# INDEX

# Mind Map-1

# RELATIONS & FUNCTIONS

## 1 Relation

A relation from a non-empty set A to itself is a subset of cartesian product $A \times A$.

Relation from a set A to set B is a subset of cartesian product $A \times B$.

## 2 Types of Relations

**Empty Relation** is the relation R in a set A, in which no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

**Universal Relation** is the relation R in a set A, in which each element of A is related to every element of A, i.e., $R = A \times A$.

Both the empty relation and the universal relation are some times called trivial relations.

**Reflexive Relation :** $(a, a) \in R$, for every $a \in A$.

**Symmetric Relation :** $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

**Transitive Relation :** $(a_1, a_2) \in R$ & $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

**Equivalence Relation :** A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric & transitive.

Equivalence class $\{a\}$ containing $a \in A$ for an equivalence relation R in A is a subset of A containing all elements b related to a.

## 3 Function

For any two non-empty sets X & Y, a function f is a rule or mapping which associates each element of set X to a unique element in set Y.

## 4 Types of Functions

**One-One Function :** A function $f : X \to Y$ is one-one (or injective) if the images of distinct elements of X under f are distinct, i.e.,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in X.$$

Otherwise, f is called many-one.

**Onto Function :** A function is onto (or surjective) if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$

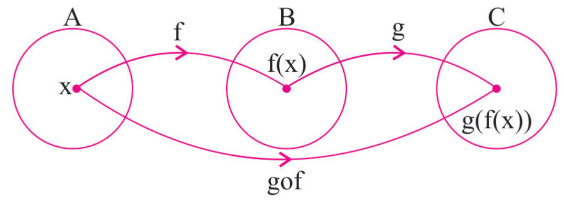
**Into Function :** A function $f : X \to Y$ is into if there exists atleast one element in Y which has no pre-image in A.

**One-One & Onto Function :** A function $f : X \to Y$ is said to be one-one & onto (or bijective) if f is both one-one & onto.

## 5 Composition of Functions

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the composition of f & g, denoted by gof, is defined as the function $gof : A \to C$ given by

$gof(x) = g(f(x)), \forall x \in A$




**Theorem :** If $f : X \to Y$, $g : Y \to Z$ & $h : Z \to S$ are functions, then

$$ho(gof) = (hog)of$$

If gof is one-one $\Rightarrow$ f is one-one

If gof is onto $\Rightarrow$ g is onto.

If $f : X \to Y$ is a function such that there exists a function $g : Y \to X$ such that

$gof = I_x$ & $fog = I_y$,

then f must be one-one and onto.

## 6 Invertible Function

A function $f : X \to Y$ is defined to be invertible, if there exists a function $g : Y \to X$ such that $gof = I_x$ & $fog = I_y$.

The function g is called the inverse of f & is denoted by $f^{-1}$.

**Note :** A function $f : X \to Y$ is invertible if & only if f is one-one & onto.

**Theorem :** Let $f : X \to Y$ & $g : Y \to Z$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

## 7 Binary Operations

A binary operation * on a set A is a function $* : A \times A \to A$. We denote $*(a, b)$ by $a * b$.

Addition, multiplication, subtraction & division are examples of binary operation, as 'binary' means 'two'.

**Types of Binary Operations**

- A binary operation * on set A is called **commutative** if, $a * b = b * a$ for every $a, b \in A$.
- A binary operation $* : A \times A \to A$ is **associative** if, $(a * b) * c = a * (b * c), \forall a, b, c \in A$.
- An element $e \in A$, if it exists, is called an **identity** element for binary operation $* : A \times A \to A$ if, $a * e = a = e * a, \forall a \in A$.
- An element $a \in A$ is said to be invertible with respect to the operation $* : A \times A \to A$ if there exists an element b in A such that $a * b = e = b * a$.
  Then, b is called the inverse of a & is denoted by $a^{-1}$.

# Mind Map-2

## INVERSE TRIGONOMETRIC FUNCTIONS

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| Inverse Function | Domain | Principal Value Branch |
|---|---|---|
| $y=\sin^{-1}x$ | $[-1, 1]$ | $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ |
| $y=\cos^{-1}x$ | $[-1, 1]$ | $[0, \pi]$ |
| $y=\text{cosec}^{-1}x$ | $R-(-1, 1)$ | $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ |
| $y=\sec^{-1}x$ | $R-(-1, 1)$ | $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ |
| $y=\tan^{-1}x$ | $R$ | $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ |
| $y=\cot^{-1}x$ | $R$ | $[0, \pi]$ |

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### Properties Of Inverse Trigonometric Functions

**Property-1**

(i) $\sin^{-1}(\sin\theta)=\theta$, if $\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$

(ii) $\cos^{-1}(\cos\theta)=\theta$, if $0\le\theta\le\pi$

(iii) $\tan^{-1}(\tan\theta)=\theta$, if $\frac{-\pi}{2}<\theta<\frac{\pi}{2}$

(iv) $\cot^{-1}(\cot\theta)=\theta$, if $0<\theta<\pi$

(v) $\sec^{-1}(\sec\theta)=\theta$, if $0\le\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$

(vi) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta$, if $-\frac{\pi}{2}\le\theta<0$ or $0<\theta\le\frac{\pi}{2}$

**Property-2**

(i) $\sin(\sin^{-1}x)=x$, if $-1\le x\le 1$

(ii) $\cos(\cos^{-1}x)=x$, if $-1\le x\le 1$

(iii) $\tan(\tan^{-1}x)=x$, if $-\infty<x<\infty$

(iv) $\cot(\cot^{-1}x)=x$, if $-\infty<x<\infty$

(v) $\sec(\sec^{-1}x)=x$, if $-\infty<x\le-1$ or $1\le x<\infty$

(vi) $\text{cosec}(\text{cosec}^{-1}x)=x$, if $-\infty<x\le-1$ or $1\le x<\infty$

**Property-3**

(i) $\sin^{-1}(-x)=-\sin^{-1}x$, if $-1\le x\le 1$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x$, if $-1\le x\le 1$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x$, if $-\infty<x<\infty$

(iv) $\cot^{-1}(-x)=\pi-\cot^{-1}x$, if $-\infty<x<\infty$

(v) $\sec^{-1}(-x)=\pi-\sec^{-1}x$, if $-\infty<x\le-1$ or $1\le x<\infty$

(vi) $\text{cosec}^{-1}(-x)=-\text{cosec}^{-1}x$, if $-\infty<x\le-1$ or $1\le x<\infty$

**Property-4**

(i) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, $x\in[-1,1]$

(ii) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$, $x\in R$

(iii) $\sec^{-1}x+\text{cosec}^{-1}x=\frac{\pi}{2}$, $x\in(-\infty,-1]\cup[1,\infty)$

**Property-5**

(i) $\sin^{-1}x=\text{cosec}^{-1}\left(\frac{1}{x}\right)$, $-1\le x\le 1$

(ii) $\text{cosec}^{-1}x=\sin^{-1}\left(\frac{1}{x}\right)$, $x\in R-(-1,1)$

(iii) $\cos^{-1}x=\sec^{-1}\left(\frac{1}{x}\right)$, $-1\le x\le 1$

(iv) $\sec^{-1}x=\cos^{-1}\left(\frac{1}{x}\right)$, $x\in R-(-1,1)$

(v) $\tan^{-1}x=\cot^{-1}\left(\frac{1}{x}\right)$, $x\in R$

(vi) $\cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)$, $x\in R$

**Property-6**

(i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy<1$

(ii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy>-1$

(iii) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z$

$=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$,

if $x>0, y>0, z>0$

and $(xy+yz+zx)<1$

**Property-7**

(i) $\sin^{-1}x+\sin^{-1}y$

$=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$,

if $-1\le x,y\le 1$

and $x^2+y^2\le 1$

or if $xy<0$ and $x^2+y^2>1$

(ii) $\sin^{-1}x-\sin^{-1}y$

$=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$,

if $-1\le x,y\le 1$

and $x^2+y^2\le 1$

or if $xy>0$

and $x^2+y^2>1$

**Property-8**

(i) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$,

if $-1\le x, y\le 1$ and $x+y\ge 0$

(ii) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}$,

if $-1\le x, y\le 1$ and $x\le y$

**Property-9**

(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, if $\frac{-1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

(ii) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$, if $\frac{-1}{2}\le x\le\frac{1}{2}$

**Property-10**

(i) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$, if $0\le x\le 1$

(ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$, if $\frac{1}{2}\le x\le 1$

**Property-11**

(i) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, if $-1<x\le 1$

(ii) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, if $\frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

**Property-12**

(i) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, if $-1\le x\le 1$

(ii) $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, if $0\le x<\infty$

**Property-13**

(i) $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\text{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$=\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\left(\frac{1}{x}\right)=\text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\frac{1}{x}\right)$

$=\sec^{-1}\sqrt{1+x^2}=\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

# Mind Map-3

# MATRICES

## Order of a Matrix

A matrix having m rows and n columns is called a matrix of order m×n or simply m×n matrix.
or $A=[a_{ij}]_{m\times n}, 1\le i\le m, 1\le j\le n$ i, $j\in N$

$a_{ij}$ is an element lying in the $i^{th}$ row & $j^{th}$ column.
The number of elements in m×n matrix will be mn.

