Problem Solving with Generating Functions

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$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

- This series often doesn't converge for all x. Sometimes we have to worry about this, but often we don't have to worry too much about it.

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- Generating functions are a way of encoding the terms in a sequence as the coefficients of a power series.
- This allows us to perform different types of operations on the data of the sequence, and to manipulate it in interesting ways.
- This talk will be heavily driven by problems. We'll learn techniques for working with generating functions by doing problems where those techniques are applicable. Many of these problems will be from familiar past math contests.

Some warm-up generating functions

Find the ordinary generating function of each of the following sequences:

$$1, 1, 1, 1, 1, 1, 1, \ldots$$

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$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

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Find the ordinary generating function of each of the following sequences:

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$$1 + 2x + 4x^2 + 8x^3 + \cdots = \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}.$$

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$$1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

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$$1 + 4x + 6x^2 + 4x^3 + x^4 = (1 + x)^4.$$

The Fibonacci sequence

The Fibonacci sequence

- The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and for each $n \geq 2$, we have $F_n = F_{n-1} + F_{n-2}$. That is, each term is the sum of the previous two.

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- The first few terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

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- The first few terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$
- How might we find the generating function of the Fibonacci sequence?

The Fibonacci sequence

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$$f(x) = x + \sum_{n=2}^{\infty} F_n x^n = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n.$$

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We split this up as

$$f(x) = x + \sum_{n=1}^{\infty} F_n x^{n+1} + \sum_{n=0}^{\infty} F_n x^{n+2} = x + xf(x) + x^2 f(x).$$

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Solving for $f(x)$ gives

$$f(x) = \frac{x}{1 - x - x^2}.$$

The Fibonacci sequence

How can we use the generating function

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

to find an explicit (non-recursive) formula for the terms of the Fibonacci sequence?

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Let's factor the denominator. We write

$$1 - x - x^2 = (1 - \phi x)(1 + \phi^{-1} x)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

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Applying a partial fraction decomposition, we have

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{(1 - \phi x)(1 + \phi^{-1} x)} = \frac{1}{\phi + \phi^{-1}} \left(\frac{1}{1 - \phi x} - \frac{1}{1 + \phi^{-1} x} \right).$$

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These are both geometric series! We have $\phi + \phi^{-1} = \sqrt{5}$, and

$$\frac{1}{1-\phi x} = \sum_{n=0}^{\infty} \phi^n x^n$$

$$\frac{1}{1+\phi^{-1}x} = \sum_{n=0}^{\infty} (-1)^n \phi^{-n} x^n$$

So

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left[\phi^n - (-1)^n \phi^{-n}\right] x^n,$$

and we can match coefficients to get an explicit formula!

The Fibonacci sequence

Binet's Formula

The *n*-th Fibonacci number is given by the explicit formula

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

The big ideas

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- For a fun exercise, try using this generating function to prove that

$$F_1 + F_2 + \cdots + F_{n-2} = F_n - 1.$$

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Hint: The generating function of the left-hand side is $\frac{x^2 f(x)}{1-x}$ *(why?) and the generating function of the right-hand side is* $f(x) - \frac{x}{1-x}$.

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Okay, let's jump right in to some hard problems...

2018 AIME I, #12

For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

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First ingredient: we need a generating function that "generates" every one of the 2^{18} possible subsets of U. We can do this by multiplying a bunch of generating functions together. The magical function we want is

$$f(x) = \prod_{n=1}^{18}(1 + x^n) = (1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^{18}).$$

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Do you see why every one of the 2^{18} choices when this is multiplied out corresponds to a different subset of U?

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Do you see why every one of the 2^{18} choices when this is multiplied out corresponds to a different subset of U?

What does the exponent of x in a particular term signify? What does the coefficient of, say, x^{12} count?

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Second ingredient: every one of these 2^{18} subsets is equally likely to be chosen. By dividing this whole function through by 2^{18}, the coefficient of x^n goes from counting the *number of ways* to get a sum of n, to counting the *probability* that we get a sum of n. So consider

$$g(x) = \frac{1}{2^{18}} \prod_{n=1}^{18} (1 + x^n).$$

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$$g(x) = \frac{1}{2^{18}} \prod_{n=1}^{18} (1 + x^n).$$

We want the probability that $s(T)$ is a multiple of 3. This is just the sum of all the coefficients of terms of the form x^{3k} in $g(x)$. How do we get our hands on those coefficients, and only those coefficients?

