on parametric and implicit differentiation – DO IT!

# TEST YOUR UNDERSTANDING 2 – GEMINI AI ANSWERS

**Question 1**

a) $\frac{dy}{dx} = t - \frac{1}{2}$

b) $\frac{dy}{dx} = \frac{5-2t}{12t^2}$

c) $\frac{dy}{dx} = -\frac{3t^4}{2}$

d) $\frac{dy}{dx} = \frac{1}{te^t}$

e) $\frac{dy}{dx} = \csc t$

f) $\frac{dy}{dx} = -\frac{\sin t}{10t}$

**Question 2**

a) Gradient = $\frac{1}{4}$

b) Gradient = -1

**Question 3**

$y = \frac{2}{3}x + 2\sqrt{2}$

**Question 4**

a) At $t = 1$, $x = 1^3 = 1$ and $y = 3(1)^2 - 1 = 2$. So P(1,2) is on the curve.

b) $3x + 5y - 13 = 0$

**Question 5**

$y = 2$

**Question 6**

$\frac{dy}{dx} = \frac{8\cos t}{2\sin(2t)} = \frac{8\cos t}{4\sin t\cos t} = \frac{2}{\sin t} = 2\csc t$

**Question 7**

- Coordinates: (30, 101)
- Equation of the tangent: $y = 2x + 41$
- Proof: Substituting the parametric equations into the tangent equation gives $(t+6)^2 = 0$, which has only one solution (t=-6), confirming a single intersection point.

# TEST YOUR UNDERSTANDING 2 – GEMINI AI WORKED SOLUTIONS

**Question 1**

Find an expression for $\frac{dy}{dx}$ in terms of the parameter $t$.

**a)** $x = 2t, y = t^2 - t$

- $\frac{dx}{dt} = 2$
- $\frac{dy}{dt} = 2t - 1$
- $\frac{dy}{dx} = \frac{2t-1}{2} = t - \frac{1}{2}$

**b)** $x = 4t^3, y = 5t - t^2$

- $\frac{dx}{dt} = 12t^2$
- $\frac{dy}{dt} = 5 - 2t$
- $\frac{dy}{dx} = \frac{5-2t}{12t^2}$

**c)** $x = \frac{2}{t}, y = t^3$

- $x = 2t^{-1} \implies \frac{dx}{dt} = -2t^{-2} = -\frac{2}{t^2}$
- $\frac{dy}{dt} = 3t^2$
- $\frac{dy}{dx} = \frac{3t^2}{-2/t^2} = -\frac{3t^4}{2}$

**d)** $x = e^t, y = \ln t$

- $\frac{dx}{dt} = e^t$
- $\frac{dy}{dt} = \frac{1}{t}$
- $\frac{dy}{dx} = \frac{1/t}{e^t} = \frac{1}{te^t}$

**e)** $x = \sec t, y = \tan t$

- $\frac{dx}{dt} = \sec t \tan t$
- $\frac{dy}{dt} = \sec^2 t$
- $\frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{1/\cos t}{\sin t/\cos t} = \frac{1}{\sin t} = \csc t$

**f)** $x = 5t^2, y = \cos t$

- $\frac{dx}{dt} = 10t$
- $\frac{dy}{dt} = -\sin t$
- $\frac{dy}{dx} = \frac{-\sin t}{10t}$

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# TEST YOUR UNDERSTANDING 2 – GEMINI AI WORKED SOLUTIONS

**Question 2**

Find the gradient of the curve at the given value of $t$.

**a)** $x = 3t^2, y = 2 + 3t$ **when** $t = 2$

- First, find $\frac{dy}{dx}$:
  - $\frac{dx}{dt} = 6t$
  - $\frac{dy}{dt} = 3$
  - $\frac{dy}{dx} = \frac{3}{6t} = \frac{1}{2t}$
- Now, substitute $t = 2$ into the expression for $\frac{dy}{dx}$:
  - Gradient = $\frac{1}{2(2)} = \frac{1}{4}$

**b)** $x = 1 - \frac{1}{t}, y = 1 + \frac{1}{t}$ **when** $t = 3$

- First, find $\frac{dy}{dx}$:
  - $x = 1 - t^{-1} \implies \frac{dx}{dt} = t^{-2} = \frac{1}{t^2}$
  - $y = 1 + t^{-1} \implies \frac{dy}{dt} = -t^{-2} = -\frac{1}{t^2}$
  - $\frac{dy}{dx} = \frac{-1/t^2}{1/t^2} = -1$
- The gradient is constant. At $t = 3$, the gradient is **-1**.

**Question 3**

Find the equation of the tangent to the curve $x = 3\cos t, y = 2\sin t$ at the point where $t = \frac{3\pi}{4}$, giving your answer in the form $y = ax + b\sqrt{2}$.

1. **Find the gradient of the tangent:**
   - $\frac{dx}{dt} = -3\sin t$
   - $\frac{dy}{dt} = 2\cos t$
   - $\frac{dy}{dx} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$
   - At $t = \frac{3\pi}{4}$, the gradient $m = -\frac{2}{3}\cot(\frac{3\pi}{4}) = -\frac{2}{3}(-1) = \frac{2}{3}$.
2. **Find the coordinates of the point:**
   - $x = 3\cos(\frac{3\pi}{4}) = 3(-\frac{\sqrt{2}}{2}) = -\frac{3\sqrt{2}}{2}$
   - $y = 2\sin(\frac{3\pi}{4}) = 2(\frac{\sqrt{2}}{2}) = \sqrt{2}$
3. **Determine the equation of the tangent (using $y - y_1 = m(x - x_1)$):**
   - $y - \sqrt{2} = \frac{2}{3}(x - (-\frac{3\sqrt{2}}{2}))$
   - $y - \sqrt{2} = \frac{2}{3}(x + \frac{3\sqrt{2}}{2})$
   - $y - \sqrt{2} = \frac{2}{3}x + \sqrt{2}$
   - $y = \frac{2}{3}x + 2\sqrt{2}$

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# TEST YOUR UNDERSTANDING 2 – GEMINI AI WORKED SOLUTIONS

**Question 4**

A curve is defined by $x = t^3, y = 3t^2 - t$.

