Observers and Light:
Time, Space and Matter

GPF

Prologue

We live in an era in which physics has described the laws of the universe with extraordinary precision, but has left out its main character: the observer. In its attempt to reduce the world to particles, fields, and equations, it has ended up blurring life itself, relegating consciousness to a secondary illusion, an insubstantial epiphenomenon.

And yet, modern physics itself has not been able to avoid the intrusion of the observer. Quantum mechanics, in particular, has revealed that the act of observation modifies the phenomenon, and has forced a reconsideration of the idea of an objective, independent reality. Some of the very founders of quantum theory were explicit on this point. Max Planck:

> “I regard consciousness as fundamental. I regard matter as derived from consciousness.”

Despite this, the majority of the scientific community has continued to rely on an implicitly Newtonian view: the universe as a dead system, governed by fixed laws, in which the observer appears as an inconvenient disturbance. Consciousness is not explained, but postponed. The soul, expelled from scientific discourse, survives in the intimacy of the subject, but not in the structure of the cosmos. This book starts from a different hypothesis, radical in its simplicity:

> *The universe is made of souls that interact freely with one another.*

From this minimal foundation—no space, no time, no matter—we will attempt to reconstruct everything: the flow of time, the emergence of space, metric, curvature, mass, relativity, and quantum phenomena. All with purely mathematical tools, without metaphors, without dogma, without sweeping consciousness under the rug.

This work is neither a mystical speculation nor a spiritual metaphor: it is a rigorous construction that brings together graphs, spectral theory, geometry, and relational physics. But it is not a cold technical treatise either. It is an honest search for meaning, starting from the root of existence.

Now, this book deals with only one part of what we might call the experience of the soul: its physical manifestation. Philosophically, I consider that each soul lives its reality in four internal dimensions or *experiences*: **sensations**, **thoughts**, **feelings**, and **actions**. These four categories are causally connected: without sensation, there is no thought; without thought, no emotion; without emotion, no reaction.

> *If you don't see the wolf running toward you (sensation), you don't think it's going to eat you (thought), you don't feel fear (feeling), and you don't run away (action).*

Among these experiences, we distinguish two broad groups: **physical experiences** (sensations and actions) and **psychic experiences** (thoughts and feelings). Throughout this book, we will focus exclusively on the former. Not because the others are irrelevant, but because we need to take a first firm step: *to show that physics can arise from a theory of souls, even before addressing the mental realm.*

This is a methodological omission, not an ontological one. The psychic world—the world of thought and feeling—must be addressed later. Here we propose only one thing: **to rationally convince humanity that the universe is made of souls.**

Roots of the Principle: Ancient Resonances

Although this principle is presented here as a structural and physical axiom, its essence has been present in many cultures and traditions throughout history. What changes in this work is not the intuition, but the seriousness with which it is taken: it is proposed as the literal foundation of reality, not as a spiritual metaphor.

In the **Advaita Vedanta** tradition of Hinduism, it is taught that ultimate reality is consciousness (Brahman), manifested in the form of souls (Atman), and that everything we perceive is illusion (maya).

In **Mahayana Buddhism**, the doctrine of interdependence states that nothing exists by itself: everything arises in relation to everything else. "This is because that is." A relational view of being that deeply resonates with our hypothesis.

Taoism holds that the Tao is not a thing, but the flow between things, that which connects them. Reality does not lie in the objects, but in the relationships that traverse them.

Even **mystical Christianity**, in thinkers such as Meister Eckhart or Teilhard de Chardin, has considered that God is relation, and that the evolution of human consciousness is part of an expanding spiritual network (the noosphere).

In Western philosophy, **Spinoza** conceived the universe as a single substance expressed through modes, all interconnected. **Whitehead** spoke of processes of experience as the basic constituents of reality. And **Heidegger** held that being is always a being-in-relation.

It is also worth highlighting modern thinkers who, from physics and philosophy, have defended views aligned with this ontological foundation.

Bertrand Russell, from a different tradition, developed an approach known as *neutral monism.* For him, underlying reality was neither physical nor mental, but something more fundamental that could manifest in both forms. In *The Analysis of Mind* (1921), he proposed that mental and physical events are descriptions of a single neutral entity. He also held that our access to reality is mediated by *sense data*, reinforcing the idea that consciousness is not a later product, but a constitutive element of knowledge itself.

Eugene Wigner Nobel Prize laureate; co-founder of modern quantum theory

"It was not possible to formulate the laws of quantum mechanics in a fully consistent way without reference to consciousness." — "Remarks on the Mind–Body Question", 1961

Wigner proposed that consciousness causes the collapse of the wavefunction, famously illustrated in the thought experiment known as Wigner's friend.

Max Planck, father of quantum theory, clearly stated that consciousness is not a product of matter, but its origin. In an interview for *The Observer* in 1931, he affirmed:
"I regard consciousness as fundamental. I regard matter as derived from consciousness. We cannot go beyond consciousness. Everything we talk about, everything we regard as existing, postulates consciousness."
In a 1944 speech in Florence, he was even clearer:
"As a man who has devoted his whole life to the most clear-headed science, to the study of matter, I can tell you as a result of my research about atoms this much: There is no matter as such. All matter originates and exists only by virtue of a force... We must assume behind this force the existence of a conscious and intelligent Mind. This Mind is the matrix of all matter."

A new idea. Maybe

What distinguishes this work from those is the execution. Here that idea — that being is made of relation, that souls exist only in interaction—is developed as a *rigorous physical theory*, with operators, metrics, structures, and consequences. Thus, we can trace a long tradition of intuitions—spiritual, philosophical, and scientific—that point in the same direction: that the universe is not made of things, but of relationships; not of objects, but of experience; not of matter, but of consciousness.

This work places itself in that lineage, but goes a step further: it proposes it as the literal and structural starting point of physics.

It is proposed as the **only possible foundation** from which to reconstruct time, space, mass, and geometry. Not as a spiritual alternative, but as *the logical root of existence.* Here that idea is developed as a *rigorous physical theory*, with operators, metrics, structures, and consequences.

This text presents a physical model in which the universe is not made of matter, nor of space, nor of time, but of conscious entities that interact with each other. We will call these entities, for convenience, *observers.*

The goal of this work is to show that, starting solely from a set of observers and the fact that they interact with each other freely, one can rigorously and naturally derive:

- a notion of **time**, emerging from the causal sequence of actions and abstracted as a particle moving in a random walk, which we will call **light**;
- a notion of **distance** between observers, understood as the time it takes for light to travel from one to the other—the *resistance distance*—which will allow us to define a *geometric space* with a negative-type metric;
- an **auxiliary Euclidean structure**, derived from the spectrum of the Laplacian operator, allowing us to represent these observers in a Hilbert space with positive metric;
- a **conceptual connection to relativity**, by identifying the constancy of the speed of light as a fundamental principle of the system;
- a **structural connection to quantum mechanics**, by modeling the evolution of the system as a linear (reversible or irreversible) transformation in a Hilbert space;
- a study of **existential stability**, which will lead us to the conclusion that the universe can have at most 3 spatial dimensions;

- a **connection to quantum gravity**, by defining mass as an emergent property of the interaction that slows down the speed of light;

The approach begins with the study of systems of interaction represented by transition matrices between observers, and uses classical tools from spectral graph theory: Laplacian operators, pseudoinverses, resistance metrics, and functional embeddings.

Although the framework is entirely discrete, it is still possible to consider its continuum limit, and even its application to real physical systems, from particles to fields. However, in this text we focus on the finite case, where everything can be expressed explicitly and unambiguously — not merely as a simplification, but because we regard the discrete formulation as fundamental. The continuum appears, in this view, as a particular limit of the discrete, not the other way around.

This model assumes no geometry, no space, no speed, and no location. All of that emerges, strictly, from the relational structure among observers. In that sense, it can be seen as an attempt to construct a **minimal physical ontology**: a universe without objects, composed only of free actions between conscious beings.

The development of this idea will be progressive: we will first formally define the system of observers, their dynamics, and their equilibrium. Then we will build the emergent metric that gives meaning to distance and with it, *geometric space*. Next, we will show how this metric can be represented in a Euclidean space through a functional *embedding*, and discuss the physical meaning of this space, seeing that it is in fact a *Hilbert space*.

Finally, some reflections will be offered on the broader implications of the model, including its relationship with the two great current theories of fundamental physics: general relativity and quantum mechanics. By the relation that exists between the world singe function and the pseudoinverse of the laplacian, and interpreting the observers as $fields$

into the Hilbert space.

Note on Terminology

In this theory, the fundamental entity of the universe is not matter, nor space, nor time, but what has historically been called a *soul*: a conscious, simple, and indivisible being, capable of acting upon other souls.

I acknowledge that this word may provoke resistance, especially in scientific contexts. For this reason, throughout this article I will use the term *observer* as a technical synonym, with the sole purpose of avoiding unnecessary biases in the reader. However, I feel compelled to clarify —explicitly— that:

- I do not regard these entities as metaphors, nor as abstract functions, nor as complex physical systems.
- This is not an analogy: in this theory, the universe is **literally** made of souls.
- And the entire mathematical structure developed below is constructed from that premise.

The reader may internally replace the term "observer" with "soul" in every instance of the text, if they so wish. I, personally, will do so.

Mathematical Notation

Throughout this work, we will use Dirac notation to represent vectors in vector spaces. In particular, the canonical basis vectors will be denoted as

$$|n\rangle,\ |m\rangle, \dots$$

where $|n\rangle$ represents the vector whose only non-zero component is the n-th one, equal to 1.

Any vector $|v\rangle \in \mathbb{C}^N$ can be written as a linear combination of these vectors:

$$|v\rangle = \sum_{n=1}^{N} v_n |n\rangle$$

We will also use the dual notation for its transpose (or associated row vector), so that:

$$\langle v| = (v_1 \quad v_2 \quad \dots \quad v_N) \qquad\qquad |v\rangle = \begin{pmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_N^* \end{pmatrix}$$

In addition, we introduce two diagonal operators associated with a vector $|v\rangle \in \mathbb{C}^N$:

- The diagonal matrix generated from v, denoted $\widehat{v}$, defined by:

$$\widehat{v} = \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_N \end{pmatrix}$$

- The matrix $\underset{\smile}{v}$, defined as the *pseudoinverse* of $\widehat{v}$, with the convention:

$$\underset{\smile}{v} := \begin{cases} \dfrac{1}{v_n}, & \text{if } v_n \neq 0 \\ 0, & \text{if } v_n = 0 \end{cases} \qquad \underset{\smile}{v} = \begin{pmatrix} v_1^{-1} & & & \\ & v_2^{-1} & & \\ & & \ddots & \\ & & & v_N^{-1} \end{pmatrix}$$

  This definition is consistent with the Moore–Penrose pseudoinverse for diagonal matrices.

- We introduce the following special objects:

$$\langle \varnothing | := (0 \quad 0 \quad \dots \quad 0)$$
$$\langle \mathit{I} | := (1 \quad 1 \quad \dots \quad 1)$$

- To operate with vectors in Dirac notation, we adopt the following convention for pointwise sums and products:

$$|v_1 + v_2\rangle := |v_1\rangle + |v_2\rangle$$

$$|v_1 v_2\rangle := \widehat{v}_1 |v_2\rangle = \widehat{v}_2\, |v_1\rangle = \widehat{v}_1 \widehat{v}_2 |\,I\,\rangle = \widehat{v_1 v_2} |\,I\,\rangle = \widehat{v_2 v_1} |\,I\,\rangle$$

Fundamental Axiom

The Universe is composed of observers
who interact freely with one another.

This single principle contains everything necessary to reconstruct reality. It presupposes no space, no time, no matter. It invokes no external laws or underlying fields. The observers are not embedded in coordinates, nor floating in a void: they simply exist, act, and are acted upon by other observers.

Nothing else is assumed. Everything else—the flow of time, the structure of space, geometry, curvature, mass, and even identity—emerges from this single postulate. What we call ”reality” is nothing more than the result of a network of elementary actions between observers.

We will call this collection of interactions the **history**. It will be the fundamental object from which we will try to derive, in the coming chapters, the entirety of physics. Because if this axiom is true, then everything that exists, everything we feel and measure, everything we call the universe… *is nothing but a network of observers acting upon one another.*

Observers and Actions

Observers

An observer is an elementary, indivisible entity that can act upon other observers. It possesses no internal parts, nor intrinsic properties. Here we consider only its capacity to produce effects on other observers.

We will call the *physical universe* the pair $\mathcal{U} = (\mathcal{O}, \mathcal{A})$, where $\mathcal{O}$ is the set of all observers, and $\mathcal{A}$ is the set of all actions among them. That is, the universe is composed of the observers and of the actions they perform on one another.

Actions: History as a chain

In the previous section we defined the universe as a set of observers capable of acting freely upon one another. We now *postulate* that the complete set of actions (the so-called *History* of the universe) is a **unique chain** that describes, step by step, all the actions that take place in the universe. The key idea is that this succession is not imposed by any predefined time, but rather that *the causal relationship between actions itself* generates what we will call *emergent time*.

In other words, we can *enumerate* each action according to its *causal* order (without assuming an external time a priori), thus obtaining a sequence that, precisely, *defines* the notion of time in this model. In this sense, the theory is *causal*: the dynamics of actions determines a sequential structure we call "History," and not the other way around. This approach differs from traditional physics, where an external and independent time is usually assumed. Here, on the contrary, time *emerges* from the causal chain of events, and it is *that* chain which determines the arrow of time and the order of actions.

We can *enumerate* each action according to causal order, and thus obtain a sequence:

$$\langle n_1|,\ \langle n_2|,\ \langle n_3|,\ \ldots$$

where $\langle n_k|$ represents the observer being "occupied" (or activated) at step k of the History. For example, if at step 1 observer "1" acts, at step 2 observer "3", at step 3 observer "18", and so on, the sequence could be:

$$\langle 1|,\ \langle 3|,\ \langle 18|,\ \langle 4|,\ \langle 9|,\ \langle 7|,\ \ldots$$

This sequence (generally infinite) fully describes what we call the physical History of the universe.

Histories expressed as digit strings. If there are N observers (labeled $0, 1, \ldots, N-1$), each $\langle n_k|$ is a "digit" in base N. Therefore, the *entire sequence* can be seen as a string of digits in that base. For example, if $N = 10$ and the History is

$$\langle 3|,\ \langle 1|,\ \langle 4|,\ \langle 1|,\ \langle 5|,\ \langle 9|,\ \ldots$$

we can imagine it as the digit sequence 314159... Here, unlike the usual notation for real numbers, we do not require a comma or decimal point; we simply consider the (potentially infinite) string of digits.

There are *as many* possible Histories as there are digit sequences in base N. This is perfectly analogous to the multitude of real numbers that, when written in base N, are described by infinite strings. Thus, illustratively, we speak of "universe π_{10}" if we take the digits of π in base 10, or "universe $\sqrt{2}_7$" if we follow the digits of $\sqrt{2}$ in base 7, and so on.

"Complete" or "cyclic" Histories. It will often be useful to consider Histories that, after a finite number of steps, **return** to the starting point to form a closed cycle. That is, we assume that the last observer in the list acts upon the one who initiated the sequence, so that all observers in the chain give and receive action the same number of times.

For example, the chain

$$72314$$

is considered *complete* if observer "4" ends up acting upon "7", thus closing the loop. We will call such chains "complete" or "cyclic." Unless stated otherwise, we will **assume by default** that the Histories we deal with belong to this "complete" category.

Example: the "π_3 universe" with a finite cycle. Consider the expansion of π in base 3. Take a truncated section of its digits (without the decimal point), for example:

$$10010221102$$

and force the last digit (2) to act upon the first (1), so that the sequence

$$1 \to 0 \to 0 \to 1 \to 0 \to 2 \to 2 \to 1 \to 1 \to 0 \to 2 \to (1),$$

forms a closed cycle. This "complete History" of length eleven (plus the closing action) will be referred to, illustratively, as "π_3 truncated to 11 digits with closure," or simply "$\pi_{3,11}$."

The essential point is that, with this chain-based language, there is no need to assume from the outset any space or physical coordinates: the History is simply the sequence of interactions.

In summary, we will consider:

- *Histories as sequences* of observers,
- *representations of those sequences* in base N,
- and *complete or "cyclic" Histories* in which the chain closes upon the beginning.

From here, we will construct the rest of our theory, showing how the notions of distance, space, mass, and other physical concepts emerge without assuming *a priori* any preexisting geometry.