## Matrix

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements of the matrix.

For example $\begin{bmatrix}1 & 2 & 5\\ 3 & 4 & 6\end{bmatrix}$ is a matrix.

The horizontal lines of elements in the above matrix are said to constitute, **rows** of the matrix & vertical lines of elements are said to constitute **columns** of the matrix. Thus above matrix has 2 rows and 3 columns.

## Multiplication of Matrices

If A & B are any two matrices, then their product AB will be defined only when the number of columns in A is equal to the number of rows in B. If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$, then their product $AB=C=[c_{ij}]$, is a matrix of order m×p, where $(ij)^{th}$ element of $AB=C_{ij}=\sum_{r=1}^{n}a_{ir}b_{rj}$

## Properties of Matrix Multiplication

**(I) Associative Law for Multiplication :** If A, B & C are three matrices of order m×n, n×p & p×q respectively, then (AB) C = A(BC)

**(ii) Distributive Law :** For three matrices A, B & C
(a) A(B + C) = AB + AC
(b) (A + B) C = AC + BC, whenever both sides of equality are defined.

**(iii) Matrix Multiplication** is not commutative in general, i.e., AB ≠ BA (in general).

**(iv) Existence of Multiplicative Identity :** For every square matrix A, there exists an identity matrix I of same order such that IA = AI = A.

## Transpose of a Matrix

The matrix obtained from a given matrix A by changing its rows into its corresponding columns or columns into its corresponding rows is called transpose of matrix A & it is denoted by $A^T$ or A'. If the order of A is m × n, then order of $A^T$ is n×m.
In other words if $A=[a_{ij}]_{m\times n}$ then $A^T=[a_{ji}]_{n\times m}$

## Types of Matrix

(i) **Column Matrix :** A matrix is said to be a column matrix if it has only one column, i.e., $A=[a_{ij}]_{m\times 1}$ is a column matrix of order m×1.

(ii) **Row Matrix :** Row matrix has only one row, i.e., $B=[b_{ij}]_{1\times n}$ is a row matrix of order 1×n.

(iii) **Square Matrix :** Square matrix has equal number of rows and columns, i.e., $A=[a_{ij}]_{m\times m}$ is a square matrix of order m.

(iv) **Diagonal Matrix :** A square matrix is said to be diagonal matrix if all of its non-diagonal elements are zero, i.e., $B=[b_{ij}]_{m\times n}$ is said to be a diagonal matrix if $b_{ij}=0$, where i ≠ j.

(v) **Scalar Matrix :** It is a diagonal matrix with all its diagonal elements equal, i.e., $B=[b_{ij}]_{m\times n}$ is a scalar matrix if
$b_{ij}=0$, where i ≠ j
$b_{ij}=k$, when i = j & k = constant.

(vi) **Identity Matrix :** It is a diagonal matrix having all its diagonal elements equal to 1, i.e., $A=[a_{ij}]_{m\times n}$ is an identity matrix if

$$a_{ij}=\begin{cases}1, \text{if } i=j\\ 0, \text{if } i\ne j\end{cases}$$

we denoted identity matrix by $I_n$ when order is n.

(vii) **Zero Matrix :** A matrix is said to be zero or null matrix if all its elements are zero. It is denoted by O.

## Properties of Transpose of the Matrices

For any matrices A & B of suitable orders, we have:
(i) $(A^T)^T=A$
(ii) $(kA)^T=k(A)^T$ (where k is constant)
(iii) $(A\pm B)^T=A^T\pm B^T$
(iv) $(AB)^T=B^TA^T$

## Symmetric & Skew Symmetric Matrices

**Symmetric Matrix**
A square matrix $A=[a_{ij}]$ is called a symmetric matrix, if $a_{ij}=a_{ji}$ for all i, j or $A^T=A$

**Skew Symmetric Matrix**
A square matrix $A=[a_{ij}]$ is called a skew-symmetric matrix, if $a_{ij}=-a_{ji}$ for all i, j or $A^T=-A$.

**Properties of Symmetric & Skew Symmetric Matrices**
(I) For any square matrix A with real number entries, $(A+A^T)$ is a symmetric matrix & $(A-A^T)$ is a skew symmetric matrix.
(ii) Any square matrix A can be expressed as the sum of a symmetric & a skew symmetric matrix as

$$A=\left[\frac{1}{2}(A+A^T)\right]+\left[\frac{1}{2}(A-A^T)\right]$$

## Equality of Matrices

Two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ are said to be equal if
(i) they are of the same order
(ii) each element of A is equal to the corresponding element of B, i.e., $a_{ij}=b_{ij}$ for all i & j

## Multiplication of a Matrix by a Scalar

Let $A=[a_{ij}]_{m\times n}$ be a matrix & k be a number.
Then, $kA=Ak=[ka_{ij}]_{m\times n}$

**Properties**
(I) k(A + B) = kA + kB (ii) (k + t) A = kA + tA.

## Addition of Matrices

Sum of the two matrices is a matrix obtained by adding the corresponding elements of the given matrices, i.e., $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of same order m×n. Then sum of two matrics A & B is defined as $C=[c_{ij}]$, where $c_{ij}=a_{ij}+b_{ij}$ for all i & j.

**Difference of matrices :** The difference A – B is defined as $D=[d_{ij}]$, where $d_{ij}=a_{ij}-b_{ij}$ for all i & j. In order words D = A – B = A + (–B), that is the sum of matrices A & (–B).

**Properties of matrix Addition**
(i) **Commulative Law:** A + B = B + A
(ii) **Associative Law:** (A + B) + C = A + (B + C)
(iii) **Existance of Additive Identity:** Let $A=[a_{ij}]_{m\times n}$ & O = zero matrix of order m×n, then A + O = O + A = A. Here O is the additive identity for matrix addition.
(iv) **Existance of Additive Inverse**
Let $A=[a_{ij}]_{m\times n}$ be any matrix then we have another matrix as $-A=[-a_{ij}]_{m\times n}$ such that A + (–A) = (–A) + A = O. Here –A is the additive inverse of A or negative of A.

## Invertible Matrix and Inverse Matrix

If A is a square matrix and there exists another square matrix B of the same order such that AB = BA = I, then B is called the inverse matrix of A & it is denoted by $A^{-1}$.
In that case A is said to be invertible matrix.

**Properties of Invertible Matrices**
(i) Uniqueness of Inverse : Inverse of a square matrix, if it exists, is unique.
(ii) $(AB)^{-1}=B^{-1}A^{-1}$

## Inverse of a Matrix by Elementary Operations

If A is a matrix such that $A^{-1}$ exists, then to find $A^{-1}$ using elementary row operations, write A = IA & apply a sequence of row operations on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find $A^{-1}$ using column operations, we write A = AI & apply a sequence of column operations on A = AI till we get, I = AB.

## Elementary Operation (Transformation of a Matrix)

There are six operations on a matrix, three of which are due to rows & three due to columns, called elementary operations or Transformations.
(i) The interchange of any two rows or two columns symbolically, interchange of $i^{th}$ & $j^{th}$ rows is denoted by $R_i\leftrightarrow R_j$ & same will be for columns, i.e., $C_i\leftrightarrow C_j$.
(ii) The multiplication of the elements of any row or column by a non zero number. For rows it is denoted as $R_i\leftrightarrow kR_i$, $k\ne 0$ & for columns: $C_i\leftrightarrow kC_i$.
(iii) The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number. Symbolically, the addition to the elements of $i^{th}$ row, the corresponding elements of $j^{th}$ row multiplied by k is denoted as: $R_i\leftrightarrow R_i+kR_j (k\ne 0)$
For columns : $C_i\leftrightarrow C_i+kC_j$

# Mind Map-4

# DETERMINANTS

## 1 Determinant

Every square matrix associates to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det (A) or |A| or $\Delta$.