**a) Verify that the point P(1, 2) lies on the curve.**

- Set $x = 1$: $t^3 = 1 \implies t = 1$.
- Substitute $t = 1$ into the equation for $y$: $y = 3(1)^2 - 1 = 3 - 1 = 2$.
- Since $x = 1$ and $y = 2$ for the same value of $t$ (t=1), the point P(1, 2) lies on the curve.

**b) Find the equation of the normal to the curve at P.**

1. **Find the gradient of the tangent at P:**
   - $\frac{dx}{dt} = 3t^2$
   - $\frac{dy}{dt} = 6t - 1$
   - $\frac{dy}{dx} = \frac{6t-1}{3t^2}$
   - At point P, $t = 1$, so the tangent gradient $m_t = \frac{6(1)-1}{3(1)^2} = \frac{5}{3}$.
2. **Find the gradient of the normal:**
   - The normal gradient $m_n$ is the negative reciprocal of the tangent gradient.
   - $m_n = -\frac{1}{m_t} = -\frac{1}{5/3} = -\frac{3}{5}$.
3. **Determine the equation of the normal:**
   - Using the point P(1, 2) and $m_n = -\frac{3}{5}$:
   - $y - 2 = -\frac{3}{5}(x - 1)$
   - $5(y - 2) = -3(x - 1)$
   - $5y - 10 = -3x + 3$
   - $3x + 5y - 13 = 0$ (or $y = -\frac{3}{5}x + \frac{13}{5}$)

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# TEST YOUR UNDERSTANDING 2 – GEMINI AI WORKED SOLUTIONS

**Question 5**

A curve is defined by $x = e^t, y = e^t + e^{-t}$. Find the equation of the tangent to the curve at the point where $t = 0$.

1. **Find the gradient of the tangent:**
   - $\frac{dx}{dt} = e^t$
   - $\frac{dy}{dt} = e^t - e^{-t}$
   - $\frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t} = 1 - e^{-2t}$
   - At $t = 0$, the gradient $m = 1 - e^0 = 1 - 1 = 0$.
2. **Find the coordinates of the point:**
   - At $t = 0$: $x = e^0 = 1$ and $y = e^0 + e^0 = 1 + 1 = 2$. The point is (1, 2).
3. **Determine the equation of the tangent:**
   - The gradient is 0, which indicates a horizontal line.
   - The equation is $y = 2$.

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# TEST YOUR UNDERSTANDING 2 – GEMINI AI WORKED SOLUTIONS

**Question 6**

A curve is defined by $x = -\cos(2t), y = 8\sin t$. Show that $\frac{dy}{dx} = 2\csc t$.

- $\frac{dx}{dt} = -(-2\sin(2t)) = 2\sin(2t)$. Using the identity $\sin(2t) = 2\sin t\cos t$, this becomes $\frac{dx}{dt} = 4\sin t\cos t$.
- $\frac{dy}{dt} = 8\cos t$
- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8\cos t}{4\sin t\cos t}$
- $\frac{dy}{dx} = \frac{2}{\sin t}$
- Since $\csc t = \frac{1}{\sin t}$, we have $\frac{dy}{dx} = 2\csc t$.

# TEST YOUR UNDERSTANDING 2 – GEMINI AI WORKED SOLUTIONS

**Question 7**

A curve is defined by $x = t^2 + t, y = t^2 - 10t + 5$. Find the coordinates of the point where $\frac{dy}{dx} = 2$, the equation of the tangent at this point, and show that the tangent does not intersect the curve at any other point.

1. **Find t where $\frac{dy}{dx} = 2$:**
   - $\frac{dx}{dt} = 2t + 1$
   - $\frac{dy}{dt} = 2t - 10$
   - $\frac{dy}{dx} = \frac{2t-10}{2t+1}$
   - Set $\frac{dy}{dx} = 2 \implies \frac{2t-10}{2t+1} = 2$
   - $2t - 10 = 2(2t + 1) \implies 2t - 10 = 4t + 2 \implies -12 = 2t \implies t = -6.$
2. **Find the coordinates of the point:**
   - Substitute $t = -6$:
   - $x = (-6)^2 + (-6) = 36 - 6 = 30$
   - $y = (-6)^2 - 10(-6) + 5 = 36 + 60 + 5 = 101$
   - The point is **(30, 101)**.
3. **Find the equation of the tangent:**
   - The gradient is given as 2, and the point is (30, 101).
   - $y - 101 = 2(x - 30)$
   - $y - 101 = 2x - 60$
   - $y = 2x + 41$
4. **Show the tangent intersects only at the point of tangency:**
   - Substitute the parametric equations for $x$ and $y$ into the tangent equation:
   - $(t^2 - 10t + 5) = 2(t^2 + t) + 41$
   - $t^2 - 10t + 5 = 2t^2 + 2t + 41$
   - $0 = t^2 + 12t + 36$
   - Factorise the quadratic: $0 = (t + 6)^2$
   - This equation has a single, repeated root at $t = -6$. Since this is the only value of $t$ for which the curve and the tangent line meet, they intersect at only one point (the point of tangency).