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Third ingredient: a **roots of unity filter**. Let $\omega = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$ be a cube root of unity. So $\omega \neq 1$, but $\omega^3 = 1$. Roots of unity have nice properties because we have expansions like

$$(x-\omega)(x-\omega^2)(x-\omega^3)\cdots(x-\omega^n) = x^n - 1$$

if $\omega = e^{2\pi i/n}$. In this problem, we want cube roots of unity, because we want to make precisely the values n that are not multiples of 3 disappear.

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We know

$$g(x) = \sum_{n=0}^{\infty} p_n x^n,$$

where p_n is the probability that $s(T)$ equals n. The magic trick is to consider

$$\frac{g(x) + g(\omega x) + g(\omega^2 x)}{3} = \frac{1}{3} \sum_{n=0}^{\infty} p_n (1 + \omega^n + \omega^{2n}) x^n.$$

If n is a multiple of 3, then $1 + \omega^n + \omega^{2n} = 1 + 1 + 1 = 3$. Otherwise, $1 + \omega^n + \omega^{2n} = 1 + \omega + \omega^2 = 0$. (This idea is called **orthogonality**.)

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Then

$$\frac{g(x) + g(\omega x) + g(\omega^2 x)}{3} = \sum_{k=0}^{\infty} p_{3k} x^k,$$

and so the probability that $s(T)$ is divisible by 3 equals

$$\sum_{k=0}^{\infty} p_{3k} = \frac{g(1) + g(\omega) + g(\omega^2)}{3}.$$

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Then

$$\frac{g(x)+g(\omega x)+g(\omega^2 x)}{3}=\sum_{k=0}^{\infty} p_{3k}x^k,$$

and so the probability that $s(T)$ is divisible by 3 equals

$$\sum_{k=0}^{\infty} p_{3k}=\frac{g(1)+g(\omega)+g(\omega^2)}{3}.$$

Now

$$g(1)=\frac{1}{2^{18}}\prod_{n=1}^{18}(1+1)=\frac{2^{18}}{2^{18}}=1.$$

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Now

$$g(1) = \frac{1}{2^{18}} \prod_{n=1}^{18} (1+1) = \frac{2^{18}}{2^{18}} = 1.$$

$$g(\omega) = \frac{1}{2^{18}} \prod_{n=1}^{18} (1+\omega^n) = \frac{\left[(1+1)(1+\omega)(1+\omega^2)\right]^6}{2^{18}} = \frac{2^6}{2^{18}} = \frac{1}{4096}.$$

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$$g(\omega) = \frac{1}{2^{18}} \prod_{n=1}^{18} (1+\omega^n) = \frac{\left[(1+1)(1+\omega)(1+\omega^2)\right]^6}{2^{18}} = \frac{2^6}{2^{18}} = \frac{1}{4096}.$$

And $g(\omega^2) = \frac{1}{4096}$ as well.

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So the desired probability is

$$\frac{1+\frac{1}{4096}+\frac{1}{4096}}{3}=\frac{1+\frac{1}{2048}}{3}=\frac{2049}{3\cdot 2048}=\frac{683}{2048}$$

in lowest terms, so $m=\boxed{683}$.

The big ideas

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- (We've already seen this once – multiplying a generating function by $\frac{1}{1-x}$ gives you the partial sums of its terms!)

The big ideas

- Multiplying generating functions together will cause their terms to combinatorially distribute out. Often this will give you what's called a **Cauchy convolution** of the individual generating functions.
- (We've already seen this once – multiplying a generating function by $\frac{1}{1-x}$ gives you the partial sums of its terms!)
- A **roots of unity filter** is an excellent way to detect only the terms of a generating function that belong to a particular arithmetic progression. (In this example, we used it to detect multiples of 3.)

2024 HMMT February Guts Round #21

Kelvin the frog currently sits at $(0,0)$ in the coordinate plane. If Kelvin is at (x, y), either he can walk to any of $(x, y+1)$, $(x+1, y)$, or $(x+1, y+1)$, or he can jump to any of $(x, y+2)$, $(x+2, y)$ or $(x+1, y+1)$. Walking and jumping from (x, y) to $(x+1, y+1)$ are considered distinct actions. Compute the number of ways Kelvin can reach $(6, 8)$.

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Let's say $f(x, y)$ is the generating function of the number of ways for Kelvin to reach a particular point. That is,

$$f(x, y) = \sum_{m,n \geq 0} a_{m,n} x^m y^n$$

where $a_{m,n}$ is the number of ways to hop to the point (m, n).

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To build $f(x,y)$, first we're going to see what happens when Kelvin takes exactly one hop. The generating function for the number of ways he could reach a point after exactly one hop is

$$p(x,y) = x + y + xy + x^2 + y^2 + xy = (x+y)(1+x+y).$$

(Do you see why?)