## 2 Determinant of a Square Matrix of Order Three

Consider $A = [a_{ij}]_{3\times3}$

Then, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

Expansion along first Row ($R_1$)

$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$

## 3 Properties of Determinants

(i) The value of a determinant remains unchanged if its rows and columns are interchanged.

(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

(iii) If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.

(iv) If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

(v) If some or all the elements of a row or column of a determinant are expressed as a sum of two (or more) terms, then the determinant can be expressed as a sum of two (or more) determinants.

(vi) If the equimultiples of corresponding elements of other row (or column) are added to each element of any row or column of a determinant, then the value of the determinant remains the same.

(vii) $|A^T| = |A|$, where $A^T$ = transpose of A.

(viii) If $A = [a_{ij}]_{3\times3}$, then $|kA| = k^3|A|$.

(ix) The determinant of the product of matrices is equal to product of their respective determinants, i.e., |AB| = |A| |B|, where A & B are square matrices of same order

## 4 Area of a Triangle

Area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

## 5 Minor and Cofactor of an Element of a Determinant

**Minor:** The determinant that is left by cancelling the row and column intersecting at a particular element of a determinant is called the minor of that element of the determinant. Minor of an element $a_{ij}$ of a determinant is denoted by $M_{ij}$.

**Cofactor:** The cofactor of an element $a_{ij}$ of a determinant is denoted by $A_{ij}$ (or $C_{ij}$) and is equal to $(-1)^{i+j} M_{ij}$.

## 6 Adjoint of a Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

where $A_{ij}$ is the cofactor of $a_{ij}$

If A be any given square matrix of order n, then
A(adj A) = (adj A) A = |A| I
where I is the identity matrix of order n.

## 7 Singular and Non-SIngular Matrices

A square matrix A is said to be singular if |A| = 0, otherwise it is called non-singular matrix.
If A & B are non-singular matrix of same order, then AB & BA are also non-singular matrices of same order.

## 8 Inverse of a Matrix

If A and B are two matrices such that
AB = I = BA
then B is called the inverse of A and it is denoted by $A^{-1}$.

Also, $A^{-1} = \frac{\text{adj}A}{|A|}$, if $|A| \neq 0$

**Properties of Inverse Matrix**
Let A and B are two invertible matrices of the same order, then

(i) $(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

## 9 Applications of Determinants and Matrices

**Solution of System of Linear Equations using Inverse of a Matrix**
Consider the system of equations

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, we can write, AX = B i.e.,

- Unique solution of the equation AX = B is given by $X = A^{-1}B$, when $|A| \neq 0$
- A system of equations is said to be consistent or inconsistent according as its solution exists or not.
- For a square matrix A in the matrix equation AX = B
  (i) If $|A| \neq 0$, there exists a unique solution and the system of equations is consistent.
  (ii) If |A| = 0, and (adj A) B ≠ 0, then there exists no solution and the system of equations is inconsistent
  (iii) If |A| = 0 and (adj A) B = 0, then the system may or may not be consistent according as the system has either infinitely many solutions or no solution.

# Mind Map-5

## CONTINUITY AND DIFFERENTIABILITY

### 1 Continuity

**Continuity of a Function at a Point**

Suppose f is a real function on a subset of the real numbers & let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

**Continuity of a Function in an Interval**

Suppose f is a function defined on a closed interval [a, b], then for f to be continuous, it needs to be continuous at every point in [a, b] including the end points a & b.

Continuity of f at a, $\lim_{x \to a^+} f(x) = f(a)$

Continuity of f at b, $\lim_{x \to b^-} f(x) = f(b)$

A function which is not continuous at point x = c is said to be discontinuous at that point

### 2 Algebra of Continuous Functions

**Theorem 1:** Suppose f & g be two real functions continuous at a real number c, Then
(1) f + g is continuous at x = c
(2) f − g is continuous at x = c
(3) f·g is continuous at x = c
(4) f/g is continuous at x = c, (provided g (c) ≠ 0)

**Theorem 2:** Suppose f & g are real valued functions such that (fog) is defined at c. If g is continuous at c & if f is continuous at g(c), then (fog) is continuous at c.

### 3 Differentiability

A function f is said to be differentiable at a point c in its domain, if its left hand & right hand derivatives exist at c & are equal.

Here at x = c,

Left Hand Derivative,

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = Lf'(c)$$

Right Hand Derivative,

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = Rf'(c)$$

**Theorem:** If a function f is differentiable at a point c, then it is also continuous at that point. Therefore, every differentiable function is continuous, but the converse is not true.

### 4 Algebra of Derivatives

Let u, v be the functions of x.

**(1) Sum and Difference Rule**
$(u \pm v)' = u' \pm v'$

**(2) Leibnitz or Product Rule**
$(uv)' = u'v + uv'$

**(3) Quotient Rule**
$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

### 5 Chain Rule

If y is a function of u, u is a function of v & v is a function of x.

Then, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

### 6 Implicit Functions

An equation in the form f(x, y) = 0 in which y is not expressible in terms of x is called an implicit function of x & y.

**Derivative of Implicit Functions**

Let y = f(x, y), where f(x, y) be an implicit function of x & y.

- Firstly differentiate both sides of equation w.r.t x
- Then take all terms involving $\frac{dy}{dx}$ on L.H.S. & remaining terms on R.H.S. to get the required value.

### 7 Differentiation of Inverse Trigonometric Functions

| f(x) | f′(x) | Domain of f′ |
|---|---|---|
| $\sin^{-1}x$ | $\frac{1}{\sqrt{1-x^2}}$ | (−1,1) |
| $\cos^{-1}x$ | $\frac{-1}{\sqrt{1-x^2}}$ | (−1,1) |
| $\tan^{-1}x$ | $\frac{1}{1+x^2}$ | R |
| $\cot^{-1}x$ | $\frac{-1}{1+x^2}$ | R |
| $\sec^{-1}x$ | $\frac{1}{\|x\|\sqrt{x^2-1}}$ | \|x\|>1 |
| $\mathrm{cosec}^{-1}x$ | $\frac{-1}{\|x\|\sqrt{x^2-1}}$ | \|x\|>1 |

### 8 Logarithmic Differentiation

Logarithmic Differentiation is a very useful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$, where f(x) & u(x) are positive.

We apply logarithm (to base) on both sides to the above equation & then differentiate by using chain rule, in this way we can find f′(x). This process is called logarithmic

$$\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(\log x) = \frac{1}{x} \quad \& \quad \frac{d}{dx}a^x = a^x \log a$$

### 9 Derivatives of Functions in Parametric Form

The set of equations x = f(t), y = g(t) is called the parametric form of an equation.

Here, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ or $\frac{g'(t)}{f'(t)}$

Here, $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables.

### 10 Second Order Derivative

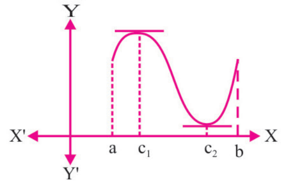
Let y = f(x), then $\frac{dy}{dx} = f'(x)$

If f′(x) is differentiable, then we may differentiate it again w.r.t. x & get the second order derivative represented by:

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $D^2y$ or $y''$ or $y_2$

### 11 Rolle's Theorem

If f : [a, b] → R is continuous on [a, b] & differentiable on (a, b) such that f(a) = f(b), then there exists some c in (a, b) such that f′(c) = 0

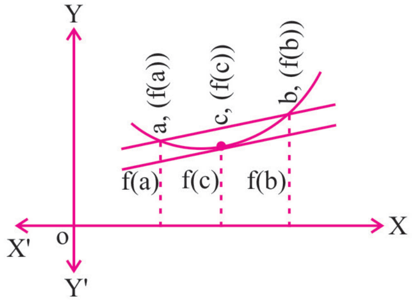



In the above graph, the slope of tangent to the curve at least at one point becomes zero. The slope of tangent at any point on the graph of y = f(x) is nothing but the derivative of f(x) at that point.

