UNIT 10.1 – PARAMETRIC EQUATIONS AND COORDINATE GEOMETRY

LEARNING OBJECTIVES: INTRODUCTION TO PARAMETRIC EQUATIONS

- To understand how curves can be defined parametrically;
- To be able to convert simple parametric equations into a Cartesian equation;
- To solve coordinate geometry problems involving curves defined parametrically.

INTRODUCTION

- Sometimes, it is useful to describe movement of x and y directions in terms of a third parameter (often t for time, or θ for angles). This is often the case in mechanics when considering the movement of an object or the distribution of a force.
- It can also be useful in pure mathematics, where to write the curve in Cartesian form (linking x and y directly) may be too complex or messy.

EXAMPLE 1

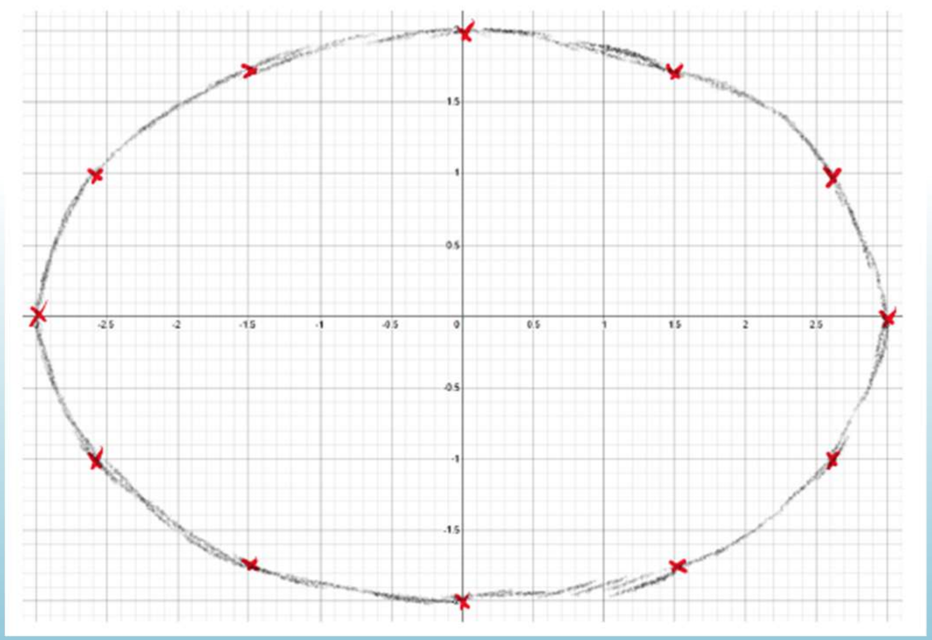
A curve is defined parametrically by the equations $x = 3\cos t, y = 2\sin t$,

where $0 \le t \le 2\pi$. Make sure you are in radians!!!

a) By completing the table of values below and plotting the graph, show that these equations represent an ellipse.

b) Using the identity $sin^2 t + cos^2 t \equiv 1$, find the Cartesian equation of the ellipse. e.g. the form y = f(x)

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$x = 3\cos t$	3	2.6	1.5	0	-1.5	-2.6	-3	-2.6	-1.5	0	1.5	2.6	3
$y = 2\sin t$	0	1	1.73	2	1.73	1	0	-1	-1.73	-2	-1.73	-1	0



b) Rearranging: $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$

Using $\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \Rightarrow 9y^2 + 4x^2 = 36$

Investigate this curve at:
https://www.desmos.com/calculator/c2wl8thyhu

EXAMPLE 2

- Where the parametric equations are non-trigonometric, we can usually find the Cartesian equation relatively simply by a combination of rearrangement and substitution.