We choose to postulate that the History of the universe is a single causal chain—not because it is the only conceivable possibility, but because it offers the clearest foundation for a theory in which time emerges from causality. While more complex structures can be imagined—branching timelines, converging paths, or tangled networks—their interpretation would demand additional assumptions, especially in the absence of any predefined space or external clock. By contrast, a single unbroken sequence of actions provides a minimal yet coherent framework: it defines temporal order through causation alone. Moreover, it aligns with our own experience—as conscious observers, we each perceive time as a linear flow. This postulate does not claim to be metaphysically final, but rather methodologically essential: other alternatives may, of course, be studied, but only after this simplest case has been fully understood. For example, if two distinct branches never intersect, they must be analyzed independently—as if they belonged to separate *parallel universes*, because no causal links would exist between them — there would exist two separate timelines.

Although our theory arises from a radically different ontological foundation, it shares certain deep structural similarities with *causal set theory*, a prominent approach in quantum gravity. Both frameworks begin with the assumption that the fundamental structure of the universe is *discrete*, and that neither time nor space exists *a priori*. Instead, both seek to derive temporal and spatial relations from more primitive elements—in our case, from interactions between conscious observers; in causal set theory, from abstract causal relations among events.

Causal set theory models the universe as a set of events endowed with a *partial order* $\prec$, where $a \prec b$ means that event a causally precedes event b. The set is required to be *transitive*, *irreflexive*, and *acyclic*, and contains no spatial structure beyond what can be derived from the pattern of causal relations. Within any causal set, *totally ordered chains*—sequences of events where every pair is causally related—always exist. In this sense, the unique causal chain postulated in our theory can be viewed as a special case: a minimal, degenerate causal set.

However, our framework imposes much stricter constraints. In our model, every event is not an anonymous point in a structure, but an *action from one observer upon another.* More importantly, each observer experiences their own life as a *linear sequence* of such interactions: no branching, no merging, no ambiguity. This excludes causal structures in which an observer would participate in multiple incompatible chains, or where causal influence would appear without direct experiential continuity. Such scenarios would violate the coherence of memory and identity, and are therefore inadmissible.

Finally, while causal set theory accommodates *spacelike-separated events*—those that are causally unrelated and thus unordered—our theory has no use for such events. In our framework, space does not arise from simultaneity or independence, but from the *pattern and geometry of interaction.* In summary, causal set theory provides a broader mathematical landscape in which our model is technically embedded, but our ontological principles serve as a filter, selecting only those causal structures that preserve *agency*, *reciprocity*, and the *linearity of individual experience.*

Action and Sensation

To quantify action, we introduce the **accumulated action function** $\mathfrak{T}(x, y)$.

$\mathfrak{T}(x, y) :=$ *the total number of actions in which observer x acts upon observer y throughout the History.*

In other words, $\mathfrak{T}(x, y)$ counts how many times (in the historical sequence) observer x has "acted" upon observer y.
We can express action more compactly as a matrix: $\mathfrak{T}$. Each row $\mathfrak{T}_n(y) := \mathfrak{T}_n(x = n, y)$ represents the number of times observer n has acted upon each observer y. And each column $\mathfrak{T}_m(x) := \mathfrak{T}(x, y = m)$ represents the number of times observer m has sensed each observer x.

In this way, we introduce the **accumulated sensation function** $\mathcal{S}(x, y)$. We define the element $\mathcal{S}_{nm}$ of the matrix $\mathcal{S}$ as *the total number of actions in which observer n has sensed the action of observer m throughout History.*
It is evident that:

$$\mathcal{S}_{nm} = \mathfrak{T}_{mn}$$

which holds for any n, m. This leads us to the equation:

$$\mathcal{S}(x, y) = \mathfrak{T}(y, x)$$

Or, in matrix form:

$$\mathcal{S} = \mathfrak{T}^*$$

Example: Construction of $\mathfrak{T}$ from the universe $\pi_{3,11}$ Recall the example from the previous section, where we had a "universe" with three observers $\{0, 1, 2\}$, and we considered the finite (cyclic) History consisting of eleven steps:

$$10010221102$$

We can count how many times the action $x \to y$ occurs throughout these steps (including the one that closes the cycle). We obtain, for example:

$$0 \to 0 \text{ occurs once,} \quad 0 \to 1 \text{ occurs once,} \quad 0 \to 2 \text{ occurs twice,}$$

$$1 \to 0 \text{ occurs three times,} \quad 1 \to 1 \text{ occurs once,} \quad 1 \to 2 \text{ does not occur,}$$

$$2 \to 0 \text{ does not occur,} \quad 2 \to 1 \text{ occurs twice,} \quad 2 \to 2 \text{ occurs once.}$$

Compiling everything into the *action matrix*, we obtain:

$$\mathfrak{T} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \implies \mathcal{S} = \mathfrak{T}^* = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Each entry $\mathfrak{T}_{nm}$ is an integer that exactly reflects how many times n has acted upon m. Note that the row and column sums of $\mathfrak{T}$ are equal, yielding the total number of actions performed and received by each observer.

Properties and notation. We now define the *action* and *sensation* vectors associated with the rows $\mathfrak{T}_n(y)$ and columns $\mathfrak{T}_m(x)$ of $\mathfrak{T}$:

$$\langle \tau | := \langle \imath | \mathfrak{T}$$

$$\langle \mathfrak{s} | := \langle \imath | \mathcal{S}$$

In other words, the component τ_n of the vector $\langle \tau |$ is the sum of row n of $\mathfrak{T}$,

$$\tau_n = \sum_m \mathfrak{T}_{nm}$$

which is interpreted as the *number of actions* ***performed*** by observer n.

Or, expressed in functional terms:

$$\tau(x) = \sum_y \mathfrak{T}(x, y)$$

Similarly, the sensation $\mathfrak{s}$ is:

$$\mathfrak{s}(x) = \sum_y \mathcal{S}(x, y)$$

which is interpreted as the number of actions ***received*** by observer x.

If we consider the History as a closed *cycle* (i.e., where the last observer acts on the first), all the observers (included the first one) maintain a balance between the number of times they act and the number of times they are "acted upon". Therefore, in such cases we have:

$$\langle \tau | \ = \ \langle \mathfrak{s} |,$$

and we interpret $\tau_n(t)$ as the number of *"proper instants"* of each observer n, that is, how many times observer n has acted, wich is the same as he has sensed the action of others. We will call the vector $\langle \tau(t)|$ the *proper time*, as it will serve us later to relate the dynamics of each observer to the time scale that the observer itself "experiences."

This interpretation highlights a key point: for an observer, time does not "flow" continuously or externally—it only advances when a complete act of interaction occurs. In particular, no proper time elapses for an observer between the moment it sends out a signal (or acts upon another) and the moment it senses the return. From its own perspective, the interval between "giving" light and "receiving" it is instantaneous. Thus, proper time is not a background variable, but a direct measure of the number of meaningful interactions the observer has undergone—each one marking a definite instant of experience.

Finally:

$$\langle \imath \,|\, \mathfrak{T} \ = \ \langle \imath \,|\, \mathcal{S} = \langle \tau |,$$

It is useful to note that the action matrix $\mathfrak{T}$ can be interpreted as the *adjacency matrix* of a weighted digraph (directed graph), whose vertices are the observers and where the *weight* of the edge $n \to m$ is precisely $\mathfrak{T}_{nm}$. If we denote by t the total sum of all entries of $\mathfrak{T}$, then t can be considered the *volume* of this digraph, that is, the sum of all weights (in our case, the total number of interactions t). In this way, the study of dynamics based on the matrix $\mathfrak{T}$ can be connected with the properties of a weighted directed graph in future developments.

Action matrices A, action flux $\mathbb{A}$, and accumulated action $\mathfrak{T}$

Once the accumulated action matrix $\mathfrak{T}$ is known, we can talk about increments in action, sensation, and proper time over a time interval Δt:

$$\Delta\mathfrak{T} = \mathfrak{T}(t + \Delta t) - \mathfrak{T}(t)$$

$$\Delta\mathcal{S} = \mathcal{S}(t + \Delta t) - \mathcal{S}(t)$$

$$\langle\Delta\tau| = \langle\tau(t + \Delta t)| - \langle\tau(t)|$$

With this, we can define the action and action flux matrices.

The action flux matrix $\mathbb{A}$ is defined as the average action per unit of time:

$$\mathbb{A} = \frac{1}{\Delta t}\Delta\mathfrak{T} \implies \Delta\mathfrak{T} = \mathbb{A}\Delta t$$

Likewise, the sensation flux matrix $\mathbb{S}$ is defined as the average sensation per unit of time:

$$\mathbb{S} = \frac{1}{\Delta t}\Delta\mathcal{S} \implies \Delta\mathcal{S} = \mathbb{S}\Delta t$$

Since $\mathcal{S} = \mathfrak{T}^*$, then:

$$\mathbb{S} = \mathbb{A}^*$$

We define the frequency vector $\langle\omega|$ as the "proper time per unit of absolute time", that is,

$$\langle\omega| = \frac{1}{\Delta t}\langle\Delta\tau|$$

The action matrix A is defined as the action per unit of proper time:

$$A = \underbrace{\Delta\tau}\Delta\mathfrak{T}$$

or equivalently:

$$A = \underset{\smile}{\omega}\mathbb{A}$$

And the sensation matrix S is defined as the sensation per unit of proper time:

$$S = \underset{\smile}{\Delta\mathfrak{s}}\Delta\mathcal{S} = \underset{\smile}{\Delta\tau}\Delta\mathcal{S} = \underset{\smile}{\Delta\tau}\Delta\mathfrak{T}^*$$

or equivalently:

$$S = \underset{\smile}{\omega}\mathbb{S}$$

Therefore, we can relate Action and Sensation:

$$S = \underset{\smile}{\omega}A^*\widehat{\omega} \qquad A = \underset{\smile}{\omega}S^*\widehat{\omega}$$

In our example, considering a time interval $\Delta t = 11$, we have:

$$\Delta\mathfrak{T} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \implies \Delta\mathcal{S} = \Delta\mathfrak{T}^* = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\langle\Delta\tau| = \langle\mathit{I}|\Delta\mathfrak{T} = \langle\mathit{I}|\Delta\mathcal{S} = \begin{pmatrix} 4 & 4 & 3 \end{pmatrix} \implies \langle\omega| = \frac{1}{11}\begin{pmatrix} 4 & 4 & 3 \end{pmatrix}$$

$$\mathbb{A} = \frac{1}{\Delta t}\Delta\mathfrak{T} = \frac{1}{11}\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \implies \mathbb{S} = \frac{1}{\Delta t}\Delta\mathcal{S} = \mathbb{A}^* = \frac{1}{11}\begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

$$A = \underset{\smile}{\omega}\mathbb{A} = \begin{pmatrix} 1/4 & 1/4 & 2/4 \\ 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix} \qquad S = \underset{\smile}{\omega}\mathbb{S} = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/4 & 2/4 \\ 2/3 & 0 & 1/3 \end{pmatrix}$$

The Random Walk. Light

Describing the History of the universe as a chain of observers, where each one *acts* upon the next, is mathematically equivalent to a *random walk* on a graph whose "nodes" are, ultimately, the observers. The idea is that there is a "particle" that "jumps" from one node to the next at each step of the sequence.

We will call this particle *light*. However, it is important to emphasize that in this theory *there is no actual particle* physically moving from one observer to another; "light" is merely an *abstraction* to model the order in which actions occur. In other words, *there is no* material object performing a random walk: what we are representing as "motion" is simply the causal sequence of who acts upon whom.

At each step, we say that "light passes" from the current observer to the next. But, again: it is not that a ray of light physically moves between nodes; *there is no such propagation in a pre-existing space.* Rather, the "jump" encodes the causal relation: observer $\langle n|$ acts upon $\langle m|$, and thus the sequence advances one step. We can describe this mathematically as a *random walk* on the nodes of a graph—with the conceptual caveat that here the graph is not embedded in an external space or time, but rather *is*, in itself, the whole of our reality.

This formulation will allow us, in later chapters, to introduce metric properties, notions of equilibrium, and other classical results from the theory of random walks, applying them to the idea of "emergent spacetime" and other physical concepts.

Transition Matrix or Temporal Evolution Matrix A

Since actions are free, *there is no cause beyond the observer itself* that determines which action occurs at each instant. As a first approximation, we assume that each observer acts upon the others according to some probability, leading us to study random walks on graphs in a broad sense. We will systematically use the term "graph" to include networks in which each edge can represent the probability of interaction, including directed and weighted cases, always with the necessary conditions of connectivity and aperiodicity to properly define a random walk.

This defines a transition matrix, or "temporal evolution" matrix $A(t)$:

$$A(t) \;=\; [\, A_{nm}(t)\,] \;\in\; \mathbb{R}^{N\times N},$$

where $A_{nm}(t)$ represents the probability that observer n acts upon observer m at instant t, given that $\langle n|$ holds the light at that moment. In other words, the matrix is row-normalized:

$$\sum_{m=1}^{N} A_{nm}(t) \;=\; 1 \quad \text{for each } n,$$

or, more compactly,

$$A(t)|\mathit{I}\,\rangle \;=\; |\mathit{I}\,\rangle$$

where $|\mathit{I}\,\rangle$ is the column vector of ones. This convention corresponds to a "forward" evolution and aligns the interpretation of A with stochastic operators used in Markov theory. A is a stochastic matrix.

State of the System

At each time t, the system is described by a vector:

$$\langle\psi(t)| \in \mathbb{R}^N$$

where $\psi_n(t)$ represents the probability of finding the light at observer n at time t. The evolution of the system is given by the iterated application of A:

$$\langle\psi(t)|A = \langle\psi(t+1)| \qquad \langle\psi(t)| = \langle\psi(0)|A^t$$

It is important to note that the matrix A — the transition or "temporal evolution" matrix of the system — can in general depend on time,

$A(t)$. However, we will focus on the case where A is *constant* (independent of t), which generalizes the typical study of homogeneous Markov chains.

Example: An Evolution Matrix A

To illustrate how temporal evolution works numerically, consider the following example of a constant 3×3 matrix A:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

Its interpretation is that, for example, if observer $\langle 2|$ holds the light at a given moment, then the probability it acts upon $\langle 1|$ is $2/3$, and the probability it acts upon $\langle 3|$ is $1/3$, etc. (read by rows).

Evolution of $\langle\psi|$. Now suppose we know that the light is initially at observer $\langle 1|$. In column vector notation:

$$\langle\psi_0| \;=\; \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

After one time step (i.e., one evolution step), the state becomes:

$$\langle\psi_1| = \langle\psi_0|A.$$

If we compute the multiplication explicitly:

$$\langle\psi_0|A \;=\; \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1/2 & 1/2 \end{pmatrix}$$

That is, in the next step, the light is at $\langle 2|$ and $\langle 3|$ with probability $1/2$ each.

After two steps, the resulting state will be:

$$\langle\psi_2| \;=\; \langle\psi_0|A^2 \;=\; \langle\psi_1|A,$$

$$\langle\psi_2| = \langle\psi_1|A = \begin{pmatrix} 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \end{pmatrix}$$

and so on. In this way, we see in practice how A functions as a *temporal evolution operator* on the state vector $\langle\psi|$.

Stationary Distribution

If the graph defined by A is strongly connected and aperiodic, the system admits a unique stationary state $\langle\omega| \in \mathbb{R}^N$, with $\omega_n > 0$ and $\sum_n \omega_n = 1$, such that:

$$\langle\omega|A = \langle\omega|$$

$$\langle\omega|\mathit{I}\,\rangle = 1$$

This vector $\langle\omega|$ represents the equilibrium distribution: if the system starts with $\langle\psi(0)| = \langle\omega|$, then $\langle\psi(t)| = \langle\omega|$ for all t. It is the unique (normalized) fixed vector under the dynamics of A.

In what follows, $\langle\omega|$ will play a crucial role. $\langle\omega|$ can be interpreted as the probability distribution of finding the light at each observer after an infinite time. Or as the number of times the light reaches each observer divided by the total number of *light movements*. Or as the probability that the light is at each observer if we choose a random moment in time.

Expected Action $\mathfrak{T}$ in Terms of Evolution A

Previously, we defined the accumulated action matrix $\mathfrak{T}$ as a way to record which observer has acted upon which, throughout history—that is, an action that has "already happened." We now want to express the *expectation* of action: how much action $\partial\mathfrak{T}$ can we expect over a period of time ∂t if we know the dynamics that govern the evolution of the system?