### 12 Mean Value Theorem

If f : [a, b] → R is continuous on [a, b] & differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$




The Mean value Theorem states that there is a point c in (a, b) such that the slope of the tangent at (c, f(c)) is same as the slope of the secant between (a, f(a)) and (b, f(b)) or there is a point c in (a, b) such that the tangent at (c, f(c) is parallel to the secant between (a, f(a)) & (b, f(b)).

# Mind Map-6

# APPLICATION OF DERIVATIVES

## 1 Rate of Change of Quantities

The rate of change of y with respect to x at a point $x = x_0$ is given by $\left(\frac{dy}{dx}\right)_{x=x_0}$

Note that $\frac{dy}{dx}$ is positive if y increases with increase in x and is negative if y decreases with increase in x.

## 2 Increasing and Decreasing Functions

(1) (I) Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)\ \forall\ x_1, x_2 \in I$.

(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in I$.

(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \ge f(x_2)\ \forall\ x_1, x_2 \in I$.

(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in I$.

(II) A function f is said to be increasing at $x_0$ if there exists an interval $I = (x_0 - h, x_0 + h), h > 0$ such that for $x_1, x_2$

$x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)$

Similarly, the other cases i.e., strictly increasing, decreasing and strictly decreasing can be clarified.

(2) A function f(x) defined in the interval [a,b] will be

Monotonic increasing $\Leftrightarrow f'(x) \ge 0\ x \in (a, b)$

Monotonic decreasing $\Leftrightarrow f'(x) \le 0\ x \in (a, b)$

Constant function $\Leftrightarrow f'(x) = 0\ x \in (a, b)$

Strictly increasing $\Leftrightarrow f'(x) > 0\ x \in (a, b)$

Strictly decreasing $\Leftrightarrow f'(x) < 0\ x \in (a, b)$

**Properties of Monotonic Functions**

(1) If f (x) and g(x) are monotonically (strictly) increasing (decreasing) functions on [a, b], then gof (x) is a monotonically (strictly) increasing function on [a, b].

(2) If one of the two functions f (x) and g(x) is strictly (monotonically) increasing and other is strictly (monotonically) decreasing, then gof (x) is strictly (monotonically) decreasing on [a, b].

## 3 Tangents and Normals

- The equation of the tangent at $(x_0, y_0)$ is given below:

  $y - y_0 = m(x - x_0)$,

  where m = slope of tangent $= \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$ or $f'(x_0)$

- The equation of the normal at $(x_0, y_0)$ is given below:

  $y - y_0 = -\frac{1}{m}(x - x_0)$,

  where m = slope of tangent at $(x_0, y_0)$

## 4 Approximations

Let y = f(x), $\Delta x$ be a small increment in x & $\Delta y$ be the increment in y corresponding to the increment in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then approximate value of

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

## 5 Maxima and Minima

1. Let f be a funciton defined on an interval I. Then

(a) f is said to have a maximum value in I. if there exists point c in I such that $f(c) \ge f(x)$, for all $x \in I$.

f(c) is the maximum value and point c is a point of maximum value of f in I.

(b) f is said to have a minimum value in I. if there exists a point c in I such that $f(c) \le f(x)$, for all $x \in I$.

f(c) is the minimum value and point c is a point of minimum value of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that f(c) is either a maximum value or a minimum value of f in I.

f(c) is an extreme value and point c is called an extreme point.

2. Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of local maxima if there is an h > 0 such that

$f(c) \ge f(x)$, for all x in (c – h, c + h)

The value f(c) is called the local maximum value of f.

(b) c is called a point of local minima if there is an h > 0 such that

$f(c) \le f(x)$, for all x in (c – h, c + h)

The value f(c) is called the local minimum value of f.

3. Let f be a function defined on an open interval I. Suppose $c \in I$ be any point. If f has a local maxima or a local minima at x = c, then either f'(c) = 0 or f is not differentiable at c.

## 6 Test of Local Maxima & Minima

**First Derivative Test:**

Let f(x) be a function differentiable at x = a. Then

**(a)** x = a is a point of local maximum of f (x), if

(i) f'(a) = 0 and

(ii) f '(x) changes sign from positive to negative as x increases through a

**(b)** x = a is a point of local minimum of f (x), if

(i) f'(a) = 0 and

(ii) f '(x) changes sign from negative to positive as x increases through a

**(c)** If f '(a) = 0, but f '(x) does not change sign as x increases through a, that is f '(a) has the same sign in the complete neighbourhood of a, then a is neither a point of local maximum nor a point of local minimum. In this case, x = a is a point of inflection.

**Second Derivative Test:**

Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

(i) x = c is a point of local maxima if f '(c) = 0 and f "(c) < 0

The value f (c) is local maximum value of f.

(ii) x = c is a point of local minima if f '(c) = 0 and f "(c) > 0

In this case, f (c) is local minimum value of f.

(iii) The test fails if f '(c) = 0 and f "(c) = 0

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflection.

## 7 Absolute Maxima & Absolute Minima

Let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) f'(c) = 0 if f attains its absolute maximum value at c.

(ii) f'(c) = 0 if f attains its absolute minimum value at c.

## 8 Steps for Finding Absolute Maxima and/or Absolute Minima

(i) Find all critical points of f in the interval, i.e., find value of x where either f '(x) = 0 or f is not differentiable.

(ii) Take the end points of the interval.

(iii) At all the above points (in step (i) and (ii)) calculate the value of f.

(iv) Identify the maximum and minimum values of f out of the values calculated in step (iii). The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

# Mind Map-7

# INDEFINITE INTEGRAL

## 1 Standard Integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{1}{x} dx = \log|x| + C$

(iii) $\int e^x dx = e^x + C$

(iv) $\int a^x dx = \frac{a^x}{\log a} + C$

(v) $\int \sin x\, dx = -\cos x + C$

(vi) $\int \cos x\, dx = \sin x + C$

(vii) $\int \sec^2 x\, dx = \tan x + C$

(viii) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(ix) $\int \sec x \tan x\, dx = \sec x + C$

(x) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

(xi) $\int \cot x\, dx = \log|\sin x| + C$

(xii) $\int \tan x\, dx = \log|\sec x| + C$

(xiii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(xiv) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C$

(xv) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xviii) $\int -\frac{1}{a^2 + x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(xix) $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xx) $\int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$

## 2 Methods of Integration

When integration cannot be reduced into some standard form, then integration is performed using following methods :

(i) Integration by Substitution
(ii) Integration using Partial Fractions
(iii) Integration by Parts

## 3 Integration by Substitution

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method by which we change the variable of integration to some other variable is known as the method of substitution.

Consider $I = \int f(x)dx$

Put $x = g(t)$, so $\frac{dx}{dt} = g'(t)$

i.e., $dx = g'(t)\, dt$

Thus, $I = \int f(x)dx = \int f(g(t))g'(t)dt$

**Some Important Substitutions are:**

| Function | Substitutions |
|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a \sin\theta$ or $x = a\cos\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta$ |

## 4 Integration Using Partial Fractions

Consider a rational function of the form $\frac{P(x)}{Q(x)}$

where P(x) & Q(x) are polynomials in x & $Q(x) \neq 0$. If degree of P(x) is greater than the degree of Q(x), then we may divide P(x) by Q(x) such that

$\frac{P(x)}{Q(x)} = T(x) + \frac{R(x)}{Q(x)}$

where, T(x) is a polynomial in x & degree of R(x) is less than the degree of Q(x).
T(x) being a polynomial can be easily integrated.

$\frac{R(x)}{Q(x)}$ can be integrated by expressing $\frac{R(x)}{Q(x)}$ as the sum of partial fractions of the following types:

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$

(ii) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ cannot be factorised further.

