Some more generating function problems to think about

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1. Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m+n$. (2010 AIME I, #4)

2. Let $\{a_n\}$ be a sequence with $a_0 = 1$, and for all $n > 0$, $a_n = \dfrac{1}{2}\displaystyle\sum_{i=0}^{n-1} a_i$. Compute the greatest value of n for which $a_n < 2017$. (2017 ARML, Team #6)

3. Alex, Justin, and Pierce each randomly choose a positive integer from 1 to 100 inclusive. They add up their numbers and notice that they sum to 100. Given this information, compute the expected value of the product of their numbers. (2022 Texas Team Selection, Combinatorics #4, by Matthew Kroesche)

4. Let T_n be the number of tilings of a $3 \times 2n$ rectangle with identical 2×1 dominoes. Compute

$$\prod_{n=1}^{\infty} \left(1 + \frac{2}{T_n^2}\right).$$

(2024 Texas Team Selection, General #10, by Ellie Breslau)

5. A doubly-indexed sequence $a_{m,n}$, for m and n nonnegative integers, is defined as follows:

- $a_{m,0} = 0$ for all $m > 0$ and $a_{0,0} = 1$.
- $a_{m,1} = 0$ for all $m > 1$, $a_{1,1} = 1$, and $a_{0,1} = 0$.
- $a_{0,n} = a_{0,n-1} + a_{0,n-2}$ for all $n \geq 2$.
- $a_{m,n} = a_{m,n-1} + a_{m,n-2} + a_{m-1,n-1} - a_{m-1,n-2}$ for all $m > 0$, $n \geq 2$.

Then there exists a unique value of x so $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_{m,n}x^m}{3^{n-m}} = 1$. Find $\lfloor 1000x^2 \rfloor$. (2019 PUMaC Algebra A, #7, by Frank Lu)

6. Kelvin the Frog has a pair of standard fair 8-sided dice (each labelled from 1 to 8). Alex the sketchy Kat also has a pair of fair 8-sided dice, but whose faces are labelled differently (the integers on each Alex's dice need not be distinct). To Alex's dismay, when both Kelvin and Alex roll their dice, the probability that they get any given sum is equal! Suppose that Alex's two dice have a and b total dots on them, respectively. Assuming that $a \neq b$, find all possible values of $\min\{a, b\}$. (2016 HMMT February, Combinatorics #7, by Alex Katz)

7. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1, a_2, a_3, a_4) to some position (x_1, x_2, x_3, x_4) with integer coordinates satisfying

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 + (x_4 - a_4)^2 = 4 \qquad \text{and} \qquad |(x_1 + x_2 + x_3 + x_4) - (a_1 + a_2 + a_3 + a_4)| = 2.$$

In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk? (2019 HMMT February, Combinatorics #10, by Brice Huang)

8. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

(2018 Putnam, B6)

9. Consider the sequence defined by $a_1 = 2$, $a_2 = 3$, and

$$a_{2k+1} = 2 + 2a_k, \quad a_{2k+2} = 2 + a_k + a_{k+1},$$

for all integers $k \geq 1$. Determine all positive integers n such that

$$\frac{a_n}{n}$$

is an integer. (2025 India National Olympiad, #1, by Niranjan Balachandran, SS Krishnan, and Prithwijit De)

10. Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \ldots, 2p\}$ such that

- A has exactly p elements, and
- the sum of all elements of A is divisible by p.

(1995 IMO, #6)