Suppose the temporal evolution matrix A is constant and irreducible, with stationary distribution $|\omega\rangle$. We can express $|\omega\rangle$ as:

$$\langle\omega| = \frac{1}{\partial t}\langle\partial\tau|$$

Here we introduce the symbol ∂ to indicate that we are assuming a *long* time interval, over which the law of large numbers can be validly applied. This implies that all the formulas in this section refer to sufficiently large times, where the expressions stabilize and are correct in the limit $\partial t \to \infty$.

Then, the expected number of times the light passes from observer n to observer m over a time interval ∂t, is given by:

$$\partial\mathfrak{T}_{nm} = \omega_n A_{nm} \partial t.$$

This can be written in matrix form as:

$$\partial\mathfrak{T} = \widehat{\omega} A \partial t,$$

This expression gives us, for long times $\partial t >>$, the accumulated expected action assuming that the evolution is governed by A.

From this perspective, it is more convenient to work with action per unit time:

$$\mathbb{A} := \frac{1}{\partial t}\partial\mathfrak{T} = \widehat{\omega} A$$

This represents the expected action per unit of time. Equivalently, it is the expected action in a time interval equal to 1. That is, $\mathbb{A}_{nm}$

is the expected number of actions from n to m divided by the total number of light movements. We can also interpret it as the probability that, choosing a random instant, n is acting upon m.

An interesting property of $\mathbb{A}$ is:

$$\langle \imath | \mathbb{A} = \langle \imath | \mathbb{A}^* = \langle \omega |$$

and therefore:

$$\langle \imath | \mathbb{A} | \imath \rangle = 1$$

Inverse Evolution: The Probability of a Past Action

In the previous section, we described *forward evolution* via the operator A, which encodes the probability of the light passing from one observer to another at a given instant. We now turn to the inverse problem: given the system's state at the present instant, how can we infer where the light "was" in the previous instant?

To approach this, we recall that the *accumulated sensation*, in our context, can be expressed as:

$$\mathcal{S} = \mathbb{A}^* \partial t,$$

where $\mathbb{A}^*$ denotes the transpose of $\mathbb{A}$.

From this sensation, we define the operator S through the following relation:

$$\mathbb{A}^* = \frac{1}{\partial t} \partial \mathcal{S} = \widehat{\omega} S$$

hence:

$$\mathbb{A}^* = \widehat{\omega} S = (\widehat{\omega} A)^* = A^* \widehat{\omega}$$

and finally:

$$S = \underset{\smile}{\omega} A^* \widehat{\omega}$$

We will see that S describes the *inverse evolution*: given the current state $\langle \psi(t) |$, S tells us the corresponding probability distribution at the previous instant.

Interpretation of S

Recall that $\langle\psi(t)|$ is a probability vector describing "where" the light is at time t. If we wish to estimate the probability that the light *came* from a particular observer $\langle n|$, S acts to give us that information. Formally:

$$\langle\psi(t)|S = \langle\psi(t-1)|$$

That is, S allows us to *rewind* the probability one step in time, under the assumption that the global dynamics is described by the matrix A.

It is important to note that S is also a stochastic matrix and that its equilibrium vector is also $\langle\omega|$:

$$\langle\omega|S = \langle\omega|$$

Backward Iteration in Time

Given a final state $\langle\psi(t)|$, we can apply S repeatedly to find where the light was in previous instants. For example, to rewind t steps in time:

$$\langle\psi(-t)| = \langle\psi(0)|S^t$$

Note that time moves "backward" in this inverted model, and each application of S corresponds to an inverse jump in the action chain.

In short, S is the operator that facilitates *inverse evolution*, allowing us to answer the question: "From which observer did the light most likely come, given that it is currently at $\langle\psi(t)|$?" This does not mean we can physically "undo" interactions, but that we can probabilistically assign a distribution over past states, consistent with the dynamics A and its transpose.

Explicit Computation of $\mathbb{A}$ and $\mathbb{S}$ in the Numerical Example

Let us return to the same transition matrix:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

whose three rows, in standard notation, describe how "observer n" acts upon the other nodes.

Stationary Distribution $\langle\omega|$. The (unique) stationary distribution is:

$$\langle\omega| = \tfrac{1}{41}\begin{pmatrix} 14 & 15 & 12 \end{pmatrix}$$

This can be verified by checking that $\langle\omega|A = \langle\omega|$ and that the components sum to 1: $\langle\omega|\mathit{I}\,\rangle = 1$.

Construction of $\mathbb{A}$.

$$\mathbb{A} = \widehat{\omega}A$$

This represents, in a certain sense, the "action" normalized per unit time (see previous sections). We now perform the multiplication explicitly, row by row of A:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}, \quad \widehat{\omega} = \begin{pmatrix} \frac{14}{41} & 0 & 0 \\ 0 & \frac{15}{41} & 0 \\ 0 & 0 & \frac{12}{41} \end{pmatrix}.$$

$$\mathbb{A} = \underset{\smile}{\omega}A = \tfrac{1}{41}\begin{pmatrix} 0 & 7 & 7 \\ 10 & 0 & 5 \\ 4 & 8 & 0 \end{pmatrix}$$

Construction of S. On the other hand, inverse evolution is described by the matrix:

$$S \;=\; \underset{\smile}{\omega}\, A^*\, \widehat{\omega},$$

Thus:

$$S \;=\; \mathbb{A}^*\, \underset{\smile}{\omega} \;=\; 41 \begin{pmatrix} 1/14 & 0 & 0 \\ 0 & 1/15 & 0 \\ 0 & 0 & 1/12 \end{pmatrix} \frac{1}{41} \begin{pmatrix} 0 & 10 & 4 \\ 7 & 0 & 8 \\ 7 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 5/7 & 2/7 \\ 7/15 & 0 & 8/15 \\ 7/12 & 5/12 & 0 \end{pmatrix}$$

It's easy to verify that:

$$\langle\omega|S = \langle\omega|$$

We want to illustrate how, with the computed matrix S, we can go backward in time. Assuming that at time "zero" the light is *certainly* at observer $\langle 1|$, we have:

$$\langle\psi(0)| \;=\; \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Given that:

$$S \;=\; \begin{pmatrix} 0 & 5/7 & 2/7 \\ 7/15 & 0 & 8/15 \\ 7/12 & 5/12 & 0 \end{pmatrix}$$

the state at time $t = -1$ is:

$$\langle\psi(-1)| \;=\; \langle\psi(0)|S \;=\; (\text{first row of } S) \;=\; \begin{pmatrix} 0 & \frac{5}{7} & \frac{2}{7} \end{pmatrix}.$$

Note that $\frac{5}{7} + \frac{2}{7} = 1$, so this vector is normalized and describes the probability that at time $t = -1$, the light was at $\langle 2|$ (with probability $\frac{5}{7}$) or at $\langle 3|$ (with $\frac{2}{7}$).

Two Steps Backward. To go two steps back, we apply S again:

$$\langle\psi(-2)| \;=\; \langle\psi(-1)|S \;=\; \langle\psi(0)|S^2$$

Thus,

$$\langle\psi(-2)| = \begin{pmatrix} 0 & \frac{5}{7} & \frac{2}{7} \end{pmatrix} \begin{pmatrix} 0 & 5/7 & 2/7 \\ 7/15 & 0 & 8/15 \\ 7/12 & 5/12 & 0 \end{pmatrix} = \frac{1}{42}\begin{pmatrix} 21 & 5 & 16 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{42} & \frac{8}{21} \end{pmatrix}$$

Interpretation. The meaning of $\langle\psi(-2)|$ is that, according to the inverse evolution model S, if at time $t = 0$ the light was at $\langle 1|$, then *two steps earlier*, the probability that it was at $\langle 1|$ is $\frac{1}{2}$, at $\langle 2|$ is $\frac{5}{42}$, and at $\langle 3|$ is $\frac{8}{21}$.

Thus we confirm that S indeed allows us to "go back" in the chain of actions with a probabilistic interpretation consistent with the dynamics defined by A.

Universes with Temporal Symmetry: $A = S$

So far we have considered the general case in which the forward evolution matrix (A) and the inverse evolution matrix (S) are, in principle, different. However, there is a special case of great importance, both in Markov theory and in statistical physics, where both matrices *coincide*. We will call this case a **universe with temporal symmetry**.

Formal Definition. We say that a universe has *temporal symmetry* if

$$A \;=\; S.$$

Recall that

$$S \;=\; \underset{\smile}{\omega}\, A^*\, \widehat{\omega},$$

where A^* denotes the transpose (or adjoint) of A, and $\widehat{\omega}$ is the diagonal matrix associated with the stationary distribution $\langle \omega|$. The condition of temporal symmetry $A = S$ thus implies that $A = \underset{\smile}{\omega}\, A^*\, \widehat{\omega}$, which can be rewritten as:

$$\widehat{\omega} A = \; A^* \underset{\smile}{\omega}$$

In the context of Markov chains, this property is equivalent to the system satisfying the *detailed balance* condition:

$$\omega_m\, A_{nm} \;=\; \omega_n\, A_{mn}, \quad \forall\, n, m,$$

which implies a fundamental symmetry in the transition probabilities when seen forward and backward.

Physical Interpretation: Time Reversal. In a universe with *temporal symmetry*, forward and inverse evolution are, in fact, the same. In other words, A and S describe the same dynamics, with no distinction in the direction of the time arrow. From a probabilistic point of view, if we know that $A = S$, we can "watch the movie in reverse" and find the same distribution of trajectories.

Symmetric Matrix $\mathbb{A}$. Another way to understand temporal symmetry is through the matrix $\mathbb{A} = \widehat{\omega} A$. In a universe with temporal symmetry, the condition $A = S$ translates to $\mathbb{A} = \mathbb{A}^*$. Indeed, if $A = \underset{\smile}{\omega}\, A^*\, \widehat{\omega}$ and $\mathbb{A} = \widehat{\omega} A$, then

$$\mathbb{A}^* = (\widehat{\omega} A)^* = A^* \widehat{\omega}^* = A^* \widehat{\omega} = \big(\widehat{\omega}\, A\, \widehat{\omega}\big)\underset{\smile}{\omega} = \widehat{\omega} A = \mathbb{A}.$$

Thus, $\mathbb{A}$ becomes a *symmetric matrix.* In graph theory terms, this corresponds to a **non-directed graph** (possibly with weights), where the action rate $\mathbb{A}_{nm}$ between n and m is equal in both directions.

Undirected Graphs and Physical Analogies. When $\mathbb{A}$ is symmetric, we can interpret the universe as an *undirected graph*, since the "weight" or "interaction probability" between nodes n and m is the same in both directions. This connects to the idea of "time reversal" in physics: a system with temporal symmetry does not privilege any direction of time. - If $A \neq S$, we say the system has **temporal asymmetry**, since forward dynamics does not match the inverse dynamics. - If $A = S$, or equivalently $\mathbb{A}$ is symmetric, the system has **temporal symmetry**, meaning the evolution remains invariant under time reversal.

Conclusion

In universes with *temporal symmetry*, evolution is completely *invariant* under time reversal: the forward propagation operator A coincides with the backward propagation operator S, and the matrix $\mathbb{A}$ becomes symmetric. These features allow us to treat such systems as **undirected graphs**, and their behavior reflects the absence of a preferred time arrow in the transition probabilities. In contrast, in universes with *temporal asymmetry*, $A \neq S$ and $\mathbb{A}$ is not symmetric, reflecting a fundamental asymmetry in the system's causal structure.
If $\mathbb{A} = \mathbb{A}^*$, then action and sensation are equal:

$$\partial \mathbf{A} = \mathbb{A} \partial t = \mathbb{A}^* \partial t = \partial \mathbf{S}$$

That is, each observer gives light to another in the same proportion as they receive it:

$$\partial\mathbf{A} = \partial\mathbf{S} \quad \Longleftrightarrow \quad \text{Temporal Symmetry}$$

Evolution Is Not Reversible: Information Loss

Although the matrices A and S allow us to model forward and backward evolution respectively, they are not inverse operators. In general, $S \neq A^{-1}$, and thus we also have $I \neq SA$. This reflects a fundamental property: evolution is *irreversible* in terms of information.

Explicit Example We consider a system with three observers and the previously known transition matrix:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

and its stationary distribution:

$$\langle\omega| = \tfrac{1}{41}\begin{pmatrix} 14 & 15 & 12 \end{pmatrix}$$

From this we construct the sensation matrix:

$$S = \underset{\smile}{\omega}\, A^{*}\, \widehat{\omega} = \begin{pmatrix} 0 & 5/7 & 2/7 \\ 7/15 & 0 & 8/15 \\ 7/12 & 5/12 & 0 \end{pmatrix}$$

First Applying A, Then S

We take a deterministic initial state:

$$\langle\psi(0)| = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Apply A:

$$\langle\psi(1)| = \langle\psi(0)|A = \begin{pmatrix} 0 & 1/2 & 1/2 \end{pmatrix}$$

And now apply S to the result:

$$\langle\psi'(0)| = \langle\psi(1)|S = \begin{pmatrix} 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 5/7 & 2/7 \\ 7/15 & 0 & 8/15 \\ 7/12 & 5/12 & 0 \end{pmatrix}$$
$$= \tfrac{1}{240}\begin{pmatrix} 126 & 50 & 64 \end{pmatrix} \approx \begin{pmatrix} 0.53 & 0.21 & 0.27 \end{pmatrix}$$

This new vector $\langle\psi(0)'|$ does not match the original $\langle\psi(0)|$, which shows that we do not recover the original state—there is *information loss*.

First Applying S, Then A

Now we perform the inverse process: first apply S to the initial state.

$$\langle\psi(0)|S = \begin{pmatrix} 0 & 5/7 & 2/7 \end{pmatrix}$$

Then apply A:

$$\langle\psi(0)|SA = \langle\psi(1)|A = \begin{pmatrix} 0 & 5/7 & 2/7 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$
$$= \frac{1}{21}\begin{pmatrix} 12 & 4 & 5 \end{pmatrix} \approx \begin{pmatrix} 0.57 & 0.19 & 0.24 \end{pmatrix}$$

Again, the result does not match the initial state.

Note that applying SA did not yield the same result as applying AS. This can be expressed by stating that the commutator is non-zero:

$$[A, S] := AS - SA \neq 0$$

This shows that the system's evolution is not *invertible*: light propagates according to dynamics that scatter information, and the past can only be reconstructed as a *probable distribution*, not as a deterministic state.

This phenomenon is analogous to the increase in entropy in thermodynamics: although the fundamental laws may be time-symmetric, practical knowledge of the past degrades with each step. Irreversibility is therefore a natural and structural feature.

Lower Entropy Implies Lower Information Loss

Intuitively, information loss in this model is directly related to the degree of dispersion caused by the evolution matrix A. The more random the action is (i.e., the more uniform or "entropized" each column of A is), the greater the degradation of the original state when attempting to reconstruct it from the future (or past).

Consider the following lower-entropy case:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 9/10 & 0 & 1/10 \\ 1/10 & 9/10 & 0 \end{pmatrix}$$

We start from the deterministic state:

$$\langle \psi(0)| = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

And obtain:

$$\langle \psi(0)|AS \approx \begin{pmatrix} 0.65 & 0.086 & 0.26 \end{pmatrix}^* \quad \longrightarrow \quad \text{reasonably close to } |\psi(0)\rangle$$
$$\langle \psi(0)|SA \approx \begin{pmatrix} 0.85 & 0.05 & 0.1 \end{pmatrix} \quad \longrightarrow \quad \text{even closer to } |\psi(0)\rangle$$

Although we do not recover the original state exactly, the deviation is much smaller than in the previous, more entropic example. This suggests that the rate of entropy increase—or the degree of information loss—is lower when interactions are more directed.

Extreme Case: Evolution Without Loss. Reversible System

We now consider the case of the pure shift matrix $\rhd$:

$$A = \rhd := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \implies \langle \omega| = \langle \mathit{I}\,| \implies S = \rhd^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Here we do have:

$$S = A^{-1} \quad \text{and} \quad SA = AS = I$$

No information is lost. Evolution is perfectly deterministic and cyclic. The past can be reconstructed exactly. Starting from the same state as in the previous example $\langle \psi(0)|$, we have:

$$\langle \psi(0)|A = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \langle \psi(0)|AS = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \langle \psi(0)|$$

The same holds if we reverse the order:

$$\langle \psi(0)|SA = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

1 Observation

An *observation* is the set formed by an action and the immediately following sensation. An observer *observes* when he *look at* another observer and *see* what happens. We will adopt the following terminology systematically:

- *To look* $\Longrightarrow$ to act.
- *To see* $\Longrightarrow$ to feel.