## 5 Integration by Parts

$\int u.v dx = u\int v dx - \int\left[\frac{du}{dx}.\int v dx\right]dx$

Here, u is the first function & v is the second function.
**Selection of first function :** For applying integration by parts, we choose the first function as the function which comes first in the word **ILATE**, where

I stands for the inverse trigonometric function ($\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ etc)
L stands for the logarithmic function
A stands for the algebraic functions
T stands for the trigonometric functions
E stands for the exponential functions

## 6 Integrals of Some Special Functions

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

## 7 Two Standard Forms of an Integral

(i) $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

(ii) $\int [xf'(x) + f(x)]dx = xf(x) + C$

## 8 Some Special Types of Integrals

(i) $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

(ii) $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

(iii) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(iv) Integrals of the types $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

can be transformed into standard form by expressing

$ax^2 + bx + c = a\left[x^2 + \frac{bx}{a} + \frac{c}{a}\right]$

$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$

(v) Integrals of the types $\int \frac{px+q}{ax^2+bx+c}dx$

or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

can be transformed into standard form by

expressing $px + q = A\frac{d}{dx}(ax^2+bx+c) + B$

$= A(2ax + b) + B$ where A & B can be determined by comparing coefficients on both sides.

# DEFINITE INTEGRAL

## 1 Definite Integral

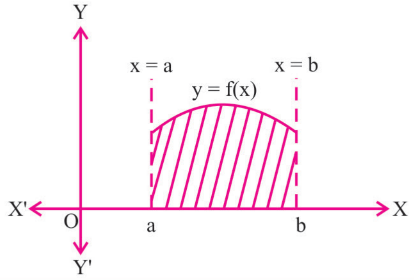
The definite integral of f(x) between the limits a to b i.e., in the interval [a, b] is denoted by

$\int_a^b f(x)dx$ and is defined as follows:

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

where, $\int f(x)dx = F(x)$

The definite integral $\int_a^b f(x)dx$ is also defined as the area bounded by the curve y = f(x), the ordinates x = a, x = b and the x-axis




## 2 Definite Integral as the Limit of a Sum

$\int_a^b f(x)dx = \lim_{h\to 0}[f(a) + f(a+h) + ---- + f(a+(n-1)h)]$

or

$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + ---- + f(a+(n-1)h]$

where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

The above expression is known as the definite integral as the limit of a sum.

## 3 Fundamental Theorem of Calculus

**Theorem 1 :** Let f be a continuous function on the closed interval [a, b] and let A(x) be area function. Then $A'(x) = f(x), \forall x \in [a, b]$
**Theorem 2 :** Let f be a continuous function defined on the closed interval [a, b] & F be the anti-derivative of f.

Then $\int_a^b f(x)dx = [F(x) + c]_a^b = F(b) - F(a)$

This is called the definite integral of f over the range [a, b], where a & b are called the limits of integration, a being the lower limit & b the upper limit.

## 4 Evaluations of Definite Integrals by Substitution

Consider a definite integral of the following form

$\int_a^b f(g(x))g'(x)dx$

To evaluate this integral we proceed as following
**Step 1 :** Substitute
**Step 2 :** Find the limits of integration in new system of variable, i.e. the lower limit is g(a) and

the upper limit is g(b), and the integral is now $\int_{g(a)}^{g(b)} f(t)dt$

**Step 3 :** Evaluate the integral so obtained by usual method.

## 5 Properties of Definite Integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$, in particular $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(vi) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if f(x) is an even function i.e., $f(-x) = f(x)$

$\int_{-a}^a f(x)dx = 0$, if f(x) is an odd function i.e., $f(-x) = -f(x)$

(vii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(viii) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$

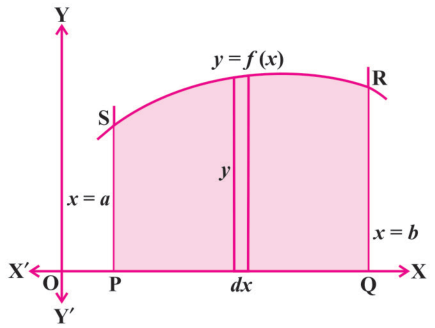
$= 0$, if $f(2a-x) = -f(x)$

# Mind Map-8

# APPLICATION OF INTEGRALS

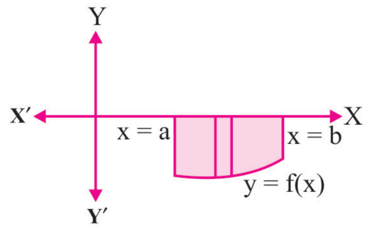
## 1 Area under simple curves

### i Area of the region bounded by a Curve y = f(x) & x-axis between the two ordinates




Area, $A = \int_a^b dA = \int_a^b y\,dx = \int_a^b f(x)\,dx$

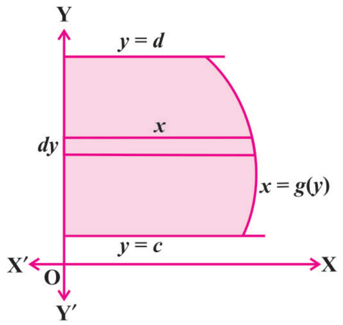
**Remark :**



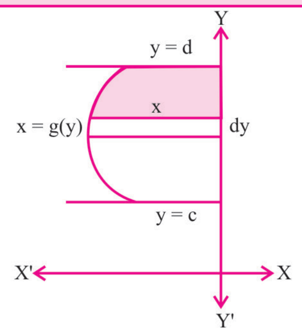

If the position of the curve under consideration is below the x-axis. Then, area is negative. So, we take its absolute value, i.e.,

Area (A) = $\left|\int_a^b f(x)\,dx\right|$

### ii Area of the region bounded by a curve x = f(y) and y-axis between two abscissae.




Area, $A = \int_c^d x\,dy = \int_c^d g(y)\,dy$

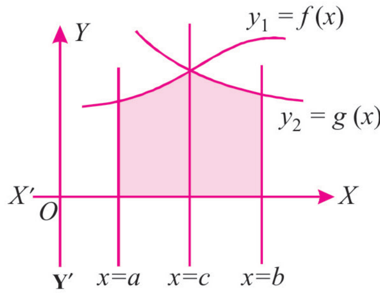



If the position of the curve under consideration is on the left side of y-axis. Then, area is negative. So, we take its absolute value, i.e.,

Area (A) = $\left|\int_c^d g(y)\,dy\right|$

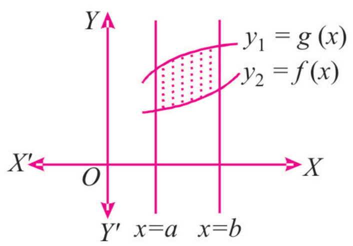
## 2 Area between different Curves

### i Case-I



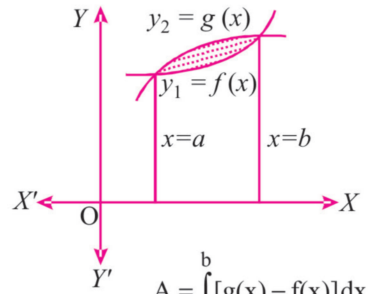

$A = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx$

### ii Case-II



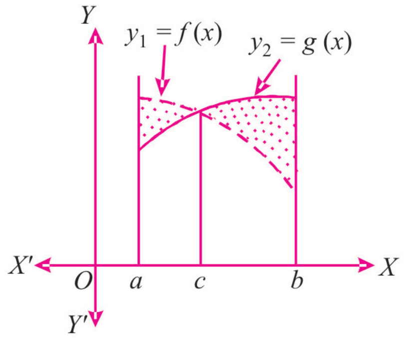

$A = \int_a^b [g(x) - f(x)]\,dx$

### iii Case-III



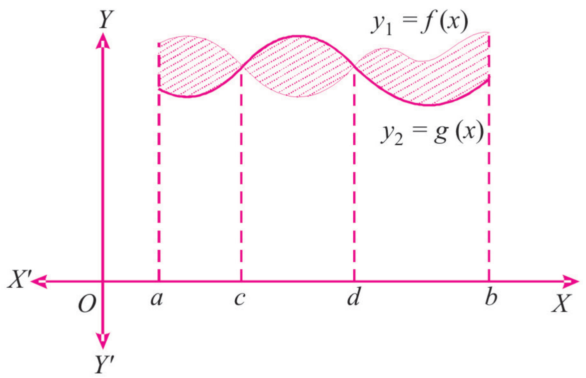

$A = \int_a^b [g(x) - f(x)]\,dx$

### iv Case-IV




$A = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$

### v Case-V




$A = \int_a^c (y_1 - y_2)\,dx + \int_c^d (y_2 - y_1)\,dx + \int_d^b (y_1 - y_2)\,dx$

# Mind Map-9

## DIFFERENTIAL EQUATIONS

### 1 Differential Equation

An equation containing an independent variable, dependent variable & differential coefficients of dependent variable w.r.t. independent variable is called a differential equation.