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Now, every time Kelvin hops somewhere, it's like we're multiplying by $p(x, y)$ again. For example, the number of ways for him to reach a point in two hops is given by $p(x, y) \cdot p(x, y)$.

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Now, every time Kelvin hops somewhere, it's like we're multiplying by $p(x, y)$ again. For example, the number of ways for him to reach a point in two hops is given by $p(x, y) \cdot p(x, y)$. So in fact,

$$f(x, y) = \sum_{k=0}^{\infty} [p(x, y)]^k = \frac{1}{1 - p(x, y)} = \frac{1}{1 - x - y - x^2 - 2xy - y^2}.$$

2024 HMMT February Guts Round #21

Do you see the Fibonacci generating function hiding here?

$$f(x, y) = \frac{1}{1 - (x + y) - (x + y)^2} = \sum_{k=0}^{\infty} F_{k+1}(x + y)^k$$

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$$f(x, y) = \frac{1}{1 - (x + y) - (x + y)^2} = \sum_{k=0}^{\infty} F_{k+1}(x + y)^k$$

and so in particular, if we want the coefficient of $x^m y^n$, we have to go to the $k = m + n$ term in this sum. Then by the binomial theorem, the number of ways to hop to (m, n) is

$$F_{m+n+1}\binom{m+n}{m}.$$

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So the answer for $(6,8)$ is

$$F_{15}\binom{14}{6} = 610 \cdot 3003 = \boxed{1831830}.$$

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- Generating functions can have more than one variable!
- Walking problems (in any number of dimensions) where we repeatedly move from one lattice point to another, often lend themselves well to generating function approaches.
- If something is changing in the same way repeatedly, it might correspond to multiplying a generating function by itself repeatedly.

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For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

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First ingredient: let's define the indicator function of the set S by

$$f(x) = \sum_{n \in S} x^n.$$

Then we want to get our hands on some kind of generating function of $r_S(n)$.

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In fact, we have

$$\sum_{n=0}^{\infty} r_S(n)x^n = \sum_{n=0}^{\infty} \sum_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2 \\ s_1 + s_2 = n}} x^n = \sum_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} x^{s_1+s_2} = \sum_{s_1, s_2 \in S} x^{s_1+s_2} - \sum_{n \in S} x^{2n},$$

and this is just $[f(x)]^2 - f(x^2)$.

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Now since A, B are a partitioning of the nonnegative integers, let's say f and g are the indicator functions associated to them. Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

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$$f(x) + g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

and we would like A, B to be such that $r_A(n) = r_B(n)$ for all n. This means

$$[f(x)]^2 - f(x^2) = [g(x)]^2 - g(x^2).$$

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So we're trying to solve the system

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One sensible thing to do might be to let $h(x) = f(x) - g(x)$. Then

$$[f(x)]^2 - [g(x)]^2 = f(x^2) - g(x^2)$$

$$\frac{h(x)}{1-x} = h(x^2)$$

and so

$$h(x) = (1-x)h(x^2) = (1-x)(1-x^2)h(x^4) = (1-x)(1-x^2)(1-x^4)h(x^8) = \cdots$$

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Then we see that

$$h(x) = \prod_{n=0}^{\infty} \left(1 - x^{2^n}\right)$$

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Since we are multiplying a $(-1)^n$ together for every 1 in the binary representation of n, we actually just have

$$h(x) = \sum_{n=0}^{\infty} (-1)^{c(n)} x^n,$$

where $c(n)$ counts the number of 1's in the binary representation of n.

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Then

$$f(x) = \frac{\frac{1}{1-x} + h(x)}{2} = \sum_{\substack{n \geq 0 \\ c(n) \text{ even}}} x^n$$

$$g(x) = \frac{\frac{1}{1-x} - h(x)}{2} = \sum_{\substack{n \geq 0 \\ c(n) \text{ odd}}} x^n$$

and so partitioning the integers based on whether they have an even or odd number of 1's in their binary representation will accomplish $r_A(n) = r_B(n)$ for all n.

The big ideas

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- Anytime we see something like $s_1 + s_2 = n$, we might want to think about letting s_1, s_2 be exponents in a generating function, since multiplying them together would give a sum over s_1, s_2 such that $s_1 + s_2 = n$.

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- Anytime we see something like $s_1 + s_2 = n$, we might want to think about letting s_1, s_2 be exponents in a generating function, since multiplying them together would give a sum over s_1, s_2 such that $s_1 + s_2 = n$.
- Generating functions often give rise to functional equations which we have to solve.