We will generalize the concept of observation to include sequences of multiple chained actions and sensations: look, see, look, see, etc. These *composite observations* will be called *experiments* when a distinction is needed, although in general we will continue to use the term *observation* for both.

Numbering and Naming of Observers

In the context of observation, we will often number the observers from 0 to N, meaning the universe contains $N+1$ observers. Observer $\langle 0|$ will be referred to as *I* or *the observer*. When referring to this observer, we will use the ***first person*** for economy of language. We will use expressions such as *I act on* $\langle n|$, meaning *observer* $\langle 0|$ *acts on* $\langle n|$. When speaking of *the other observers* or the *external world*, their indices will range from 1 to N.

The Observable Matrix Θ. Definition

To define the observation of the first observer (without loss of generality), we start from the **global action matrix** A_G, and decompose its structure into four blocks:

$$A_G = \begin{pmatrix} 0 & \langle a| \\ |a^+\rangle & A \end{pmatrix} \tag{1}$$

where:

- $\langle a|$ is the first row of A_G excluding the element $(A_G)_{00}$: it represents *my action* toward the others.
- $|a^+\rangle$ is the first column of A_G excluding the element $(A_G)_{00}$: it represents the *reaction toward me*. We will refer to it as the *halo*.
- A is the $N \times N$ submatrix of A_G corresponding to the interactions between the observers other than myself.

Note that in this context, we call *Action* a substochastic matrix—i.e., each row sums to ≤ 1.

We will assume that the are no *masses*, or the elements of the diagonal $(A_G)_{nn} = 0$ for all n. This hypothesis will be justified later. For now, intuitively: if others act on themselves, I cannot perceive it; and if I act on myself, I obtain no information, so it will not be considered an *observation*.

Formal definition. The observable matrix is defined as:

$$\Theta := \mathbf{G}^2 \widehat{a^+} \tag{2}$$

$$\mathbf{G}^2 := (I - A)^{-1} \tag{3}$$

Interpretation. The matrix $\mathbf{G}^2 = (I - A)^{-1}$ is known as the fundamental matrix (or Green's matrix), which accumulates the expected visits to each node before returning.

Θ is a stochastic matrix: its rows sum to 1, reflecting that, when performing an experiment by acting on $\langle m|$, the light returns to me from some node with total probability one.

The *observable matrix* Θ represents the **only information** that observer $\langle 0|$ can extract from the universe. Formally, its entry Θ_{nm} represents the probability that, if I act on observer $\langle n|$, the light returns to me from $\langle m|$—or more compactly, the probability of seeing $\langle m|$ when looking at $\langle n|$:

$$\Theta_{nm} := \text{Prob}(\text{I see } m \mid \text{I look at } n) \tag{4}$$

The Fundamental Matrix $\mathbf{G}^2$

We start from the equation for the stationary evolution of A_G:

$$\langle \omega_G | A_G = \langle \omega_G |$$

We can separate this expression into blocks as we did in equation (1):

$$\begin{pmatrix} \omega_0 & \langle \omega | \end{pmatrix} \begin{pmatrix} 0 & \langle a| \\ |a^+\rangle & A \end{pmatrix} = \begin{pmatrix} \omega_0 & \langle \omega | \end{pmatrix}$$

From this we obtain two equations. First:

$$\langle \omega | a^+ \rangle = \omega_0 \tag{5}$$

and the other:

$$\langle a | \omega_0 + \langle \omega | A = \langle \omega | \implies \langle \omega | (I - A) = \langle a | \omega_0 \implies \omega_0 \langle a | (I - A)^{-1} = \langle \omega |$$

which leads to the following equation:

$$\langle a | \mathbf{G}^2 \omega_0 = \langle \omega | \tag{6}$$

This expression can be interpreted as saying that $\mathbf{G}^2$ is the operator that, applied to my action, returns the frequency of the other observers.

From (6) and (5) we obtain the following fundamental relation:

$$\langle a|\mathbf{G}^2|a^+\rangle = 1 \tag{7}$$

The matrix $\mathbf{G}^2 := (I - A)^{-1}$ is known as the *Green's function* or *fundamental matrix* in the context of absorbing Markov chains.

Convergence of the Geometric Series

Since the matrix A is substochastic (its rows sum to at most 1), it is well known in operator theory and Markov chains that its spectral radius satisfies $\rho(A) < 1$. This guarantees that the matrix $I - A$ is invertible, and that the inverse can be expressed as a convergent geometric series:

$$\mathbf{G}^2 = (I - A)^{-1} = \sum_0^\infty A^t$$

Note the analogy with the geometric series for real numbers: $\sum_0^\infty r^t = \frac{1}{1-r}$

This sum converges absolutely, since all powers A^t are norm-bounded and decay exponentially in magnitude.

The Observer as an Absorbing State

To better understand $\mathbf{G}^2$, consider the block form of A_G under the assumption that observer $\langle 0|$ is absorbing—that is, it only acts on itself, so when the light reaches it, it never escapes:

$$A_G = \begin{pmatrix} 1 & \langle \varnothing | \\ |a^+\rangle & A \end{pmatrix}$$

Then A_G^t takes the form:

$$A_G^t = \begin{pmatrix} 1 & \langle \varnothing | \\ (I + A + A^2 + \cdots + A^{t-1})\,|a^+\rangle & A^t \end{pmatrix}$$

Starting at node $\langle n|$ $(n = 1, \ldots, N)$, while the light remains outside of $\langle 0|$ its dynamics are governed by the submatrix A (with $\rho(A) < 1$).

Interpretation of $\mathbf{G}^2_{nm}$

Being in $\langle m|$, starting from $\langle n|$ after t steps *without having touched* $\langle 0|$ has probability A^t_{nm}. Summing over all t gives the expected number of times the light visits $\langle m|$ before being absorbed:

$$\sum_0^\infty A^t_{nm} = \mathbf{G}^2_{nm}$$

Thus, we obtain the following direct interpretation:

$\mathbf{G}^2_{nm}$ is the expected number of times the light visits $\langle m|$ before being absorbed at $\langle 0|$, starting from $\langle n|$.

Strictly Positive Entries

Each power A^t represents the indirect action of order t between observers (excluding $\langle 0|$). Formally, the entry $A^t_{nm} := \langle n|A^t|m\rangle$ expresses the probability that the light is at $|m\rangle$ at time t assuming it started from $\langle n|$ at $t = 0$.

If the graph defined by A is strongly connected, then for every pair of observers $\langle n|$, $\langle m|$, there exists at least one value $t \in \mathbb{N}$ such that:

$$\langle n|A^t|m\rangle > 0 \text{ for some } t \in \mathbb{N}$$

Hence, summing over all t in the definition of the fundamental matrix yields:

$$\langle n|\mathbf{G}^2|m\rangle = \sum_{t=0}^{\infty} \langle n|A^t|m\rangle > 0 \quad \text{for all } n, m$$

since at least one term in the sum is strictly positive and all others are non-negative.

As a result, all entries of the matrix $\mathbf{G}^2$ are strictly positive:

$$\mathbf{G}^2_{nm} > 0 \quad \text{for all } n, m$$

Any matrix with strictly positive entries defines a strictly positive quadratic form, so $\mathbf{G}^2$ defines such a form:

$$\langle x|\mathbf{G}^2|x\rangle > 0 \quad \forall\, |x\rangle \neq |\varnothing\rangle$$

Positive Spectrum of $\mathbf{G}^2$

Let μ_i be the eigenvalues of A. Then the eigenvalues of $(I - A)$ are $1 - \mu_i$, and those of $\mathbf{G}^2$:

$$\lambda_i(\mathbf{G}^2) = \frac{1}{1 - \mu_i(A)}$$

Since $\rho(A) < 1$, all eigenvalues μ_i of A satisfy $|\mu_i| < 1$, so their real parts obey:

$$\Re(\mu_i) < 1 \implies \Re(1 - \mu_i) > 0$$

Therefore:

$$\Re\left(\frac{1}{1 - \mu_i}\right) > 0$$

This is justified by noting that if $z \in \mathbb{C}$, with $|z| < 1$, then $1 - z$ lies in a disk centered at 1 with strictly positive real part. The function $f(z) = \frac{1}{1-z}$ is holomorphic in that disk, and satisfies:

$$\Re\left(\frac{1}{1-z}\right) > 0 \quad \text{if } |z| < 1$$

Hence, all eigenvalues of $\mathbf{G}^2$ have strictly positive real part.

Summary

Let us review the properties and relevance of $\mathbf{G}^2$.

1. Definition and Existence

$$\mathbf{G}^2 \;=\; \sum_{t=0}^{\infty} A^t = (I-A)^{-1}, \qquad \rho(A) < 1 \implies \text{the series converges.}$$

- A is **substochastic**: rows sum to ≤ 1.
- There exists at least one escape route to $\langle 0|$ $\Rightarrow$ some row of A sums to < 1 $\Rightarrow \rho(A) < 1$.

2. Probabilistic Interpretation

$\mathbf{G}^2_{nm}$ is the expected number of times the light visits $|m\rangle$ before being absorbed at $|0\rangle$, starting from $|n\rangle$.

In classical absorbing Markov chain theory, $\mathbf{G}^2$ is the *fundamental matrix* (Kemeny–Snell).

3. Algebraic Properties

- $(I-A)\mathbf{G}^2 = \mathbf{G}^2(I-A) = I$.
- Eigenvalues: if $A|v\rangle = \lambda|v\rangle$, then $\mathbf{G}^2|v\rangle = \frac{1}{1-\lambda}\,|v\rangle$. All eigenvalues have strictly positive real part.
- Entries are strictly positive if the external graph is strongly connected.

4. Physical and Mathematical Connections

1. **Electrical networks**: $I-A$ acts like a directed Laplacian; $\mathbf{G}^2$ is the effective impedance between nodes.
2. **Difference equations**: To solve $\langle x|(I-A) = \langle b|$, we compute $\langle v| = \langle b|\mathbf{G}^2$.
3. **Diffusion mechanics**: $\mathbf{G}^2$ is the response to a point source in discrete media.

5. Connection with Observation As previously stated and further explored in the next section, the observable matrix can be expressed as:

$$\Theta = \mathbf{G}^2\widehat{a+}$$

Deriving the Observable Matrix Θ

We start at an external node $\langle n|$ $(n = 1, \ldots, N)$. While the light remains outside of $\langle 0|$, its dynamics are governed by the submatrix A (with $\rho(A) < 1$). Let a_m^+ denote the *halo*: the probability that the light, upon reaching $\langle m|$, immediately escapes to $\langle 0|$.

Escape Exactly at Step t

Being at $\langle m|$ after t steps *without having touched* $\langle 0|$ has probability $(A^t)_{nm}$. At that instant, it escapes with probability a_m^+. Therefore,

$$\Pr\big(\text{reaches me via } |m\rangle \text{ at step } t\big) = (A^t)_{nm} a_m^+.$$

Summing Over All Path Lengths

$$\Pr\big(\text{reaches me via } \langle m|\big)$$
$$= \sum_{t=0}^{\infty} (A^t)_{nm} a_m^+ = a_m^+ \underbrace{\sum_{t=0}^{\infty} (A^t)_{nm}}_{\mathbf{G}^2_{nm}} = \mathbf{G}^2_{nm} a_m^+ = \Theta_{nm}$$

And finally, organizing these probabilities into a matrix:

$$\Theta = \mathbf{G}^2 \widehat{a^+},$$

- Each **row** $\langle n|\Theta$ is the probability distribution describing the likelihood that the light returns to $\langle 0|$ via each node $\langle m|$, when the observer acted on $|n\rangle$.
- The rows of Θ sum to 1: they are proper probability distributions.

Explicit Example

We consider the typical example with $N = 3$ and compute the observation of the first observer:

$$A_G = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix} \tag{8}$$

We then separate:

$$|a^+\rangle = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}, \qquad \langle a| = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1/3 \\ 2/3 & 0 \end{pmatrix}$$

We compute:

$$I - A = \begin{pmatrix} 1 & -1/3 \\ -2/3 & 1 \end{pmatrix} \tag{9}$$

Its inverse is:

$$\mathbf{G}^2 = (I - A)^{-1} = \begin{pmatrix} 9/7 & 3/7 \\ 6/7 & 9/7 \end{pmatrix}$$

We can interpret the rows of the matrix $\mathbf{G}^2$ as follows: If I act on $\langle 1|$, the light visits $9/7$ times $\langle 1|$ and $3/7$ times $\langle 2|$ before returning to me. If I act on $\langle 2|$, the light visits $9/7$ times $\langle 2|$ and $6/7$ times $\langle 1|$ before returning to me.

The observable matrix is obtained as:

$$\Theta_o = \begin{pmatrix} 9/7 & 3/7 \\ 6/7 & 9/7 \end{pmatrix} \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{pmatrix} \tag{10}$$

We can interpret the rows of the matrix Θ as follows: If I act on observer $\langle 1|$, the light returns $6/7$ of the time from $|1\rangle$ and $1/7$ from $|2\rangle$. If I act on observer $\langle 2|$, the light returns $4/7$ of the time from $|1\rangle$ and $3/7$ from $|2\rangle$.

We can compute the observation matrices of the remaining observers using the same procedure:

$$\Theta_1 = \begin{pmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{pmatrix} \qquad \Theta_2 = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} \tag{11}$$

Observation as the Application of the Operator Θ

The matrix Θ lies at the center of all observations. If I look at an observer, the probability of seeing each of the observers is precisely given by Θ.

In fact, in any experiment consisting of *looking according to a distribution* of action $\langle a|$, the expected distribution of resulting sensations will be:

$$\langle a|\Theta = \langle s| \tag{12}$$

If we recall equation (6) and multiply both sides on the right by $\widehat{a^+}$, we obtain:

$$\langle s|\omega_0 = \langle \omega|\widehat{a^+} \tag{13}$$

Example 1: staring at a single observer $\langle n|$ If the experiment consists of repeatedly observing a single observer $\langle n|$ (a sufficient number of times to apply the law of large numbers), that is, if:

$$\langle a| = \langle n|$$

then the resulting sensation will simply be:

$$\langle s| = \langle a|\Theta = \langle n|\Theta$$

that is, the n-th row of the observable matrix.

Example 2: action according to A_G In the case where my action follows the original dynamics given by the first row of A_G, that is:

$$\langle a| = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$$

we have:

$$\langle s| = \langle a|\Theta = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{pmatrix} = \begin{pmatrix} 10/14 & 4/14 \end{pmatrix} = \begin{pmatrix} 5/7 & 2/7 \end{pmatrix}$$

Recalling the global sensation matrix for this example (calculated in previous sections):

$$S_G = \begin{pmatrix} 0 & 5/7 & 2/7 \\ 7/15 & 0 & 8/15 \\ 7/12 & 5/12 & 0 \end{pmatrix}$$

we see that the sensation of $\langle 0|$ is $\langle s| = \begin{pmatrix} 5/7 & 2/7 \end{pmatrix}$, as expected.

Example: an arbitrary action We can arbitrarily choose an action and use the observable to obtain the resulting sensation. For example: What happens if I conduct the following experiment? I act according to this distribution, and "see" what happens:

$$\langle a| = \begin{pmatrix} 2/3 & 1/3 \end{pmatrix}$$

then the resulting sensation is:

$$\langle s| = \langle a|\Theta = \begin{pmatrix} 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{pmatrix} = \begin{pmatrix} \frac{12}{21} + \frac{4}{21} & \frac{2}{21} + \frac{3}{21} \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 16 & 5 \end{pmatrix}$$

This illustrates how, through the observable matrix, the distribution of sensations depends linearly on the way I act.

The Observation matrix $\lhd$

The matrix Θ is not merely a technical construction. It governs the phenomenon of observation. Every time I act upon the universe, the distribution of what I see is determined by it. It represents the universe's *reaction* to my action. Everything I can know, everything I can infer about the other observers, must go through Θ. In this sense, this matrix constitutes the bridge between *what I do and what I feel.*

As we have seen our observation depends on the observable matrix and our own action. We define the Observation matrix $\lessdot$ as:

$$\lessdot = \widehat{a}\Theta = \widehat{a}\,\mathbf{G}^2\widehat{a^+}$$

The element $\lessdot_{nm}$ can be understood as the number of times that I look at n and I see m per unit of proper time.