For example,

(i) $\frac{dy}{dx} = \sin x$ (ii) $\frac{dy}{dx} + xy = \cot x$ (iii) $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = x^2$

A differential equation involving derivatives of the dependent variable w.r.t only one independent variable is called an ordinary differential equation. Above equations are all ordinary differential equations.

### 2 Order of Differential Equation

The order of a differential equation is the order of the highest derivative occuring in the differential equation.

For example

$\frac{d^2y}{dx^2} + y = 0$ is a second order differential equation.

$\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$ is a third order differential equation.

### 3 Degree of Differential Equation

The degree of a differential equation is the highest degree of the highest derivative occuring in the differential equation when it is a polynomial of the differential coefficients i.e., differential coefficients free from radicals & fractions.

For example

Since, $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$ as order = 3

$\therefore$ its degree = 1, as $\frac{d^3y}{dx^3}$ has power 1.

### 4 Solution of Differential Equations

Any relation between the dependent & independent variables (not involving the derivatives) which, when substituted in the differential equation reduces it to an identity is called a 'solution of the differential equation'.

**General Solution :** The solution which contains a number of independent arbitrary constants equal to the order of the equation is called general solution.

**Particular Solution :** Solutions obtained from the general solution by giving particular values to independent arbitrary constants are called particular solutions.

### 5 Formation of Differential Equation

Formation of a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation.

The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

### 6 Differential Equations with Variables Separable

If a first order-first degree equation can be expressed in such a manner that coefficient of dx is f(x) & coefficient of dy is g(y), then we say that variables are separable. A first order-first degree differential equation is of the form $\frac{dy}{dx} = F(x,y)$

Above equation can also be written as:

$\frac{dy}{dx} = h(y)\cdot g(x)$ [if F(x, y) can be expressed as product of g(x) & h(y)]

Separating the variables, we have $\frac{dy}{h(y)} = g(x)\cdot dx$

$\therefore$ Integrate both sides $\int \frac{dy}{h(y)} = \int g(x)\cdot dx$

which is the required solution.

### 7 Homogeneous Differential Equations

An equation in x & y is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f(x, y) & g(x, y) are homogeneous functions of the same degree in x & y.

Here, $(x - y)\frac{dy}{dx} = x + 2y$

or $\frac{dy}{dx} = \frac{x+2y}{x-y}$ is an example of homogeneous differential equation.

To solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$,

Substitute $y = vx$ & so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus $v + \frac{xdv}{dx} = F(v) \Rightarrow \frac{dx}{x} = \frac{dv}{F(v) - v}$

Therefore, solution is $\int \frac{dx}{x} = \int \frac{dv}{F(v) - v} + c$

### 8 Linear Differential Equations

A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P&Q are constants or functions of x only, is known as a First Order Linear Differential Equation.

$\frac{dy}{dx} + y = \sin x$, $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$ are some examples of Linear differential equations.

**Steps to Solve First Order Linear Differential Equation :**

(i) Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$

(ii) Find the Integrating Factor (I.F) $= e^{\int P dx}$

(iii) Write the solution of the given differential equation as

$$y(I.F) = \int (Q \times I.F)dx + c$$

Note that if the first order differential equation is in the form $\frac{dx}{dy} + P'x = Q'$ where $P'$ & $Q'$ are constants or functions of y only.

Then I.F = $e^{\int P dy}$ & the solution of the differential equation is given by

$x(I.F) = \int (Q' \times I.F)dy + c$

# Mind Map-10

## VECTOR ALGEBRA

### 1. Position Vector

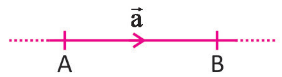
Let O be the origin & P be a point in space having coordinates (x, y, z) with respect to the origin O. Then the vector $\overrightarrow{OP}$ is called the position vector of the point P with respect to O.

$|\overrightarrow{OP}| = \sqrt{x^2+y^2+z^2}$

The angles made by $\overrightarrow{OP}$ with positive direction of x, y, & z-axes (say α, β & γ respectively) are called its direction angles, and the cosine value of these angles i.e., cos α, cos β & cos γ are called direction cosines of $\overrightarrow{OP}$, denoted by $l, m$ & $n$ respectively.

### 2. Vector Quantity

A quantity which has magnitude & also a direction in space is called a vector quantity.



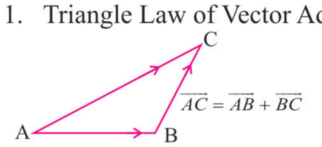

The direct line segment AB is a vector denoted as $\overrightarrow{AB}$ or $\vec{a}$. The point A from where the vector $\overrightarrow{AB}$ starts is called its initial point, & the point B where it ends is called its terminal point. The distance between these two points is called the magnitude of the vector denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a.

### 3. Types of Vectors

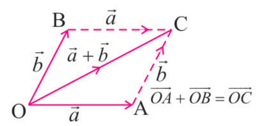
1. **Zero Vector :** A vector whose initial and terminal points coincide, is called a zero vector (or null vector) denoted as $\vec{O}$. It has zero magnitude.
2. **Unit Vector :** A vector whose magnitude is unity (i.e., 1 unit) is called unit vector. The unit vector in the direction of is denoted as .
3. **Coinitial Vectors :** Two or more vectors having the same initial point are called coinitial vectors.
4. **Collinear Vectors :** Two or more vectors are called collinear, if they are parallel to the same line, irrespective of their magnitude.
5. **Equal Vectors :** Two vectors are said to be equal, if they have same magnitude & direction regardless of the position of their initial points.
6. **Negative of a vector :** A vector whose magnitude is the same as that of the given vector, but the direction is opposite to that of it, is called negative of the given vector.

### 4. Addition of Vectors

1. Triangle Law of Vector Addition




2. Parallelogram Law of Vector Addition




**Properties of Vector Addition :**

(i) For any two vectors $\vec{a}$ & $\vec{b}$,
$\vec{a}+\vec{b}=\vec{b}+\vec{a}$
(commutative property)

(ii) For any three vectors $\vec{a}, \vec{b}, \& \vec{c}$,
$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
(Associative property)

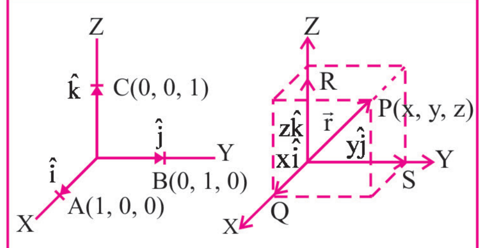
### 5. Multiplication of a Vector by a Scalar

Multiplication of any vector $\vec{a}$ by a scalar λ is denoted as $\lambda\vec{a}$ & its magnitude is given as:

$|\lambda\vec{a}| = |\lambda||\vec{a}|$.

**Unit Vector in the Direction of a given vector :** A unit vector in the direction of given vector $(\vec{a})$ is given as: $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.

### 6. Component of Vector




$\overrightarrow{OA}, \overrightarrow{OB}$ & $\overrightarrow{OC}$ are unit vectors along x, y & z axes respectively, denoted by $\hat{i}, \hat{j}$ & $\hat{k}$

Position Vector of P with reference to O is given by:

$\overrightarrow{OP}(\text{or } \vec{r}) = x\hat{i}+y\hat{j}+z\hat{k}$.

This form of any vector is called its component form.

Also, $|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

Consider two vectors $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then;

(i) $\vec{a}\pm\vec{b} = (a_1\pm b_1)\hat{i}+(a_2\pm b_2)\hat{j}+(a_3\pm b_3)\hat{k}$.

(ii) Vector $\vec{a}$ & $\vec{b}$ are equal if & only if : $a_1 = b_1$, $a_2 = b_2$ & $a_3 = b_3$.

(iii) $\lambda\vec{a} = \lambda a_1\hat{i}+\lambda a_2\hat{j}+\lambda a_3\hat{k}$.