A curve is defined by parametric equations $x = 3 - t$, $y = 2t^2$, for $-3 \le t \le 3$.
a) Find the coordinates of the point where $t = 1$.
b) Find a Cartesian equation of the curve, stating the domain and range.

a) When $t=1$, $x = 3-1 = 2$ $\quad y = 2(1)^2 = 2$

$\therefore (2,2)$

b) $x = 3 - t \Rightarrow t = 3 - x$

$\therefore y = 2t^2 \Rightarrow y = 2(3-x)^2 = 2(x^2 - 6x + 9) = 2x^2 - 12x + 18$

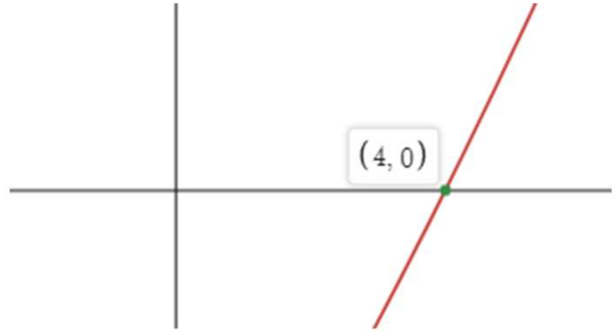
For domain, consider x for $-3 \le t \le 3$
$0 \le x \le 6$

For range, consider y for $-3 \le t \le 3$
$0 \le y \le 18$

Key Point: To find the domain and range of a parametric curve, we can just consider the possible values of the x (domain) and y (range) parameters - it arguably makes life easier!

EXAMPLE 3

The curve shown in the image is defined parametrically by equations $x = pt^2 - t, y = t^3 - 8,$ where *p* is a constant.



a) Given that the curve passes through *(4, 0)*, find the value of *p*.
b) Find the coordinates of the points where the curve intersects the *y*-axis.

Key Point: A coordinate is of the form (x, y) – if we know a coordinate we can substitute the x and/or y values into the respective parametric equation.

a) $(\overset{x}{4}, \overset{y}{0})$ $\quad x = pt^2 - t \quad y = t^3 - 8$

$0 = t^3 - 8 \Rightarrow t = 2$

$4 = p(2)^2 - 2$

$6 = 4p \Rightarrow p = 1.5$

Key Point: A curve intersects the y-axis when $x = 0$ – again, we can substitute this into our parametric equation for x to find the value(s) of t which this is true for!

b) Curve intersects y-axis when $x = 0$

$\therefore 0 = 1.5t^2 - t \quad \Rightarrow 3t^2 - 2t = 0$

$t(3t - 2) = 0$

$t = 0, \ t = \frac{2}{3}$

When $t = 0$, $y = -8$, when $t = \frac{2}{3}$, $y = -7\frac{19}{27}$

$\therefore (0, -7\frac{19}{27})$ and $(0, -8)$

EXAMPLE 4

The curve C is defined by parametric equations $x = 3t, y = t^2$. The line with equation $x + y + 2 = 0$ meets C at the points A and B. Find the coordinates of A and B.

Key Point: At points of intersection, both equations must be satisfied. This means we can substitute our parametric equations for x and y into the Cartesian equation to solve!

If solving points of intersection, the equation must be satisfied for x and y. Thus

$$x + y + 2 = 0$$

$$\Rightarrow (3t) + (t^2) + 2 = 0$$

(Substitute both parametric equations in)

$$t^2 + 3t + 2 = 0$$

$$(t+2)(t+1) = 0$$

$$t = -1, \quad t = -2$$

Solve to find the values of t for the two coordinates.

At $t = -1$, $x = 3t$ $y = t^2$ $\therefore (-3, 1)$

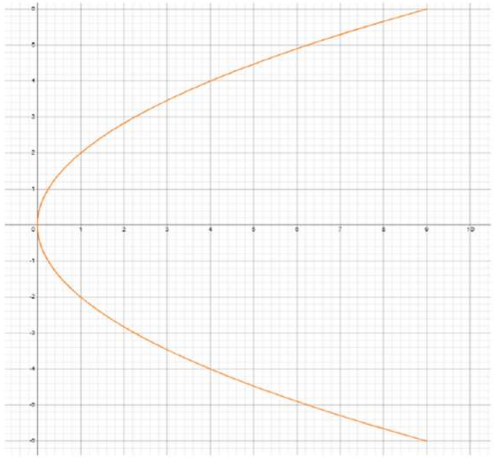
At $t = -2$, $x = 3(-2)$, $y = (-2)^2$ $\therefore (-6, 4)$

MathEYmatics

TEST YOUR UNDERSTANDING 1

Complete TYU 1 from your pack.