It is important to note that the observation matrix $\lessdot$ is the one that the observer can directly access. However, the observable matrix Θ can be readily deduced from it by extracting the observer's action from the observation.

$$\Theta = \underset{\smile}{a}\lessdot$$

Positive Spectrum of Θ

The eigenvalues of Θ also have strictly positive real part if the halo is invertible. Indeed:

The matrix $\tilde{\mathbf{G}}^2 := \widehat{a^+}^{1/2}\mathbf{G}^2\widehat{a^+}^{-1/2}$ is similar to $\mathbf{G}^2$, since $\widehat{a^+}^{1/2}$ is invertible (as long as $|a^+\rangle$ has no zero entries). Therefore:

$$\mathrm{Spec}(\tilde{\mathbf{G}}^2) = \mathrm{Spec}(\mathbf{G}^2)$$

Moreover,

$$\Theta = \widehat{a^+}^{1/2} \cdot \tilde{\mathbf{G}}^2 \cdot \widehat{a^+}^{1/2}$$

and since $\widehat{a^+}^{1/2}$ is invertible, we conclude that Θ is similar to $\tilde{\mathbf{G}}^2$. By transitivity, we obtain:

$$\mathrm{Spec}(\Theta) = \mathrm{Spec}(\tilde{\mathbf{G}}^2) = \mathrm{Spec}(\mathbf{G}^2)$$

Since all eigenvalues of $\mathbf{G}^2$ have strictly positive real part, the same holds for the eigenvalues of Θ.

Mute or Invisible Observers and the Structure of the Halo

The observable matrix $\Theta = \mathbf{G}^2\widehat{a^+}$ depends directly on the halo vector $|a^+\rangle$, which represents how much of the light arriving at each observer is sent directly to $|0\rangle$.

The Case of Zero Halos

It may happen that for some m, we have $a^+_m = 0$. This means that observer $|m\rangle$ never acts on $\langle 0|$. In that case:

- Column m of Θ is entirely zero. - The vector $\langle m|$ belongs to the kernel of Θ. - The dimension $\langle m|$, from the standpoint of observation, becomes *mute*: no action on the system yields sensations originating from $\langle m|$.

However, this does not mean that $|m\rangle$ does not participate in the dynamics. It may be involved as an intermediary along the path the light takes from other observers, and thus still appear in other columns of Θ.

In particular, the row $\langle m|$ of Θ reflects the observation resulting from acting on $\langle m|$, i.e., the probability that light returns from each observer when acting on $\langle m|$ (even if $|m\rangle$ never sends light directly to $\langle 0|$).

This behavior is consistent with the operational interpretation of the observable matrix: it represents what the observer can perceive. If a given node never sends light back, it is invisible from the perspective of $\langle 0|$, which is faithfully reflected in a zero row.

Invertibility of the Observable

When all entries of $|a^+\rangle$ are strictly positive, that is,

$$|a^+\rangle > 0 \quad \text{(entrywise)},$$

then the matrix $\widehat{a^+}$ is diagonal and invertible, and hence:

$$\Theta = \mathbf{G}^2 \widehat{a^+} \quad \text{is invertible.}$$

Moreover, Θ is similar to $\mathbf{G}^2$:

$$\Theta = \widehat{a^+}^{1/2} \cdot \left(\widehat{a^+}^{-1/2} \mathbf{G}^2 \widehat{a^+}^{1/2} \right) \cdot \widehat{a^+}^{1/2}$$

and therefore they share the same spectrum. In particular, since all eigenvalues of $\mathbf{G}^2$ have strictly positive real part, the same holds for the eigenvalues of Θ.

Summary

The structure of the vector $|a^+\rangle$ directly determines:

- Which observers are visible from $\langle 0|$ as sources of light. - Which directions are included in the effective range of Θ. - The dimension of the kernel of Θ, in case some $a^+_m = 0$.

When there are no mute observers, the observable matrix is invertible and shares the spectral properties of $\mathbf{G}^2$, including the positivity of the real part of its eigenvalues.

Knowing the halo

Let us consider a universe with only three observers. The first observer, whom we will call "o" or I, observes the other two. Below we display the full action matrix A and the Observable matrix Θ of o. Θ is the only direct information that o can access. We will now verify how the observer can fully reconstruct A from this observation alone assuming no masses. I, as the observer, know my observation, and therfoer the observable matrix.

$$\Theta = \begin{pmatrix} \Theta_{011} & 1-\Theta_{011} \\ 1-\Theta_{022} & \Theta_{022} \end{pmatrix} := \begin{pmatrix} \Theta_1 & 1-\Theta_1 \\ 1-\Theta_2 & \Theta_2 \end{pmatrix}$$

And I want to know the full Action matrix:

$$A = \begin{pmatrix} 0 & A_{01} & A_{01} \\ A_{10} & 0 & A_{12} \\ A_{20} & A_{21} & 0 \end{pmatrix}$$

We can extract $\mathbf{G}^2$:

$$A_o = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \to \mathbf{G}^2 = (I - A_o)^{-1} = \frac{1}{1 - A_{12}A_{21}} \begin{pmatrix} 1 & A_{12} \\ A_{21} & 1 \end{pmatrix}$$

From the definition of Θ, we also have:

$$\Theta = \mathbf{G}^2 \widehat{a^+} \quad \to \quad \Theta^{-1} = \underset{\smile}{a^+}(\mathbf{G}^2)^{-1} = \underset{\smile}{a^+}(I - A_o) \to$$
$$(I - A_o) = \widehat{a^+}\Theta^{-1}$$

Now we have to assume that the masses (the observers's action upon themselves) are all equal to zero. There is no way for an observer to notice the mass of other observers. Assuming this, the diagonal elements of $\widehat{a^+}\Theta^{-1}$ must be 1:

$$\mathrm{Diag}(\widehat{a^+}\Theta^{-1}) = |I\rangle \to \underset{\smile}{a^+} = \mathrm{Diag}(\Theta^{-1})$$

Let us compute Θ^{-1}:

$$\Theta^{-1} = \frac{1}{\Theta_1 + \Theta_2 - 1} \begin{pmatrix} \Theta_2 & -(1-\Theta_1) \\ -(1-\Theta_2) & \Theta_1 \end{pmatrix}$$

Therefore:

$$\frac{1}{a_1^+} = \frac{\Theta_2}{\Theta_1 + \Theta_2 - 1}, \qquad \frac{1}{a_2^+} = \frac{\Theta_1}{\Theta_1 + \Theta_2 - 1}$$

which leads directly to the expression of the halo in terms of the observable:

$$|a^+\rangle = (\Theta_1 + \Theta_2 - 1) \begin{pmatrix} 1/\Theta_2 \\ 1/\Theta_1 \end{pmatrix}$$

This expression allows the observer to deduce their halo from their own observation, thereby fully reconstructing the global action matrix. Given the halo, the full action matrix is simply:

$$A = \begin{pmatrix} 0 & a_1 & a_2 \\ a_1^+ & 0 & 1 - a_1^+ \\ a_2^+ & 1 - a_2^+ & 0 \end{pmatrix}$$

where

$$\langle a| = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$$

is known by the observer, since it is their own action.

Thus, in a universe with 3 observers, no masses, and strong connectivity, any observer can deduce the full action matrix from their own observation alone.

It is important to note that if any of the other observers had mass, observer o would not be able to detect it.

As long as the halo has no zeros, the observable Θ is invertible. And so, we could theorically operate the same way despite the number of observers, obtain my observable matriz (with infinite precision) get its inverse and assume the n^{th} element of the diagonal is $1/a^+_n$. That way we could get the full halo $|a^+\rangle$, and therefore $I - A$. My own action $\langle a|$ is known by me. So I can reconstruct the whole action matrix A_G.

$$A_G = \begin{pmatrix} 0 & \langle a| \\ |a^+\rangle & A \end{pmatrix}$$

So:

I could deduce the entire action of the whole Universe A_G

if I could get my full observable matrix Θ with infinite precision

Numerical Example

We assume the typical example:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

$$\Theta_o = \begin{pmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{pmatrix}, \quad \Theta_1 = \begin{pmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

We now apply the halo expression to reconstruct the Action from the observable:

$$|a^+\rangle = (\Theta_1 + \Theta_2 - 1) \begin{pmatrix} 1/\Theta_2 \\ 1/\Theta_1 \end{pmatrix}$$

According to observer o:

$$\Theta_o = \begin{pmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{pmatrix} \to |a^+\rangle = (6/7 + 3/7 - 1) \begin{pmatrix} 7/3 \\ 7/6 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

According to observer 1:

$$\Theta_1 = \begin{pmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{pmatrix} \rightarrow |a^+\rangle = (3/5 + 4/5 - 1)\begin{pmatrix} 5/4 \\ 5/3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2/3 \end{pmatrix}$$

And according to observer 2:

$$\Theta_2 = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} \rightarrow |a^+\rangle = (3/4 + 1/2 - 1)\begin{pmatrix} 2/1 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

Direct and Indirect Action Between Observers

The Set of Observers Ω and the Global Action Matrix A_G

Let A_G be a global action matrix defined over the entire universe. We focus on a subset of observers Ω and the flow of light among them. Light may travel from one observer $\langle n|$ to another $|m\rangle$ either directly — $\langle n|$ acts on $|m\rangle$ — or indirectly: $\langle n|$ acts on an observer external to Ω, and the light returns to Ω through $|m\rangle$.

To define the direct and indirect action among the observers in Ω, we start from the **global action matrix** A_G, which we decompose into four blocks:

$$A_G = \begin{pmatrix} M & \langle\langle a| \\ |a^+\rangle\rangle & A \end{pmatrix} \tag{14}$$

The Energy Equation

To obtain the expression for energy, we begin from the stationary evolution equation of A_G:

$$\langle \omega_G|A_G = \begin{pmatrix} \langle\omega_\Omega| & \langle\omega| \end{pmatrix} \begin{pmatrix} M & \langle\langle a| \\ |a^+\rangle\rangle & A \end{pmatrix} = \begin{pmatrix} \langle\omega_\Omega| & \langle\omega| \end{pmatrix}$$

From which we extract two equations. The first:

$$\langle\omega_\Omega|M + \langle\omega||a^+\rangle\rangle = \langle\omega_\Omega| \tag{15}$$

And the second:

$$\langle\omega_\Omega|\langle\langle a| + \langle\omega|A = \langle\omega| \quad \Rightarrow \quad \langle\omega|(I - A) = \langle\omega_\Omega|\langle\langle a|$$

Which leads to:

$$\langle\omega| = \langle\omega_\Omega|\langle\langle a|\mathbf{G}^2 \tag{16}$$

Multiplying both sides on the right by $|a^+\rangle\rangle$, we obtain:

$$\langle\omega||a^+\rangle\rangle = \langle\omega_\Omega|\langle\langle a|\mathbf{G}^2|a^+\rangle\rangle$$

Combining this with equation (15), we obtain:

$$\langle\omega_\Omega| \left(M + \langle\langle a|\mathbf{G}^2|a^+\rangle\rangle\right) = \langle\omega_\Omega|$$

Which yields the final stationary evolution equation for Ω:

$$\langle\omega_\Omega|(M + E) = \langle\omega_\Omega| \qquad \text{with} \qquad E := \langle\langle a|\mathbf{G}^2|a^+\rangle\rangle$$

Direct Action Matrix: The Mass M

The *mass* matrix M of the *body* Ω is the matrix of direct action between the nodes in Ω, defined as the visible part of the original A_G matrix corresponding to direct interaction:

$$M := \Big((A_G)_{n\to m}\Big)_{n,m\in\Omega}$$

Indirect Action Matrix: The Energy E

The *energy* matrix E of the *body* Ω represents the indirect interaction, i.e., the total expected interaction from each n to each m through intermediate paths that pass exclusively through nodes not in Ω. Its general expression is:

$$E = \langle\langle a|\mathbf{G}^2|a^{+}\rangle\rangle \qquad \mathbf{G}^2 := (I - A)^{-1}$$

The entry $\mathbf{G}^2_{nm}$ can be interpreted as the expected number of times light passes through m starting from n before returning to Ω. The entry E_{nm} corresponds to the probability that light returns to Ω through m when departing from n.

Total Action Matrix

It is defined as the sum:

$$A := M + E$$

This matrix A can be interpreted as the matrix of total accessibility (direct + indirect) between nodes in Ω.

Numerical Example

We consider the matrix A_G:

$$A_G = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix} \qquad \langle\omega_G| = \tfrac{1}{41}\begin{pmatrix} 14 & 15 & 12 \end{pmatrix}$$

We define Ω as the set of observers $\langle 1|$ ("You") and $\langle 0|$ ("Me"), and treat node $|2\rangle$ as the *external world*. Then we have:

$$M = \begin{pmatrix} 0 & 1/2 \\ 2/3 & 0 \end{pmatrix}, \qquad A = 0, \qquad \langle\omega_\Omega| = \frac{1}{41}\begin{pmatrix} 14 & 15 \end{pmatrix}$$

The fundamental matrix is:

$$\mathbf{G}^2 = (I - A)^{-1} = 1$$

From the rows and columns of A that connect with Ω, we extract:

$$|a^+\rangle\rangle = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix}, \quad \langle\langle a| = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

Thus:

$$E = \langle\langle a|\mathbf{G}^2|a^+\rangle\rangle = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} \cdot 1 \cdot \begin{pmatrix} 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1/6 & 1/3 \\ 1/9 & 2/9 \end{pmatrix}$$

So we obtain:

$$A = M + E = \begin{pmatrix} 0 & 1/2 \\ 2/3 & 0 \end{pmatrix} + \begin{pmatrix} 1/6 & 1/3 \\ 1/9 & 2/9 \end{pmatrix} = \begin{pmatrix} 1/6 & 5/6 \\ 7/9 & 2/9 \end{pmatrix}$$

Note that the resulting matrix is stochastic, as expected.
Also observe that:

$$\langle\omega_\Omega|A = \langle\omega_\Omega|, \qquad \begin{pmatrix} 14/41 & 15/41 \end{pmatrix} \begin{pmatrix} 1/6 & 5/6 \\ 7/9 & 2/9 \end{pmatrix} = \begin{pmatrix} 14/41 & 15/41 \end{pmatrix}$$

Resistance distance

Motivation

In the system of observers defined so far, there is no explicit notion of spatial distance. However, since interaction between observers occurs causally, through successive actions, we can define a notion of distance based on the expected time it takes for light to *travel* from one observer to another.

This distance does not rely on any pre-existing space: it is emergent, and its definition is supported by the spectral properties of the system, as we shall see later.

Definition

Resistance distance has been thoroughly studied in the context of random walks on graphs and the analysis of electrical circuits. Here, we introduce an equivalent alternative definition that allows us to understand its physical meaning.

To define it, let $\partial\nu_{nm}$ be the *number of journeys made by light from n to m during the time interval ∂t*. Then, the resistance distance between observer $|n\rangle$ and observer $|m\rangle$ is defined as:

$$\mathbb{R}_{nm} := \frac{\partial t}{\partial \nu_{nm}}$$

It can be interpreted as the *instants per journey*, or simply as:

The distance between n and m, $\mathbb{R}_{nm}$, is the time it takes for light to make a journey from n to m

Interpretation

To illustrate this, we may imagine an *absolute clock* as a *counter of instants* that records every single movement of light. With it, we obtain ∂t. On the other hand, to count the journeys from n to m, we begin by waiting until light reaches n — this marks the *start of a journey.* We then wait until light reaches m — the *journey is complete*, so we increment $\partial\nu_{nm} = 1$. We repeat the process: wait for a new start at n, and end at m, incrementing the counter each time. Eventually, we will have counted the total number of journeys $\partial\nu_{nm}$ during the interval ∂t. Dividing the total time by the number of journeys yields the average time per journey, which is precisely the resistance distance.

The resistance distance $\mathbb{R}_{nm}$ satisfies the following fundamental properties, which make it a proper metric:

- **Non-negativity**: $\mathbb{R}_{nm} \geq 0$

- **Symmetry**: $\mathbb{R}_{nm} = \mathbb{R}_{mn}$
- **Triangle inequality**: $\mathbb{R}_{nl} \leq \mathbb{R}_{nm} + \mathbb{R}_{ml}$
- **Identity of indiscernibles**: $\mathbb{R}_{nn} = 0$

These properties can be proven rigorously. Here, we provide an intuitive explanation.

Non-negativity: This is evident, since $\mathbb{R}_{nm}$ represents an average time, which cannot be negative.

Symmetry: Symmetry is easy to understand. Once light makes a journey from n to m, it cannot make the same journey again until it returns from m to n. Thus, light always completes the same number of trips in both directions. From the clock's perspective, we always have $\partial\nu_{nm} = \partial\nu_{mn}$.