### 7. Vector Joining Two Points

Let A$(x_1, y_1, z_1)$ & B$(x_2, y_2, z_2)$ be any two points in the space, then $\overrightarrow{OA} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ & $\overrightarrow{OB} = x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

### 8. Section Formulae

The position vector of a point R dividing a line segment joining the points P & Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively, in the ratio $m : n$

(i) internally, is given by $\frac{m\vec{b}+n\vec{a}}{m+n}$

(ii) externally, is given by $\frac{m\vec{b}-n\vec{a}}{m-n}$

The position vector of the middle point of PQ is given by $\frac{1}{2}(\vec{a}+\vec{b})$

### 9. Scalar (or dot) Product of Two Vectors

Let $\vec{a}$ & $\vec{b}$ be the two non-zero vectors inclined at an angle θ, then scalar product is defined as :

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0\le\theta\le\pi$

**Observations:**

- $\vec{a}\cdot\vec{b}$ is a real number
- $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\perp\vec{b}$
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ & $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ or $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$
- The scalar product is commutative $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

### 10. Projection of Vector Along a Directed Line

Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

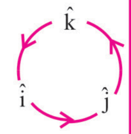
$\vec{a}\cdot\hat{b} = \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right) = \frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|}$.

### 11. Vector (or Cross) Product of Two Vectors

Let $\vec{a}$ & $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then, vector product is defined as : $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where, $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ & $\vec{b}$, such that $\vec{a}, \vec{b}$ & $\hat{n}$ form a right handed system.

**Observations:**

- $\vec{a}\times\vec{b}$ is a vector
- $\vec{a}\times\vec{b} = \vec{0} \Rightarrow \vec{a} \parallel \vec{b}$
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$
- $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$




- Vector product is not commutative.
  $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
  $\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}$ & $\hat{i}\times\hat{k} = -\hat{j}$
- If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a triangle, then its area is given by $\frac{1}{2}|\vec{a}\times\vec{b}|$
- If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a parallelogram then its area is given by $|\vec{a}\times\vec{b}|$

**Cross Product of Vectors in Component Form**

Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, Then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

### 12. Properties Regarding Scalar and Vector Product

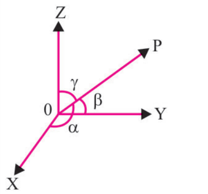
1. **Scalar Product Property :**
   (i) For three vectors $\vec{a}, \vec{b}$ & $\vec{c}$,
   $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (distributive property)
   (ii) For two vector $\vec{a}$ & $\vec{b}$ & any scalar λ,
   $(\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$
2. **Vector Product Property:**
   (i) For any three vectors $\vec{a}, \vec{b}$ & $\vec{c}$,
   $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distribution property)
   (ii) For any two vector $\vec{a}$ & $\vec{b}$ and any scalar λ,
   $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$.

# Mind Map-11

# THREE DIMENSIONAL GEOMETRY

## 1 Direction Cosines of a Line (DC's)

The direction cosines are generally denoted by $l, m, n$.




Hence, $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$

Note that $l^2 + m^2 + n^2 = 1$

## 2 Direction Ratios of a Line (DR's)

Any three numbers $a, b$ and $c$ proportional to the direction cosines $l, m$ and $n$, respectively are called direction ratios of the line.

- The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$
- $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
- $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

## 3 Equation of a Line

**1. Equation of a line through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ :**

**In vector form,** $\vec{r} = \vec{a} + \lambda\vec{b}$

**In cartesian form,**

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Here, a, b, c are also the direction ratios of the line.

**2. Equation of a line passing through two given points with position vectors $\vec{a}$ and $\vec{b}$ :**

**In vector form,** $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

**In cartesian form,**

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ & $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

## 4 Angle Between Two Lines

**In vector form,**

The angle between two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

**In cartesian form,**

The angle between two lines :

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is :

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

- If two lines are perpendicular, then $\vec{b}_1 \cdot \vec{b}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If two lines are parallel, then $\vec{b}_1 = \lambda\vec{b}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

## 5 Shortest Distance Between Two Lines

**1. Distance Between Parallel Lines**

The shortest distance between parallel lines $L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}$ is

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

**2. Distance Between Two Skew Lines**

**In vector form,**

The distance between two skew lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

**In cartesian form,**

The distance between two skew lines :

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is :

$$d = \left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

## 6 Equation of a Plane in Normal Form

**Vector Form**

$\vec{r} \cdot \hat{n} = d$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{n}$ is the unit vector along the normal from origin to the plane.

d is perpendicular distance of the plane from the origin.

**Cartesian Form**

$lx + my + nz = d$

where $l, m, n$ are the direction cosines of $\hat{n}$ (unit vector along the normal from origin to the plane).

## 7 Equation of a Plane Perpendicular to a Given Vector and Passing Through a Given Point

**Vector Form**

Let a plane pass through a point with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$. Then its equation is given as: $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

**Cartesian Form**

Let a plane pass through a point $(x_1, y_1, z_1)$ & the direction ratio of the vector perpendicular to the plane be A, B, C. Then its equation is given as:

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

## 8 Equation of a Plane Passing Through Three Non-Collinear Points

**Vector Form**

$[\vec{r}\,\vec{b}\,\vec{c}] + [\vec{r}\,\vec{a}\,\vec{b}] + [\vec{r}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]$

or $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

where, $\vec{a}, \vec{b}, \vec{c}$ are the position vector of three given non-collinear points through which the plane passes.

**Cartesian Form**

The equation of plane passing through three non-collinear points Y with coordinates $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ & $(x_3, y_3, z_3)$ is given as:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

## 9 Intercept Form of the Equation of a Plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Where a, b, c are the intercepts made by the plane on x, y & z axes respectively.

## 10 Plane Passing Through the Intersection of Two Given Planes

**Vector Form**

Equation of plane passing through the point of intersection of two planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given as :

$$\vec{r} \cdot (\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$$

**Cartesian Form**

Let $\vec{n}_1 = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$

$\vec{n}_2 = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

therefore its cartesian equation is:

$$(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0$$

## 11 Coplanarity of Two Lines

**Vector Form**

Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

are coplanar, if $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$

**Cartesian Form**

Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$

are coplanar, if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

## 12 Angle Between Two Planes

**Vector Form :** The angle between two planes $\vec{r} \cdot \vec{n} = d_1$ & $\vec{r} \cdot \vec{n} = d_2$ is given as:

$$\cos\theta = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$$

**Cartesian Form** The angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- If two planes are perpendicular, then $\vec{n}_1 \cdot \vec{n}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If two planes are parallel, then $\vec{n}_1 = \lambda\vec{n}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

## 13 Distance of a Point from a Plane

**Vector Form**

Distance of a point with position vector $\vec{a}$ from a plane $\vec{r} \cdot \vec{n} = d$ is given as:

$$\left|\frac{\vec{a} \cdot \vec{n} - d}{|\vec{n}|}\right|$$

**Cartesian Form**

Distance of a point $(x_1, y_1, z_1)$ from a plane : $ax + by + cz = d$ is given as :

$$\left|\frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

## 14 Angle Between a Line and a Plane

**Vector Form**

Angle between a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and a plane $\vec{r} \cdot \vec{n} = d$ is

$$\cos\theta = \left|\frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right|$$

**Cartesian Form**

Angle between a line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and a plane $a_2x + b_2y + c_2z = d$ is given as:

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- If line is perpendicular to the plane, then $\vec{n} = \lambda\vec{b}$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- If line is parallel to the plane, then $\vec{n} \cdot \vec{b} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

# Mind Map-12

# LINEAR PROGRAMMING

## 1 Linear Programming Problems

A linear programming problem is concerned with finding the minimum or maximum value of a linear function Z (called objective function) of several variables (say x & y), subject to certain conditions that the variables are non-negative & satisfy a set of linear inequalities (called linear constraints).

## 2 Mathematical form of Linear Programming Problems

The general mathematical form of a linear programming problem may be written as follow.

**Objective Function :** $Z = C_1x + C_2y$

**Subject to constraints are:**

$a_1x + b_1y \le d_1$

$a_2x + b_2y \le d_2$ etc

and non-negative restrictions are $x \ge 0, y \ge 0$

(1) **Objective Function :** A linear function Z = ax + by, where a & b are constants, which has to be maximized or minimized according to a set of given conditions, is called a linear objective function.

(2) **Decision Variables :** In the objective function Z = ax + by, the variables x, y are said to be decision variables.

(3) **Constraints :** The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \ge 0$, $y \ge 0$ are known as non-negative restrictions.

In the constraints given in the general form of a LPP there may be anyone of the 3 signs $\le, =, \ge$.