TEST YOUR UNDERSTANDING 1 – GEMINI AI ANSWERS



Question 1

a) Table of values:

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	-6	-4	-2	0	2	4	6

The plot is a parabola on its side, opening right.

b) Cartesian equation: $y^2 = 4x$. Domain: $0 \leq x \leq 9$. Range: $-6 \leq y \leq$

Question 2

a) (12, 33)
b) (3, -3) and (3, 5)

Question 3

a) $x^2 + \frac{y^2}{4} = 1$

b) $\frac{x^2}{9} + y = 1$

c) $y = -\frac{3}{2}x + \frac{1}{2}$

d) $y = 1 + \frac{1}{x^2}$

Question 4

A = (8, 0), B = (0, 8)

Question 5

(0, 0) and (0, -2)

Question 6 (Corrected)

(0, 2) and (-1, 0)

Question 7

(1/2, 3/2)

Question 8

(4, -6) and (1, 3)

Question 9 (Corrected)

a) Set $y = 0 \implies \cos\theta = 1 \implies \theta = 0$. Then $x = 2 + 3\sin(0) = 2$. Point is (2, 0).

b) (0, -1.35), (0, -1.92), (0, -0.65), (0, -0.08) (approx)

Question 10

$k > \frac{4}{3}$

Question 11

(0, 0) and (12, 36)

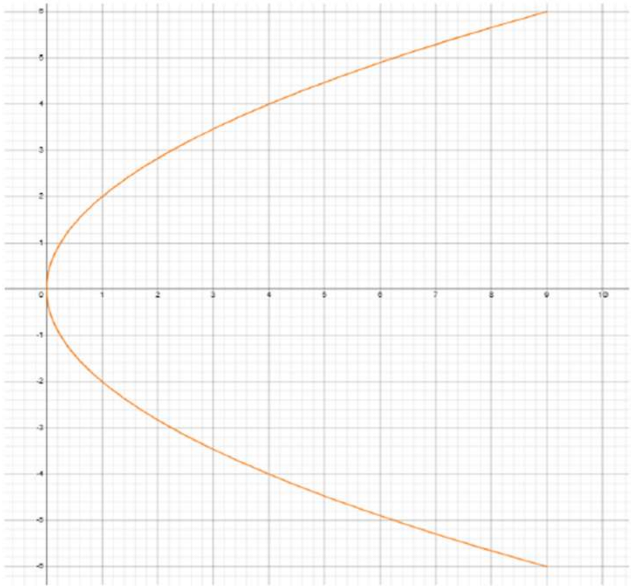
TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 1

A curve is defined parametrically by the equations $x = t^2, y = 2t$, for $-3 \leq t \leq 3$.

a) By creating a table of values for t, x, and y, plot the curve on graph paper.

t	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
x	9	4	1	0	1	4	9
y	-6	-4	-2	0	2	4	6



The plot of these points reveals a parabola opening to the right, with its vertex at the origin (0, 0).

b) Find a Cartesian equation for the curve in the form $y = f(x)$, stating the domain and range.

- From $y = 2t$, we can write $t = \frac{y}{2}$.
- Substitute this into the equation for x: $x = (\frac{y}{2})^2 = \frac{y^2}{4}$.
- This gives the Cartesian equation $y^2 = 4x$.
- **Domain:** The minimum value of x occurs at $t = 0$ ($x = 0$) and the maximum occurs at $t = \pm 3$ ($x = 9$). So, the domain is $0 \leq x \leq 9$.
- **Range:** The minimum value of y occurs at $t = -3$ ($y = -6$) and the maximum occurs at $t = 3$ ($y = 6$). So, the range is $-6 \leq y \leq 6$.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 2

A curve is defined by $x = 3t^2, y = 1 - 4t^3$.

a) Find the coordinates of the point on the curve where $t = -2$.

- Substitute $t = -2$ into the equations:
  - $x = 3(-2)^2 = 3(4) = 12$
  - $y = 1 - 4(-2)^3 = 1 - 4(-8) = 1 + 32 = 33$
- The coordinates are **(12, 33)**.

b) Find the coordinates of the points on the curve where $x = 3$.

- Set $x = 3$: $3t^2 = 3 \implies t^2 = 1 \implies t = 1$ or $t = -1$.
- For $t = 1$:
  - $y = 1 - 4(1)^3 = 1 - 4 = -3$. Point: **(3, -3)**.
- For $t = -1$:
  - $y = 1 - 4(-1)^3 = 1 - 4(-1) = 1 + 4 = 5$. Point: **(3, 5)**.