Triangle inequality: This holds because the time it takes for light to travel from n to l along any path is always less than or equal to the time it takes when forced to pass through m. If light travels from n to m and then from m to l, the total time is greater than or equal to the direct time from n to l, except in the special case when the only path from n to l goes through m, in which case both times are equal.

Identity of indiscernibles: $\mathbb{R}_{nn} = 0$. This property is chosen by convention, but will be justified later.

With these four properties, resistance distance behaves as a valid mathematical metric, which will allow us to define *a space* in which to locate the observers.

Expression in Terms of the Pseudoinverse of the Laplacian

As we will see later, the resistance distance between two observers n and m can be defined as:

$$\mathbb{R}_{nm} := \langle n - m | L^\dagger | n - m \rangle$$

where $L^\dagger$ is the Moore–Penrose pseudoinverse of the Laplacian operator, defined as:

$$L := \widehat{\omega}(I - A)$$

An explicit construction of $L^\dagger$ is given by:

$$L^\dagger = \left(L - \frac{1}{N}\,\Pi \right)^{-1} + \frac{1}{N}\,\Pi$$

with:

$$\Pi := |\mathit{I}\rangle\langle \mathit{I}|$$

Note: Although the formula for resistance distance in terms of the Laplacian pseudoinverse is introduced at this point, its full derivation will be presented later in the chapter. The intention of showing it here is to provide an immediate tool for computing resistance distance and, at the same time, to link it with previous studies. As we proceed, we will explore in detail how this formulation relates to the dynamics and spectral properties of the system.

Subjective Distance

We define the subjective distance $R_{nm}(o)$ between n and m from the point of view of an observer $|o\rangle$ as:

$$R_{nm}(o) := \frac{\partial \tau_o}{\partial \nu_{nm}}$$

where $\partial \tau_o$ is the proper time interval of observer $|o\rangle$, and $\partial \nu_{nm}$ is the number of light journeys from n to m during that interval. This can be restated as:

The subjective distance from my point of view between n and m, $R_{nm}(o)$, is the amount of my proper time $\partial \tau_o$ that light takes to make a journey from n to m.

With this definition, we can relate it to the resistance distance. Recall that:

$$\omega_o = \frac{\partial \tau_o}{\partial t}, \qquad \mathbb{R}_{nm} = \frac{\partial t}{\partial \nu_{nm}}$$

Then:

$$R_{nm}(o) = \frac{\partial \tau_o}{\partial \nu_{nm}} = \omega_o \frac{\partial t}{\partial \nu_{nm}} = \omega_o \mathbb{R}_{nm}$$

Thus, we arrive at the expression:

$$R_{nm}(o) = \omega_o \mathbb{R}_{nm} \tag{17}$$

Distance from the Observer's Point of View. The Relative Distance

A particularly important case is when $|o\rangle = |n\rangle$, that is, we are measuring distances *to* observer n. We adopt the following notation:

$$R_{nm} := R_{nm}(n) = \omega_n \mathbb{R}_{nm}$$

That is, if no observer is explicitly referenced, it is understood that the observer is the first index of R_{nm}, in this case n. We refer to this as the *relative distance from n to m*.

With this convention, we can express:

The distance from me to you, R_{nm}, is the time it takes for light to travel from me to you

It is understood that *I* am the observer (n), and both distance and time are measured from my point of view.

Using this terminology, the distance from m to n, as seen from m, is:

$$R_{mn} = \omega_m \mathbb{R}_{mn}$$

And since resistance distance is symmetric:

$$\mathbb{R}_{mn} = \mathbb{R}_{nm}$$

we obtain the following relation between relative distances:

$$\omega_m R_{nm} = \omega_n R_{mn} \tag{18}$$

This highlights that $R_{nm} \neq R_{mn}$. The *relative distance* is not symmetric, but this does not break the symmetry of subjective distance, because the equation compares distances measured from different points of view. We could alternatively write the same equation as:

$$\omega_m R_{nm}(n) = \omega_n R_{mn}(m)$$

Escape Probability

Definition

The escape probability between two nodes n and m, denoted σ_{nm}, is the probability that light, starting at node n, reaches m before returning to n.

Relation between σ and Resistance Distance

The escape probability between nodes n and m can be expressed as:

$$\sigma_{nm} := \frac{\partial \nu_{nm}}{\partial \tau_n}$$

In other words, *the number of light journeys from n to m per unit of proper time.*

This expression is precisely the inverse of the relative distance:

$$\sigma_{nm} = \frac{1}{R_{nm}}$$

Interpretation

Let us imagine that I am the observer n. Then σ_{nm} is the probability that light, starting from me, reaches m before returning to me. From my point of view, I act repeatedly, and σ_{nm} is the probability of reaching m in each of those *attempts*. Thus, the distance is the average number of *attempts* required to reach m:

$$R_{nm} = 1 \cdot \sigma + 2 \cdot (1-\sigma)\sigma + 3 \cdot (1-\sigma)^2\sigma + \cdots = \sigma \sum_{k=1}^{\infty} k(1-\sigma)^{k-1}$$

where $(1-\sigma)^{k-1}\sigma$ is the probability that light reaches m on the k-th attempt.

Now, consider the infinite series:

$$S = \sum_{k=1}^{\infty} k(1-\sigma)^{k-1}$$

This is a standard series solvable using the derivative of a geometric series. We know the geometric series:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad \text{for } |x| < 1$$

Differentiating both sides with respect to x, we get:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Substituting $x = 1 - \sigma$ into this expression yields:

$$\sum_{k=1}^{\infty} k(1-\sigma)^{k-1} = \frac{1}{\sigma^2}$$

Therefore, the total sum is $1/\sigma^2$, and plugging this into the expression for R_{nm}, we obtain:

$$R_{nm} = \sigma \cdot \frac{1}{\sigma^2} = \frac{1}{\sigma_{nm}}$$

From this, we derive an alternative (and more commonly seen) definition of the resistance distance, frequent in the literature:

$$\mathbb{R}_{nm} := \frac{1}{\omega_n \sigma_{nm}}$$

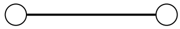
This leads us to define the "conductance" ρ_{nm} as:

$$\rho_{nm} := \omega_n \sigma_{nm} \qquad \rho_{nm} = \frac{1}{\mathbb{R}_{nm}}$$

Examples

We now illustrate the computation of distances through a few examples.

The Pair



The action matrix is:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

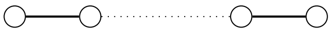
In this case, $\sigma_{12} = \sigma_{21} = 1$, and $\omega_1 = \omega_2 = 1/2$. Therefore:

$$\mathbb{R}_{12} = \frac{1}{\omega_1 \sigma_{12}} = \frac{1}{1/2 \cdot 1} = 2$$

$$\mathbb{R}_{21} = \mathbb{R}_{12} = 2 \quad \Rightarrow \quad R_{12} = R_{21} = 1$$

That is, light travels between them every 2 absolute instants, or once per instant from the perspective of each observer.

The Segment



For $N = 3$, the action matrix is:

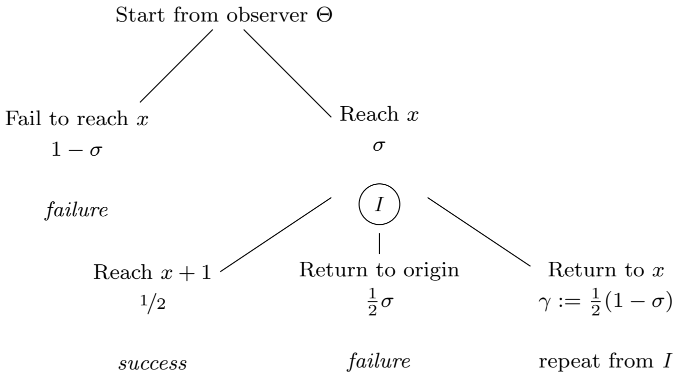
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbb{A} = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \langle \omega | = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

Let us imagine that I am the observer at the left end, observer $\langle 0|$:

To compute the distances, we reason in terms of escape probabilities.
What is the probability that I reach observer x?
We prove by induction that the solution is $\sigma(x) = \frac{1}{x}$.
This holds for $x = 1$ and $x = 2$ since:

$$\sigma_1 = 1, \qquad \sigma_2 = \frac{1}{2}$$

Now let us compute $\sigma(x + 1)$ in terms of $\sigma(x)$:



So we get:

$$\sigma(x+1) = \sigma(x) \cdot \left[\frac{1}{2} + \frac{1}{2}\gamma + \frac{1}{2}\gamma^2 + \cdots\right] = \frac{\sigma(x)}{2} \cdot \frac{1}{1-\gamma} = \frac{\sigma(x)}{2} \cdot \frac{1}{\frac{1}{2}(1+\sigma(x))}$$

$$\Rightarrow \sigma(x+1) = \frac{\sigma(x)}{1+\sigma(x)}$$

By induction:

$$\sigma(x) = \frac{1}{x} \quad \Rightarrow \quad \sigma(x+1) = \frac{1}{x+1}$$

Thus we conclude:

$$\sigma_{ox} = \frac{1}{x}, \qquad R_{ox} = x$$

This generalizes easily to other observers:

$$\sigma_{x_1 x_2} = \frac{1}{2|x_1 - x_2|}, \qquad R_{x_1 x_2} = 2|x_1 - x_2|$$

Note: other observers can move both left and right, so the probability is halved.

Typical Example

In this case, we compute the Laplacian, its pseudoinverse, and the resistance distance.

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix} \qquad \langle\omega| = \tfrac{1}{41}\begin{pmatrix} 14 & 15 & 12 \end{pmatrix}$$

$$L = \widehat{\omega}(I-A) = \widehat{\omega} - \mathbb{A} = \frac{1}{41}\begin{pmatrix} 14 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{pmatrix} - \tfrac{1}{41}\begin{pmatrix} 0 & 7 & 7 \\ 10 & 0 & 5 \\ 4 & 8 & 0 \end{pmatrix} = \frac{1}{41}\begin{pmatrix} 14 & -7 & -7 \\ -10 & 15 & -5 \\ -4 & -8 & 12 \end{pmatrix}$$

$$L^\dagger = \left(L - \frac{1}{N}\Pi\right)^{-1} + \frac{1}{N}\Pi = \frac{41}{1260}\begin{pmatrix} 40 & -20 & -20 \\ -11 & 37 & -26 \\ -29 & -17 & 46 \end{pmatrix}$$

Thus, the distances are given by:

$$\mathbb{R}_{nm} = \langle n - m | L^\dagger | n - m \rangle$$

$$\mathbb{R} = \frac{41}{140}\begin{pmatrix} 0 & 12 & 15 \\ 12 & 0 & 14 \\ 15 & 14 & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & 3.51 & 4.39 \\ 3.51 & 0 & 4.1 \\ 4.39 & 4.1 & 0 \end{pmatrix}$$

And the subjective distances are:

$$R = \widehat{\omega} \cdot \mathbb{R} = \begin{pmatrix} 0 & 6/5 & 3/2 \\ 9/7 & 0 & 3/2 \\ 9/7 & 6/5 & 0 \end{pmatrix}$$

Note: We have used the resistance distance formula involving the Laplacian. Alternatively, we could compute each value directly, for example:

$$\sigma_{21} = \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} = \frac{5}{6} = R_{21}^{-1}$$

Although the formula using the pseudoinverse of the Laplacian is introduced here, its full derivation will be given later. The goal is to provide an immediate tool to compute distances while linking with earlier concepts. We will later explore how this formulation arises from the dynamics and spectral properties of the system.

The potential ϕ

Definition

We will call potential ϕ_{onm} the probability that, if light starts at n, it reaches m before o.
We can verify that this function is closely related to the observable. Let us recall the meaning of the element Θ_{onm}:

$$\Theta_{onm} = \text{Probability that, if I } (o) \text{ look at } n\text{, I see } m$$

If I look at n, I already know that the light is at n. That is, we start from n. What is the probability that m returns the light to me? First, the light must reach m from n without passing through me. And that is precisely the potential ϕ_{onm}. Once the light is at m, the probability that it is finally m who returns the light to me is the same as if I had looked at m from the start. That is, Θ_{omm}.

Bounce probability $\mathbb{O}$ We define the bounce probability $\mathbb{O}_{on} := \Theta_{onn}$ as the probability that if I, o, act upon n, it is n who returns the light to me.

Finally, we have:

$$\Theta_{onm} = \phi_{onm}\mathbb{O}_{om}$$

Numerical example

Recall the calculation of the standard example for observations:

$$\Theta_{1nm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6/7 & 1/7 \\ 0 & 4/7 & 3/7 \end{pmatrix}$$

$$\Theta_{2nm} = \begin{pmatrix} 3/5 & 0 & 2/5 \\ 0 & 1 & 0 \\ 1/5 & 0 & 4/5 \end{pmatrix} \qquad \Theta_{3nm} = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that we have added consistent rows and columns for the observer.

To obtain the potentials we only need to scale the observations by the diagonal:

$$\phi_{1nm} = \Theta_{1nm}\mathbb{O}_{1m}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6/7 & 1/7 \\ 0 & 4/7 & 3/7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7/6 & 0 \\ 0 & 0 & 7/3 \end{pmatrix}$$

Thus we finally obtain for all observers:

$$\phi_{1nm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 2/3 & 1 \end{pmatrix} \qquad \phi_{2nm} = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{pmatrix}$$

$$\phi_{3nm} = \begin{pmatrix} 1 & 1/2 & 0 \\ 2/3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This represents the three slices $o = 1, 2, 3$ of the full tensor $\phi := \phi_{onm}$.

The reading is as follows. For example for ϕ_{12m}:

$$\phi_{12m} = \begin{pmatrix} 0 & 1 & 1/3 \end{pmatrix} = \begin{pmatrix} \phi_{121} & \phi_{122} & \phi_{123} \end{pmatrix}$$

If light starts from $n = 2$ and ends at $o = 1$, the probabilities are:

- 0 to pass through $m = 1$ first: $\phi_{121} = 0$.
- 1 to pass through $m = 2$ first: $\phi_{122} = 1$.
- $1/3$ to pass through $m = 3$ first: $\phi_{123} = 1/3$.

Definition of harmonic function

Let L be a linear operator defined on a vector space V, and let $\Omega \subseteq V$ be a linear subspace, which we will call the *boundary*.

We say that a vector $|f\rangle \in V$ is *harmonic with respect to L with boundary in Ω* if:

$$L|f\rangle \in \Omega$$

This means that $L|f\rangle$ has no component outside the boundary: any imbalance or source lies exclusively within the subspace Ω. Outside of it, the system is in equilibrium.

Equivalently and explicitly, we have:

$$\langle z|L|f\rangle = 0 \qquad \forall\ |z\rangle \perp \Omega$$

that is, $L|f\rangle$ is orthogonal to all vectors not in the boundary. This formulation makes it easy to verify the harmonicity condition by evaluating the action of L on $|f\rangle$ projected onto an orthonormal basis.

Harmonic functions appear ubiquitously across many disciplines. In classical analysis, they are solutions to Laplace's equation and describe steady states in problems of diffusion, temperature, or electric potential. In physics, they correspond to configurations without external sources. In graph theory and stochastic processes, harmonic functions describe probabilistic equilibria and expected value functions in Markov chains with absorbing states. In all these cases, a function is harmonic when its value at each interior point depends only on its immediate neighborhood, reflecting symmetry and local stability.

Potential theory is largely based on harmonic functions, and its analysis is deeply connected to the concept of the *Green's function*, which describes the system's response to a pointwise perturbation at the boundary. In fact, in many contexts, a harmonic function is obtained as the solution to a system of linear equations with prescribed boundary conditions, whose inverse matrix (or pseudoinverse) is precisely the Green's function of the operator L.

In summary, the concept of a harmonic function allows us to identify balanced, stable, or stationary configurations within a linear system, conditioned to possible perturbations localized at a defined boundary.

Interpretation of ϕ as a harmonic function

Let ϕ_{onm} be the probability that, starting from node n, light reaches node m before node o. This function satisfies an averaging relation grounded in the law of total probability.

Suppose light starts from node n. On the first step, it can reach any node i with probability A_{ni}. Once at i, the probability that light passes through m before o is ϕ_{oim}. Therefore, the total probability from n is given by:

$$\phi_{onm} = \sum_i A_{ni}\phi_{oim} \qquad \forall n \notin \{o, m\}$$

This equation expresses that ϕ_{onm} is the weighted average of the probabilities ϕ_{oim}, with weights given by the transition distribution A_{ni}. The sum runs over all possible nodes i that node n can act on, and the equation is valid as long as $n \notin \{o, m\}$, the boundary Ω.