## 3 Some Important Terms Related to LPP

1. **Feasible Region :** The common region determined by all the constraints including non-negative constraints $x, y \ge 0$ of linear programming problem is known as feasible region (or solution region). If we shade the region according to the given constraints, then the shaded area is the feasible region which is the common area of the regions drawn under the given constraints.
2. **Feasible Solution :** Each point within & on the boundary of the feasible region represents feasible solution of constraints. Note that in the feasible region there are infinitely many points which satisfy the given condition.
3. **Optimal Solution :** Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

## 4 Theorems for Solving Linear Programming Problems

**Theorem 1**

Let R be the feasible region (convex polygon) for a linear programming problem and let Z = ax + by be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point of the feasible region.

**Theorem 2**

Let R be the feasible region for a linear programming problem, and let Z = ax + by be the objective function. If R is bounded then the objective function Z has both maximum and minimum value on R and each of these occurs at a corner point of R.

## 5 Corner Point Method of Solving LPP

**Steps Involved :**

(1) Find the feasible region of the LPP & determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

(2) Evaluate the objective function Z = ax + by at each corner point. Let M & m, respectively be the largest & smallest values of these points.

(3) (i) When the feasible region is bounded, M & m are the maximum & minimum values of Z.

(ii) In case the feasible region is unbounded, we have:

(a) M is the maximum value of Z, if the open half plane determined by ax + by > M has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum value of Z, if the open half plane determined by ax + by < m has no point in common with the feasible region. Otherwise, Z has no minimum value.

## 6 Types of Linear Programming Problems

(i) **Manufacturing Problems**

In such problem, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed man power, machine hours, labour hour per unit of product, ware house space per unit of the output etc., in order to make maximum profit.

(ii) **Diet Problems**

We determine the amount of different types of constituents or nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituent / nutrients.

(iii) **Transportation Problems**

In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

## 7 Mathematical Formulation of Linear Programming Problems

The following Algorithm will be helpful in the mathematical Formation of L.P.P.

**Algorithm**

**Step-1** In every LPP certain decision are to be made. These decision are represented by decision variables. These decision variable are those quantities whose values are to be determined. Identify the variables and denote them by $x_1, x_2, x_3$...

**Step-2** Identify the objective function and express it as a linear function of the variables introduced in step 1.

**Step-3** In a L.P.P the objective function may be in the form of maximizing profits or minimizing costs, so after expressing the objective function as a linear function of the decision variables, we must find the type of optimization i.e. maximization or minimization identify the type of objective function.

**Step-4** Identify the set of constraints, stated in terms of decision variables and express them as linear inequations or equations as the case may be.

## 8 Solution of the Linear Programming Problems

(i) First of all formulate the given problem in terms of mathematical constraints and an objective function.

(ii) The constraints would be inequations which shall be plotted and relevant area shall be shaded and check that feasible region is bounded or unbounded.

(iii) The corner points of common shaded area shall be identified and the coordinates corresponding to these points shall be substituted in the objective function.

(iv) The coordinates of one corner point which maximize or minimize the objective function shall be optimal solution of the given problem.

Note that if feasible region is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of feasible region.

# Mind Map-13

# PROBABILITY

## 1. Conditional Probability

If E and F are two events associated with the sample space of a random experiment, the conditional probability of the event E given that F has occured is given as:

$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{n(E \cap F)}{n(F)}, P(F) \neq 0$

**Properties of Conditional Probability**

1. Let E & F be events of sample space S of an experiment, then we have $P(S/F) = P(F/F) = 1$.
2. If A and B are any two events of a sample space S & F is an event of S such that $P(F) \neq 0$, then $P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$
   In particular if A and B are disjoint events, then $P((A \cup B)/F) = P(A/F) + P(B/F)$
3. $P(E'/F) = 1 - P(E/F)$

## 2. Multiplication Theorem on Probability

For two events E & F associated with a sample space S, we have

$P(E \cap F) = P(E)\ P(F/E)$
$= P(F)\ P(E/F)$

provided $P(E) \neq 0$ & $P(F) \neq 0$

The above result is known as Multiplication Rule of Probability.

## 3. Independent Events

Two or more events are said to be independent if occurrence or non-occurrence of any of them does not affect the probability of occurrence or non-occurrence of other events. For example, when two cards are drawn from a pack of 52 playing cards with replacement (the first card drawn is put back in the pack & then the second card is drawn).

(i) If E & F are independent, then

$P(E \cap F) = P(E)\,P(F)$

$P(E/F) = P(E), P(F) \neq 0$

$P(F/E) = P(F), P(E) \neq 0$

(ii) Three events A, B & C are said to be mutually independent, if

$P(A \cap B) = P(A)\,P(B)$

$P(A \cap C) = P(A)\,P(C)$

$P(B \cap C) = P(B)\,P(C)$

& $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

If at least one of the above is not true for three given events, we say that the events are not independent.

## 4. Baye's Theorem

- **Partition of a Sample Space**
  A set of events $E_1, E_2, \ldots, E_n$ is said to represent a partition of the sample space S if
  (a) $E_i \cap E_j = \phi, i \neq j, i, j = 1, 2, 3, \ldots n$
  (b) $E_1 \cup E_2 \cup \ldots \cup E_n = S$
  (c) $P(E_i) > 0$ for all $i = 1, 2, \ldots n$
- **Theorem of Total Probability**
  Let $\{E_1, E_2, \ldots, E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, \ldots, E_n$ has nonzero probability of occurence. Let A be any event associated with S, then
  $$P(A) = \sum_{j=1}^{n} P(E_j)P(A/E_j)$$
- **Baye's Theorem:** If $E_1, E_2, \ldots, E_n$ are non-empty events which constitute a partition of sample space S & A is any event of non-zero probability.
  $$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)} \quad \text{for any } i = 1, 2, 3, \ldots, n$$

## 5. Random Variable & its Probability Distributions

A random variable is a real valued function whose domain is the sample space of a random experiment.

The probability distribution of a random variable X is the system of numbers.

| | | | | |
|---|---|---|---|---|
| X : | $x_1$ | $x_2$ | … | $x_n$ |
| P(X): | $p_1$ | $p_2$ | … | $p_n$ |

where, $p_i > 0$, $\sum_{i=1}^{n} p_i = 1, i = 1, 2, \ldots, n$

The real numbers $x_1, x_2, \ldots, x_n$ are the possible values of the random variable X and $p_i$ (i = 1, 2, ..., n) is the probability of the random variable X taking the value $x_i$ i.e., $P(X = x_i) = p_i$

## 6. Mean of a Random Variable

The mean ($\mu$) of a random variable X is also called the expectation of X, denoted by E(X)

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i$$

Here $x_1, x_2, \ldots, x_n$ are possible values of random variable X, occuring with probabilities $p_1, p_2, \ldots, p_n$ respectively.

## 7. Variance of a Random Variable

Let X be a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p(x_1), p(x_2), \ldots, p(x_n)$ respectively. Also let $\mu = E(X)$ be the mean of X, then the variance of X is given as:

$$\text{Var}(X) \text{ or } \sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i) = E(X - \mu)^2 = E(X^2) - [E(X)]^2$$

The non-negative number, $\sigma_x = \sqrt{\text{Var}(X)}$ is called the Standard Deviation of random variable X.

## 8. Bernoulli Trials & Binomial Distribution

**Bernoulli Trials :**

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:

(i) There should be a finite number of trials.
(ii) The trials should be independent.
(iii) Each trial has exactly two outcomes: success or failure.
(iv) The probability of success remains same in each trial.

**Binomial Distribution :**

The probability distribution of number of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion $(q + p)^n$, where p is probability of success in each trial and $p + q = 1$. Hence, this distribution (also called Binomial distribution B(n, p)) of number of successes X can be written as:

| X | 0 | 1 | 2 | – – – | x | n |
|---|---|---|---|---|---|---|
| P(x) | ${}^nC_0q^n$ | ${}^nC_1q^{n-1}p^1$ | ${}^nC_2q^{n-2}p^2$ | | ${}^nC_xq^{n-x}p^x$ | ${}^nC_np^n$ |

The probability of x successes $P(X = x)$ is also denoted by P(x) is given as:

$P(x) = {}^nC_xq^{n-x}p^x$, x = 0, 1, ----, n $\quad (q = 1 - p)$.

This P(x) is called the probability function of the binomial distribution.