Evil Team Selection Problem

Alexei is playing a game in which he throws darts at a target. When throwing a regular, unweighted dart, Alexei has a 50% chance of hitting the target on each throw. However, the game is rigged, and so each time after he hits the target, for his next throw the host gives him a weighted dart that will hit the target with probability only 25%. (If he misses the target, the host will give him a regular dart for his next throw, and his first throw is with a regular dart.) If Alexei throws a total of ten darts, compute the expected number of times he hits the target *as a decimal to the nearest hundredth*.

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So we are interested in the value of A_{10}.

Evil Team Selection Problem

If Alexei throws a fair dart, there's a $\frac{1}{2}$ chance he hits the target (so his next throw is with an unfair dart) and a $\frac{1}{2}$ chance he misses. So

$$A_n = \frac{1}{2}\left(1 + B_{n-1}\right) + \frac{1}{2}A_{n-1}.$$

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If Alexei throws an unfair dart, there's a $\frac{1}{4}$ chance he hits the target (so his next throw is with an unfair dart) and a $\frac{3}{4}$ chance he misses. So

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We also have $A_0 = B_0 = 0$ – if he throws zero darts he's not going to hit the target at all.

Evil Team Selection Problem

Let's define the generating functions of A_n and B_n: let

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

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Then our recursions from the previous slide transform into

$$f(x) = \frac{1}{2}\left(\frac{x}{1-x} + xg(x)\right) + \frac{1}{2}xf(x)$$

$$g(x) = \frac{1}{4}\left(\frac{x}{1-x} + xg(x)\right) + \frac{3}{4}xf(x)$$

Evil Team Selection Problem

This is a system of two linear equations in two unknowns $f(x)$ and $g(x)$! We clear denominators to write it as

$$(1-x)(2-x)f(x) - x(1-x)g(x) = x$$

$$(1-x)(4-x)g(x) - 3x(1-x)f(x) = x$$

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We're really only interested in $f(x)$, so we eliminate $g(x)$ and write

$$g(x) = \frac{(1-x)(2-x)f(x) - x}{x(1-x)} = \frac{3x(1-x)f(x) + x}{(1-x)(4-x)}$$

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$$(1-x)(2-x)(4-x)f(x) - x(4-x) = 3x^2(1-x)f(x) + x^2$$

Evil Team Selection Problem

Solving for $f(x)$ gives

$$f(x) = \frac{x^2 + x(4-x)}{(1-x)(2-x)(4-x) - 3x^2(1-x)} = \frac{4x}{(1-x)(8-6x-2x^2)} = \frac{2x}{(1-x)^2(4+x)}.$$

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Then we write this as partial fractions:

$$f(x) = \frac{2}{5(1-x)^2} - \frac{8}{25(1-x)} - \frac{8}{25(4+x)}.$$

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Expanding out into geometric series gives

$$f(x) = \sum_{n=0}^{\infty} \left[\frac{2}{5}(n+1) - \frac{8}{25} - \frac{2}{25}\left(-\frac{1}{4}\right)^n \right] x^n.$$

Evil Team Selection Problem

So

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$$A_n = \frac{2}{5}(n+1) - \frac{8}{25} - \frac{2}{25}\left(-\frac{1}{4}\right)^n,$$

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$$A_n = \frac{2}{5}(n+1) - \frac{8}{25} - \frac{2}{25}\left(-\frac{1}{4}\right)^n,$$

and

$$A_{10} = \frac{102}{25} - \frac{2}{25 \cdot 4^{10}}.$$

We want A_{10} as a decimal to the nearest hundredth, so the answer is $\boxed{4.08}$.

The big ideas

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The big ideas

- Recursions can be transformed into functional equations of the generating functions.
- Multiplying a generating function by a power of x shifts all its coefficients over. This goes hand in hand with recursive sequences.
- Sometimes if you're playing with more than one sequence, it makes sense to work with more than one generating function.
- All the usual algebra tricks (solving for a function and partial fraction decomposition) that worked so well in the Fibonacci problem, can be used in a lot of other problems too!

Some other big ideas we probably don't have much time for...

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and **Dirichlet generating functions** which look like

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Exponential generating functions tend to be useful if a_n is counting permutations of some sort, and Dirichlet generating functions are useful if a_n is something number-theoretic (perhaps a divisor function).

Some other big ideas we probably don't have much time for...

Sometimes, calculus comes into the equation too! If

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is an ordinary generating function, its derivative is

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

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Multiplying the terms in a sequence by their index thus corresponds to differentiating... and similarly, dividing by the index corresponds to integrating.
When this comes up, often the generating function will give rise to a *differential equation* that needs to be solved.

Thank you!