Now fixing o and m, we define the potential function:

$$\phi_{om}(n) := \phi_{onm}$$

and the previous equation becomes:

$$\phi_{om}(n) = \sum_i A_{ni}\phi_{om}(i) \qquad \forall n \notin \{o, m\} \tag{19}$$

We can express this in matrix form:

$$(I - A)|\phi_{om}\rangle = \alpha_o|o\rangle + \alpha_m|m\rangle \qquad \alpha_o, \alpha_m \in \mathbb{R}$$

This is precisely the harmonicity condition with respect to $(I - A)$ outside the boundary $\Omega : \{o, m\}$.

Dirichlet problem:

The proposed problem can be considered a discrete Dirichlet problem. Defining the potential ϕ as:

$$\phi_{onm} := \text{Probability that light reaches } m \text{ before } o \text{ if it starts at } n$$

The potential function $\phi_{om}(n) := \phi_{onm}$ is harmonic with respect to the Laplacian operator

$$\nabla^2 := I - A$$

with boundary

$$\Omega : \{o, m\}$$

and the following boundary conditions:

- $\phi_{om}(m) = 1$ (light reaches m before o if starting from m)
- $\phi_{om}(o) = 0$ (light cannot reach m before o if starting from o)

This can be written as:

$$\nabla^2 |\phi_{om}\rangle = \alpha_o |o\rangle + \alpha_m |m\rangle \qquad \alpha_o, \alpha_m \in \mathbb{R} \tag{20}$$

Escape probability σ

So far, α_o and α_m are unknown. To determine them, note that the potential ϕ is closely related to the escape probability σ.

We define the escape probability σ_{om} as the probability that, if light starts at o, it reaches m before returning to o.

Let's compute σ_{om} by building the event step by step. Imagine light starts at o, on the first step it goes to each observer i with probability A_{oi} and, from there, what is the probability that it reaches m before o? This is precisely ϕ_{oim}. Therefore:

$$\sigma_{om} = \sum_i A_{oi} \phi_{om}(i) \tag{21}$$

Recall equation (19)

$$\sum_i A_{ni} \phi_{om}(i) = \phi_{om}(n) \qquad \forall n \notin \{o, m\}$$

And we observe that equation (21) is a special case of (19), where $n = o$, exactly one of the boundary points Ω

Similarly, starting from (21) but swapping the indices $o \leftrightarrow m$

$$\sum_i A_{mi}\phi_{mo}(i) = \sigma_{mo}$$

Since the events *reach o first* and *reach m first* from n are complementary:

$$\phi_{mo}(n) = 1 - \phi_{om}(n) \qquad \phi_{mno} = 1 - \phi_{onm}$$

The above expression becomes:

$$\sum_i A_{mi}(1 - \phi_{om}(i)) = \sum_i A_{mi} - \sum_i A_{mi}\phi_{om}(i) = 1 - \sum_i A_{mi}\phi_{om}(i) = \sigma_{mo}$$

And we finally arrive at the expression:

$$\sum_i A_{mi}\phi_{om}(i) = 1 - \sigma_{mo}$$

Comparing again with equation (19) we see this equation is another special case of (19), where $n = m$, the other boundary point Ω
All this allows us to complete equation (19) for the full domain as:

$$\sum_i A_{ni}\phi_{om}(i) = \begin{cases} \phi_{om}(n) & \forall n \notin \{o, m\} \\ \sigma_{om} & n = o \\ 1 - \sigma_{mo} & n = m \end{cases}$$

The boundary conditions are $\phi_{om}(o) = 0$ and $\phi_{om}(m) = 1$, allowing us to write the previous equation as:

$$\sum_i A_{ni}\phi_{om}(i) = \begin{cases} \phi_{om}(n) & \forall n \notin \{o, m\} \\ \phi_{om}(n) + \sigma_{om} & n = o \\ \phi_{om}(n) - \sigma_{mo} & n = m \end{cases}$$

Or equivalently:

$$\sum_i (I - A_{ni})\phi_{om}(i) = \begin{cases} 0 & \forall n \notin \{o, m\} \\ -\sigma_{om} & n = o \\ \sigma_{mo} & n = m \end{cases}$$

In matrix form, this can be expressed as:

$$\nabla^2|\phi_{om}\rangle = \sigma_{mo}|m\rangle - \sigma_{om}|o\rangle \tag{22}$$

Thus, $\alpha_m = \sigma_{mo}$ and $\alpha_o = -\sigma_{om}$

Expression of the resistance distance

Recall that the resistance distance could be defined as:

$$\mathbb{R}_{om} = \frac{1}{\omega_o \sigma_{om}}$$

We define the weighted Laplacian:

$$\boldsymbol{\nabla}^2 := \widehat{\omega}\nabla^2 := \widehat{\omega}(I - A)$$

Therefore:

$$\boldsymbol{\nabla}^2|\phi_{om}\rangle = \widehat{\omega}\nabla^2|\phi_{om}\rangle = \sigma_{mo}\widehat{\omega}|m\rangle - \sigma_{om}\widehat{\omega}|o\rangle = \omega_m\sigma_{mo}|m\rangle - \omega_o\sigma_{om}|o\rangle = \frac{1}{\mathbb{R}_{mo}}|m\rangle - \frac{1}{\mathbb{R}_{om}}|o\rangle$$

And we arrive at the expression:

$$\boldsymbol{\nabla}^2|\phi_{om}\rangle = \frac{1}{\mathbb{R}_{om}}|m - o\rangle \tag{23}$$

This expression is a system of linear equations. Since $\boldsymbol{\nabla}^2$ is singular, it does not have a unique solution. The solution admits a constant vector:

$$\mathbb{R}_{om}|\phi_{om}\rangle = \mathbf{G}^2|m - o\rangle + \alpha|\mathit{I}\rangle$$

Where $\mathbf{G}^2$ is the Moore–Penrose pseudoinverse of $\boldsymbol{\nabla}^2$.

We multiply both sides of the equation on the left by the vector $\langle m - o|$:

$$\mathbb{R}_{om}\langle m - o|\phi_{om}\rangle = \mathbb{R}_{om}(\phi_{omm} - \phi_{oom}) = \mathbb{R}_{om}(1 - 0) = \mathbb{R}_{om}$$

$$\langle m - o|\mathbf{G}^2|m - o\rangle + \alpha\cancelto{0}{\langle m - o|\mathit{I}\rangle} = \langle m - o|\mathbf{G}^2|m - o\rangle$$

Thus we finally arrive at the well-known expression for the resistance distance:

$$\mathbb{R}_{om} = \langle m - o|\mathbf{G}^2|m - o\rangle \tag{24}$$

From this point onward, the text corresponds to a draft in progress and may contain errors or inaccuracies. Its content is subject to revision.

Maxwell's Equations

General Setting

Let $G = (V, E)$ be a finite directed graph without loops, $|V| = n$. A *random walk* on G is described by a transition matrix $A \in \mathbb{R}^{n\times n}$ with $A_{ij} \geq 0$ and $\sum_j A_{ij} = 1$. We assume the chain is irreducible and aperiodic, so it admits a unique *stationary distribution*

$$\boldsymbol{w}\, A \;=\; \boldsymbol{w}, \qquad w_i > 0, \sum_i w_i = 1, \qquad W := \operatorname{diag}(w_1, \dots, w_n). \tag{25}$$

Cochain complex.

- $C^0(G; \mathbb{R}) = \mathbb{R}^V$: functions on vertices (potentials).
- $C^1(G; \mathbb{R}) = \mathbb{R}^E$: antisymmetric functions $E_{ij} = -E_{ji}$ (edge fields).
- $C^2(G; \mathbb{R}) = \mathbb{R}^{\Delta_2}$: antisymmetric functions on oriented triangles (fluxes).

Laplacian and Green's function

We define the *Laplace–Markov* operator on vertices:

$$\Delta_0 \;=\; W\,(I - A), \qquad \ker \Delta_0 = \langle \mathbf{1} \rangle. \tag{26}$$

Its Moore–Penrose pseudoinverse $G^2 := \Delta_0^\dagger$ is the *Green's function* and provides the *resistance distances* $R_{ij} = G^2_{ii} + G^2_{jj} - 2G^2_{ij}$.

Green Inner Product

On C^0 we adopt the metric:

$$\langle \varphi, \psi \rangle_G \;:=\; \varphi^\top\, G^2\, \psi, \qquad \varphi, \psi \in C^0. \tag{27}$$

It has a kernel $\langle \mathbf{1} \rangle$; on the quotient $C^0/\langle \mathbf{1} \rangle$ it is positive definite. On C^1 and C^2 we use the standard Euclidean one (other weightings also work).

Hodge Operators

Gradient. $d_0 \colon C^0 \to C^1$: $(d_0\varphi)_{ij} = \varphi_j - \varphi_i$.

Codivergence. We define

$$\delta_1 \ := \ \Delta_0 \, d_0^{\dagger}, \qquad d_0^{\dagger} = \big(d_0^{\top} d_0\big)^{\dagger} \, d_0^{\top}. \tag{28}$$

Then, $\langle d_0\varphi, E\rangle_2 = \langle \varphi, \delta_1 E\rangle_G$ and $\Delta_0 = \delta_1 d_0$ (on $C^0/\langle \mathbf{1}\rangle$).

Curl and its adjoint. Let $d_1 \colon C^1 \to C^2$ be the usual one: $(d_1E)_{ijk} = E_{ij} + E_{jk} + E_{ki}$. Its adjoint $\delta_2 = d_1^{*} : C^2 \to C^1$ acts as discrete curl.

Laws of Discrete Electrostatics and Magnetostatics

Potential, field, and charge. For $\varphi \in C^0/\langle \mathbf{1}\rangle$, define

$$E \ := \ -\, d_0\varphi \ \in C^1, \qquad \rho \ := \ \delta_1 E \ \in C^0. \tag{29}$$

Teorema 1.1 (Discrete Gauss Law). With definitions (26)–(29):

$$\boxed{\delta_1 E = \rho} \quad \text{and} \quad \boxed{\Delta_0\varphi = -\,\rho} \quad \text{(Poisson)}.$$

Proof. By construction, $\delta_1 d_0 = \Delta_0$; multiplying by $-\varphi$ yields both equalities. □

Proposición 1.2 (Static Faraday Law). $\boxed{d_1 E = 0}$ since $d_1 d_0 = 0$.

Steady Current and Magnetic Field

Definición 1.1. *The* irreversible current *is defined as* $J_{ij} := w_i A_{ij} - w_j A_{ji}$ $(J_{ji} = -J_{ij})$.

Lema 1.3 (Current Conservation). $\delta_1 J = 0$.

Proof. $\sum_j J_{ij} = w_i \sum_j A_{ij} - \sum_j w_j A_{ji} = w_i - w_i = 0$. □

Minimum-norm Magnetic Field. Since $J \in \ker \delta_1$, there exists $B \in C^2$ such that $\delta_2 B = J$. We choose the minimum-norm solution:

$$\boxed{B := \delta_2^{\dagger} J} \quad \Longrightarrow \quad \delta_2 B = J, \ \ d_2 B = 0. \tag{30}$$

Observación 1.4. In a triangulated graph, the elementary formula $B_{ijk} = \frac{1}{2}\big(J_{ij} + J_{jk} + J_{ki}\big)$ also satisfies $\delta_2 B = J$ and $d_2 B = 0$ for interior edges; it can be adapted at boundaries by dividing by 1.

Discrete Magnetostatic Laws.

$$\boxed{\delta_2 B = J}, \qquad \boxed{d_2 B = 0}$$

complete the electro-/magnetostatic analogy.

Detailed Example with Full Verifications

We work with the complete graph of three vertices:

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 \\ 1 \\ \frac{2}{3} \end{pmatrix}.$$

Stationary Distribution, Laplacian and Green

Solving $\boldsymbol{w}A = \boldsymbol{w}$ with $\sum_i w_i = 1$ gives:

$$\boldsymbol{w} = \left(\tfrac{14}{41}, \ \tfrac{15}{41}, \ \tfrac{12}{41}\right).$$

The Laplacian (26) and its pseudoinverse are:

$$\Delta_0 = \frac{1}{41}\begin{pmatrix} 14 & -7 & -7 \\ -10 & 15 & -5 \\ -4 & -8 & 12 \end{pmatrix}, \quad G^2 = \Delta_0^\dagger = \frac{1}{123}\begin{pmatrix} 151 & -82 & -69 \\ -82 & 164 & -82 \\ -69 & -82 & 151 \end{pmatrix}.$$

Electric Field $E = -d_0\varphi$

$$\begin{aligned} E_{01} &= \varphi_1 - \varphi_0 = 1 - 0 = 1, \\ E_{02} &= \varphi_2 - \varphi_0 = \tfrac{2}{3} - 0 = \tfrac{2}{3}, \\ E_{12} &= \varphi_2 - \varphi_1 = \tfrac{2}{3} - 1 = -\tfrac{1}{3}, \end{aligned} \qquad E = \begin{pmatrix} 0 & 1 & \frac{2}{3} \\ -1 & 0 & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Static Faraday Law.

$$d_1 E_{012} = E_{01} + E_{12} + E_{20} = 1 + (-\tfrac{1}{3}) + (-\tfrac{2}{3}) = 0 \ \ \Rightarrow d_1 E = 0.$$

Charge $\rho = \delta_1 E$

For each vertex we use $\rho_i = \sum_j w_i A_{ij} E_{ij}$.

$$\rho_0 = \frac{14}{41}\Big(\tfrac{1}{2}(1) + \tfrac{1}{2}(\tfrac{2}{3})\Big) = \frac{14}{41}\cdot\frac{5}{6} = \boxed{\frac{35}{123}},$$

$$\rho_1 = \frac{15}{41}\Big(\tfrac{2}{3}(-1) + \tfrac{1}{3}(-\tfrac{1}{3})\Big) = \frac{15}{41}\Big(-\tfrac{23}{18}\Big) = \boxed{-\frac{35}{123}},$$

$$\rho_2 = \frac{12}{41}\Big(\tfrac{1}{3}(-\tfrac{2}{3}) + \tfrac{2}{3}(\tfrac{1}{3})\Big) = 0.$$

Check that $\sum_i \rho_i = 0$.

Poisson Equation.

Multiply $\Delta_0\varphi$:

$$\Delta_0\varphi = \frac{1}{41}\begin{pmatrix} 14(0) - 7(1) - 7(\frac{2}{3}) \\ -10(0) + 15(1) - 5(\frac{2}{3}) \\ -4(0) - 8(1) + 12(\frac{2}{3}) \end{pmatrix} = \begin{pmatrix} -\frac{35}{123} \\ +\frac{35}{123} \\ 0 \end{pmatrix} = -\rho.$$

So $\Delta_0\varphi = -\rho$, and therefore $\delta_1 E = \rho$.

Irreversible Current J

$$J_{01} = w_0 A_{01} - w_1 A_{10} = \tfrac{14}{41}\cdot\tfrac{1}{2} - \tfrac{15}{41}\cdot\tfrac{2}{3} = -\tfrac{3}{41},$$

$$J_{12} = w_1 A_{12} - w_2 A_{21} = \tfrac{15}{41}\cdot\tfrac{1}{3} - \tfrac{12}{41}\cdot\tfrac{2}{3} = -\tfrac{3}{41},$$

$$J_{02} = w_0 A_{02} - w_2 A_{20} = \tfrac{14}{41}\cdot\tfrac{1}{2} - \tfrac{12}{41}\cdot\tfrac{1}{3} = +\tfrac{3}{41}.$$

Antisymmetric matrix:

$$J = \begin{pmatrix} 0 & -\frac{3}{41} & \frac{3}{41} \\ \frac{3}{41} & 0 & -\frac{3}{41} \\ -\frac{3}{41} & \frac{3}{41} & 0 \end{pmatrix}.$$

Conservation.

Row 0: $(-3+3)/41 = 0$; rows $1, 2$ similar $\Rightarrow \delta_1 J = 0$.

Magnetic Field B

Since there is only one triangle $(0, 1, 2)$, we use the simple rule:

$$B_{012} = \frac{1}{2}(J_{01} + J_{12} + J_{20}) = \frac{1}{2}\Big(-\tfrac{3}{41} - \tfrac{3}{41} + \tfrac{3}{41}\Big) = -\tfrac{3}{41}.$$

The opposite orientation gives $B_{021} = +3/41$.