Question 3

Determine Cartesian equations for the following curves defined parametrically.

a) $x = \cos t, y = 2\sin t$

- From the equations, $\cos t = x$ and $\sin t = \frac{y}{2}$.
- Using the identity $\cos^2 t + \sin^2 t = 1$:
- $x^2 + (\frac{y}{2})^2 = 1 \implies x^2 + \frac{y^2}{4} = 1$.

b) $x = 3\cos t, y = \sin^2 t$

- From the equations, $\cos t = \frac{x}{3}$ and $\sin^2 t = y$.
- Using the identity $\cos^2 t + \sin^2 t = 1$:
- $(\frac{x}{3})^2 + y = 1 \implies \frac{x^2}{9} + y = 1$.

c) $x = 2t - 3, y = 5 - 3t$

- From the first equation, $x + 3 = 2t \implies t = \frac{x+3}{2}$.
- Substitute this into the second equation:
- $y = 5 - 3(\frac{x+3}{2}) = 5 - \frac{3x+9}{2} = \frac{10-(3x+9)}{2} = \frac{1-3x}{2}$.
- $y = -\frac{3}{2}x + \frac{1}{2}$.

d) $x = t, y = 1 + \frac{1}{t^2}$

- Since $x = t$, substitute x for t in the y-equation:
- $y = 1 + \frac{1}{x^2}$.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 4

The curve $x = 5 + t, y = 3 - t$ meets the x-axis at A and the y-axis at B. Find the coordinates of A and B.

- **Intersection with x-axis (A):** This occurs when $y = 0$.
  - $3 - t = 0 \implies t = 3$.
  - $x = 5 + 3 = 8$.
  - Coordinates of A are **(8, 0)**.
- **Intersection with y-axis (B):** This occurs when $x = 0$.
  - $5 + t = 0 \implies t = -5$.
  - $y = 3 - (-5) = 8$.
  - Coordinates of B are **(0, 8)**.

Question 5

Find the coordinates of the points where the curve $x = t^2 - 1, y = \frac{1}{t} - 1$ meets the y-axis.

- The curve meets the y-axis when $x = 0$.
- $t^2 - 1 = 0 \implies t^2 = 1 \implies t = 1$ or $t = -1$.
- For $t = 1$:
  - $y = \frac{1}{1} - 1 = 0$. Point: **(0, 0)**.
- For $t = -1$:
  - $y = \frac{1}{-1} - 1 = -1 - 1 = -2$. Point: **(0, -2)**.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Corrected Solution for Question 6

A curve is defined parametrically as $x = \frac{t-1}{t+1}, y = 2t^2, t \neq -1$. Find the coordinates of any points of intersection with the x- or y-axes.

- **Intersection with y-axis (where x = 0):**
  - Set the equation for x to 0:
  - $\frac{t-1}{t+1} = 0 \implies t - 1 = 0 \implies t = 1.$
  - Now substitute $t = 1$ into the equation for y:
  - $y = 2(1)^2 = 2.$
  - The point of intersection with the y-axis is **(0, 2)**.
- **Intersection with x-axis (where y = 0):**
  - Set the equation for y to 0:
  - $2t^2 = 0 \implies t = 0.$
  - Now substitute $t = 0$ into the equation for x:
  - $x = \frac{0-1}{0+1} = -1.$
  - The point of intersection with the x-axis is **(-1, 0)**.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 7

A line L_1 is defined by $x = 3t + 2, y = 1 - t$. The line L_2 has Cartesian equation $y = 2 - x$. Find the point of intersection of L_1 and L_2.

- Substitute the parametric equations of L_1 into the equation for L_2:
- $(1 - t) = 2 - (3t + 2)$
- $1 - t = 2 - 3t - 2$
- $1 - t = -3t$
- $1 = -2t \implies t = -\frac{1}{2}$.
- Now substitute this value of t back into the equations for L_1:
  - $x = 3(-\frac{1}{2}) + 2 = -\frac{3}{2} + 2 = \frac{1}{2}$
  - $y = 1 - (-\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$
- The point of intersection is **(1/2, 3/2)**.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 8

Find the coordinates of the points of intersection of the line $y = 6 - 3x$ and the curve with parametric equations $x = t^2, y = 3t$.

- Substitute the parametric equations into the line's equation:
- $3t = 6 - 3(t^2)$
- $3t = 6 - 3t^2$
- $3t^2 + 3t - 6 = 0$
- $t^2 + t - 2 = 0$
- Factorise the quadratic: $(t + 2)(t - 1) = 0$.
- This gives $t = -2$ or $t = 1$.
- For $t = -2$:
  - $x = (-2)^2 = 4$
  - $y = 3(-2) = -6$. Point: **(4, -6)**.
- For $t = 1$:
  - $x = (1)^2 = 1$
  - $y = 3(1) = 3$. Point: **(1, 3)**.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 9

The curve is defined by $x = 2 + 3\sin(2\theta), y = \cos\theta - 1$.

a) Show that the curve meets the x-axis at the point (2, 0).

- The curve meets the x-axis when $y = 0$.
- $\cos\theta - 1 = 0 \implies \cos\theta = 1$.
- For $0 \leq \theta < 2\pi$, this occurs when $\theta = 0$.
- Substitute $\theta = 0$ into the equation for x:
- $x = 2 + 3\sin(2 \times 0) = 2 + 3\sin(0) = 2 + 0 = 2$.
- Thus, the curve meets the x-axis at **(2, 0)**.

b) Find the coordinates of the points where the curve meets the y-axis.

- Set $x = 0 \implies 2 + 3\sin(2\theta) = 0 \implies \sin(2\theta) = -2/3$.
- Since $0 \leq \theta < 2\pi$, we must solve for 2θ in the interval $0 \leq 2\theta < 4\pi$.
- Let $\alpha = \arcsin(2/3) \approx 0.7297$. The values for 2θ are $\pi + \alpha, 2\pi - \alpha, 3\pi + \alpha, 4\pi - \alpha$.
- $2\theta \approx 3.8713, 5.5535, 10.1545, 11.8367$.
- $\theta \approx 1.9357, 2.7767, 5.0772, 5.9183$.
- Substituting these into $y = \cos\theta - 1$ gives:
- $y \approx -1.354, -1.922, -0.646, -0.078$.
- The approximate coordinates are (0, -1.35), (0, -1.92), (0, -0.65), and (0, -0.08).

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 10

Given that the line $y = 2x - k$ does not intersect the curve defined by $x = 1 - t, y = 3t^2 + 1$, find the range of possible values for k.

1. Substitute the parametric equations into the line's equation:
   - $3t^2 + 1 = 2(1 - t) - k$
   - $3t^2 + 1 = 2 - 2t - k$
   - $3t^2 + 2t + k - 1 = 0$
2. For the line and curve not to intersect, this quadratic equation in t must have no real roots. This means the discriminant $(b^2 - 4ac)$ must be less than 0.
   - Here, $a = 3, b = 2, c = k - 1$.
   - $b^2 - 4ac < 0$
   - $(2)^2 - 4(3)(k - 1) < 0$
   - $4 - 12(k - 1) < 0$
   - $4 - 12k + 12 < 0$
   - $16 - 12k < 0$
   - $16 < 12k$
   - $\frac{16}{12} < k \implies \frac{4}{3} < k.$
3. The range of possible values for k is $k > \frac{4}{3}$.

TEST YOUR UNDERSTANDING 1 – GEMINI AI WORKED SOLUTIONS

Question 11

Two curves are defined as $C_1 : x = 2t, y = t^2$ and $C_2 : x = t, y = 3t$. Find the coordinates of the points where C_1 and C_2 intersect.

- To find the intersection points, we set the x-coordinates and y-coordinates equal. It is best to use different parameters for each curve to avoid confusion, for instance t for C_1 and s for C_2.
  - $x_{C1} = 2t, y_{C1} = t^2$
  - $x_{C2} = s, y_{C2} = 3s$
- At intersection points, $x_{C1} = x_{C2}$ and $y_{C1} = y_{C2}$.
  1. $2t = s$
  2. $t^2 = 3s$
- Substitute equation (1) into equation (2):
  - $t^2 = 3(2t)$
  - $t^2 = 6t$
  - $t^2 - 6t = 0$
  - $t(t - 6) = 0$
  - This gives two possible values for t: $t = 0$ or $t = 6$.
- Find the coordinates for each value of t using the equations for C_1:
  - If $t = 0$: $x = 2(0) = 0, y = (0)^2 = 0$. Point: **(0, 0)**.
  - If $t = 6$: $x = 2(6) = 12, y = (6)^2 = 36$. Point: **(12, 36)**.