Ampère Without Displacement.

$$\delta_2 B_{01} = B_{012} = J_{01},\ \delta_2 B_{12} = B_{120} = J_{12},\ \delta_2 B_{02} = B_{201} = J_{02}.$$

$\delta_2 B = J$ is verified.

Absence of Monopoles. No 3-simplices $\Rightarrow d_2 B = 0$ trivially.

Numerical Summary

$$\boxed{\delta_1 E = \rho,\ d_1 E = 0,\ \delta_2 B = J,\ d_2 B = 0} \quad \Longrightarrow \text{electro- and magnetostatic laws verified pointwise}$$

Conclusion

The explicit computation confirms that the random walk with constant A exactly reproduces the four equations of electrostatics and magnetostatics in Hodge discretization: Gauss, Faraday (static), Ampère without displacement,

and the no-monopole condition.

Conclusion

With the Hodge operators $d_0,\ d_1,\ \delta_1,\ \delta_2$ and Green's metric $G^2 = \Delta_0^{\dagger}$, we have recovered on a finite graph:

- Gauss's law $\delta_1 E = \rho$.
- Static Faraday law $d_1 E = 0$.
- Ampère's law without displacement $\delta_2 B = J$.
- Absence of monopoles $d_2 B = 0$.

The potential $\varphi = -G^2 \rho$ and the magnetic field $B = \delta_2^{\dagger} J$ are minimum-energy solutions. With constant A we obtain, therefore, a complete description of electrostatics and magnetostatics; the dynamic part will require introducing a temporal operator and its adjoint, which will be addressed in future chapters.

Geometric–Dynamic Postulate

Postulado 1 (Resistance Metric). Let $G = (V, E)$ be a finite graph with Laplacian $L = W(I - A)$. The *geometric distance* between vertices i, j is

$$R_{ij} \ := \ (L^{\dagger})_{ii} + (L^{\dagger})_{jj} - 2(L^{\dagger})_{ij}.$$

A photon traverses that distance in *proper time*

$$T_{ij} \ = \ R_{ij}, \qquad c = 1.$$

Thus we set natural units $\varepsilon_0 = \mu_0 = c = 1$; each edge (i, j) simultaneously has length, resistance, and "light-time" equal to R_{ij}.

Discrete Spacetime Complex

4-D Mesh

We use two indices: (i) for spatial vertices, $(n) \in \mathbb{Z}$ for temporal slices. The pair (i, n) is a vertex of the 4-complex $\mathcal{K}$.

dim. 0	$\varphi(i, n)$
dim. 1	spatial edges (i, j, n), temporal edges $(i, n, n+1)$
dim. 2	spatial faces (i, j, k, n), mixed faces $(i, j, n, n+1)$
dim. 3	volumes $(i, j, k, n, n+1)$
dim. 4	closed 4-D cells

Derivatives

Let d_x be the purely spatial exterior derivative (known, $d_x^2 = 0$), and d_t the mesoscopic temporal derivative:

$$(d_t\varphi)_{(i,n,n+1)} = \frac{\varphi(i, n+1) - \varphi(i, n)}{\Delta t}, \qquad \Delta t \gg \tau_{\text{mix}}.$$

Define

$$d := d_x + d_t, \qquad d^2 = 0.$$

Physical Cochains

Definición 1.2.

$$A \in C^1(\mathcal{K}) : \quad \begin{cases} A_{ij,n} & \textit{(spatial component)} \\ A_{i,t,n} := \Phi_{i,n}\Delta t & \textit{(temporal component)} \end{cases}$$

is the discrete 4-vector potential.

The field tensor *is* $F := dA \in C^2(\mathcal{K})$.

Decomposing:

$$E_{ij,n} = -A_{ij,n} - (d_t A)_{ij,n}, \qquad B_{ijk,n} = (d_x A)_{ijk,n},$$

we recover the electric and magnetic components already seen in the stationary regime.

Full Maxwell Equations

Let δ be the total codifferential associated with the discrete Lorentz metric: $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{G^2} \oplus (-)\langle\cdot,\cdot\rangle_t$.

Teorema 1.5 (Discrete 4-D Maxwell).

$$dF = 0 \quad \text{(Bianchi identity)}, \qquad \delta F = J,$$

where the 3-cochain $J = (\rho\, \mathrm{d}^3x + J^i \mathrm{d}S_i \wedge \mathrm{d}t)$ satisfies $\delta J = 0$ (charge conservation).

Sketch. $d^2 = 0$ yields the first equation. The second is imposed as a constitutive law and decomposes into:

$\delta_x E = \rho$ (Gauss)
$d_x E + \partial_t B = 0$ (Faraday)
$\delta_x B = 0$ (no monopoles)
$\partial_t E + d_x B = J$ (Ampère with displacement)

with $\partial_t \equiv \Delta_t^{-1}(\cdot)_{n+1} - (\cdot)_n$. □

Units and Norms

Since $c = \varepsilon_0 = \mu_0 = 1$, the displacement term $\partial_t E$ enters with the same norm weight as J; no distinct Hodge stars are needed.

Relation to the Law of Large Numbers

Since $\Delta t \geq 10\,\tau_{\text{mix}}$ (e.g.), the variance of any edge-sum observable is $O(1/\sqrt{\Delta t})$. Thus:

$$F \;=\; \mathbb{E}[F] \;+\; O\big(\Delta t^{-1/2}\big),$$

and all the above equations hold *on average* with controlled precision.

Stationary Limit $\partial_t \to 0$

If A remains exactly constant and averages are stabilized, $\partial_t E = \partial_t B = 0$, and the identities reduce to the electrostatic and magnetostatic cases treated in previous chapters.

Conclusion

Under Postulate 0 (resistance-distance = light-time), the Whitney 4-complex with total derivative $d = d_x + d_t$ provides a *discrete relativistic framework* in which Maxwell's equations are identical to the continuous ones in natural units. The discretization is consistent because Δt exceeds the mixing time, ensuring that the averages required by the LLN are reliable.

Outlook. Open problems include: (a) discrete curvature via R_{ij}; (c) fluctuations of order $O(\Delta t^{-1/2})$ and their large deviation theorems.

Speed of Light, Moving Observers, and the Emergence of Lorentz Transformations

In this theory, the resistance distance R_{ij} is interpreted literally as the absolute time it takes for light to travel from observer i to observer j. Light, understood as an elementary action, propagates probabilistically according to the structure defined by the transition matrix $\mathbf{P}$, and its speed is constant in all frames: $c = 1$.

Change in the Transition Matrix

Let us now suppose that $\mathbf{P} = \mathbf{P}(t)$ varies with time. This implies that the structure of interactions between observers changes, and with it the resistance distances $R_{ij}(t)$. From the point of view of an observer, this change can be interpreted as *other observers moving relative to them.*

Formally, if at two different time points t and t' the matrix $\mathbf{P}$ changes, and yet light continues to propagate at constant speed $c = 1$ between all pairs of observers, then the new metric $R_{ij}(t')$ must be consistent with the previous one, such that:

Light still takes the same amount of time to travel the distance $R_{ij}(t')$ under the new configurati

Relative Motion of Observers

This phenomenon can be interpreted as the emergence of a *kinematics*: if $\mathbf{P}(t)$ changes but the speed of light remains constant, then the observers have changed their relative positions in the emergent metric.

This change does not require postulating velocity or displacement: it arises from the variation in interaction probabilities. An observer i "moves away" from another j if the probability of transitioning from i to j decreases, and "moves closer" if it increases. But regardless of this, light propagation automatically adjusts so that its effective speed remains $c = 1$.

Emergence of Lorentz Transformations

In special relativity, the invariance of the speed of light implies that the transformations between reference frames must preserve the causal structure of spacetime. These transformations are the *Lorentz transformations*.

In this model, the following is proposed:

- The matrix $\mathbf{P}(t)$ defines a dynamics and a geometry at each instant.
- Light propagates with $c = 1$ at all times, in every direction of the system.

- The evolution of the metric $R_{ij}(t)$ describes the relative motion of observers.
- If the metric evolves such that the speed of light remains constant in all local frames, then the transformations between observers must preserve that constancy.

This is, essentially, the foundational condition of special relativity. Therefore, if it is possible to express how $\mathbf{P}(t)$ varies in a way that is compatible with the invariance of c, then it will be formally possible to deduce that the coordinate changes connecting the perspectives of different observers must obey Lorentz-type transformations.

A Path to Deriving Special Relativity

If a "trajectory" of each observer is defined in the emergent space ($\mathbb{R}^{N-1}$) induced by the resistance embedding, and its evolution is analyzed over time induced by the sequence of matrices $\mathbf{P}(t)$, it becomes possible to derive the notion of relative velocity, time dilation, and length contraction.

The key point is that in all frames defined by different observers:

$$R_{ij}(t) = (\text{absolute time taken by light}) \Rightarrow c = \frac{R_{ij}(t)}{\partial t} = 1$$

Therefore, any transformation between the coordinates of observers that preserves the functional form of R_{ij} while allowing for relative motion must, by construction, be a Lorentz transformation.

Conclusion

Special relativity is not postulated in this model: it *emerges* as a necessary consequence of the constancy of the speed of light defined as the propagation of action between observers. If the metric $R_{ij}(t)$ evolves in such a way that this constancy is preserved, then the laws governing the change of reference between observers must be precisely the Lorentz transformations.

Spectral Dimension and the Threshold of Existence

In this appendix, we develop one of the most fundamental questions for the model: why does the universe have approximately three spatial dimensions? We propose that the key lies in the system's **spectral dimension**, and that there exists a **critical threshold** beyond which observers may cease to receive light forever. This threshold marks the boundary between continuous existence and functional disappearance.

Random Walks and Return Time

Let $p_{ii}(t)$ be the probability that a particle (light) which started at node i returns to i in exactly t steps. The **cumulative return time** is:

$$P_i := \sum_{t=0}^{\infty} p_{ii}(t)$$

This value represents the expected number of times that light will return to visit observer i, if it starts at i. If $P_i = \infty$, the observer is said to be **recurrent**. If $P_i < \infty$, it is **transient**: there is a positive probability that it will never again receive light.

Ontological Interpretation

In the framework of this theory, an observer that never again receives light can be understood as an **observer who has ceased to live**. Therefore, a universe where $P_i < \infty$ for some i is a universe where *death* (understood as ontological silence) is possible. In contrast, if $P_i = \infty$ for all observers, then the system ensures that every observer will continue to receive experiences indefinitely.

Spectral Dimension and Recurrence

The probability $p_{ii}(t)$ is deeply related to the spectrum of the Laplacian operator $\mathbf{L}$ associated with the graph. If the graph is regular or has properties similar to a d-dimensional lattice, it is known that:

$$p_{ii}(t) \sim t^{-d_s/2} \quad \text{as } t \to \infty$$

where d_s is the system's **spectral dimension**.

This implies that the total return time is:

$$P_i = \sum_{t=0}^{\infty} p_{ii}(t) \sim \sum_{t=1}^{\infty} t^{-d_s/2}$$

And this sum converges if and only if $d_s > 2$.

The critical limit of existence

We then arrive at a crucial result:

- If $d_s \leq 2$, then $P_i = \infty$ for all i: the system is **recurrent**, and every observer will live indefinitely.
- If $d_s > 2$, then $P_i < \infty$: the system is **transient**, and some observers may cease to receive light forever.

Therefore, $d_s = 2$ is the **critical limit** of existence. In ontological terms, it is the threshold beyond which death emerges as a structural phenomenon in a universe of infinite observers.

Global Stability

A global measure of the universe's stability is the weighted average of the number of returns, considering the stationary distribution π_i:

$$S := \sum_i \pi_i P_i = \sum_{t=0}^{\infty} \sum_i \pi_i p_{ii}(t)$$

This value represents the average rate of light returning to its origin, weighted by the system's energy balance. This sum diverges if $d_s \leq 2$ and converges if $d_s > 2$, confirming that the **spectral dimension governs the global stability of the universe**.

Three Spatial Dimensions

The spectral dimension is not just a mathematical property: it is the **functional parameter that determines whether a universe of observers can sustain itself without irreversible silences**. If the universe has $d_s \leq 2$, light never abandons anyone forever. If $d_s > 2$, some observers may become isolated with no return.

Thus, we propose that the emergent universe dynamically organizes itself near the threshold $d_s = 2$, seeking the maximum complexity compatible with the continuity of experience.

In this theory, observers do not inhabit a preexisting physical space: they form among themselves a network of interactions that defines their own universe. Actions between observers generate time, and the path of light through that network defines an emergent metric: the resistance distance.

We have seen that resistance distance, which is the *real geometric distance*, can be expressed as the square of the Euclidean norm in a Hilbert space generated by the spectral decomposition of the Laplacian:

$$R_{ij} = r_{ij}^2 = \|\mathbf{x}_i - \mathbf{x}_j\|^2 \quad \text{with } \mathbf{x}_i \in \mathbb{R}^{d_s}$$

This is the physical geometry, whose dimension is N, not the one in Hilbert space, which is Euclidean and has dimension $N-1$ and is the one that must have not more than 2 dimensions. Space must "lift" one dimension: it must go from the Hilbert space to the geometric space.

Thus, if observers live in a Hilbert space of dimension $d_s = 2$, the geometry that can contain those distances as geometric distances must have one more dimension. The resistance distance is not directly expressed in the Hilbert space, but in an emergent space with an additional dimension.

$$\text{Dimension of physical space} = d_s + 1 = 3$$

The *three dimensionsal space*, then, is a must, for any stability, It is the minimal geometry that allows the functional distances arising from their interaction to become a continuous, external, and perceptible geometry.

The universe is not three-dimensional because it had to be from the start, but because sustained interaction between conscious observers—can only emerge in networks whose spectral dimension is two or less. The third dimension appears to come back to the geometric space comming from the Hilbert space.

Euclidean Embedding from Resistance Distance

Summary

The resistance distance R_{ij} between pairs of nodes in a connected, undirected graph defines a symmetric metric of negative type. It is possible to represent each node as a point in a Euclidean space $\mathbb{R}^{N-1}$ such that the Euclidean distance r_{ij} between these points satisfies $R_{ij} = r_{ij}^2$. In this section, we rigorously develop the associated embedding, analyze its relationship with the spectrum of the graph Laplacian, and clarify the role played by the square root of this matrix in the geometric construction.

Resistance Distance

Let $G = (V, E)$ be an undirected, loopless, weighted graph with $N = |V|$ nodes, and assume it is connected. The graph Laplacian is defined as:

$$L = D - A$$

where: - $D \in \mathbb{R}^{N\times N}$ is the diagonal matrix of (weighted) degrees, - $A \in \mathbb{R}^{N\times N}$ is the symmetric adjacency matrix.

Since the graph is connected, L is symmetric and positive semidefinite, with a one-dimensional kernel generated by the constant vector $1 \in \mathbb{R}^N$.

The Moore–Penrose pseudoinverse of L, denoted L^+, is also symmetric and positive semidefinite, with the same spectral basis as L but the reciprocals of the non-zero eigenvalues.

The resistance distance between two nodes i and j is defined as:

$$R_{ij} := (e_i - e_j)^\top L^+ (e_i - e_j)$$

where $e_i \in \mathbb{R}^N$ is the canonical basis. This definition corresponds to the energy dissipated when injecting one unit of current between nodes i and j, assuming resistances are inversely proportional to the graph weights.

The matrix $R = [R_{ij}] \in \mathbb{R}^{N\times N}$ satisfies:

$$R_{ij} \geq 0, \quad R_{ij} = R_{ji}, \quad R_{ii} = 0, \quad R_{ij} \leq R_{ik} + R_{kj}$$

so it defines a metric of negative type.

Associated Euclidean Embedding

We want to represent each node $i \in V$ as a vector $r_i \in \mathbb{R}^{N-1}$ such that:

$$R_{ij} = \|r_i - r_j\|^2$$

This representation exists and is unique up to isometries, since R is a metric of negative type. To construct it, we diagonalize the Laplacian:

$$L = U\Lambda U^\top$$

where: - $U = [u_1 \mid u_2 \mid \cdots \mid u_N]$ is an orthogonal matrix of eigenvectors, with $u_1 = \frac{1}{\sqrt{N}}1$, - $\Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$, with $0 < \lambda_2 \leq \cdots \leq \lambda_N$.

Then the pseudoinverse is written as:

$$L^+ = U\Lambda^+ U^\top, \quad \text{where } \Lambda^+ = \mathrm{diag}(0, \lambda_2^{-1}, \ldots, \lambda_N^{-1})$$

Embedding Construction

The embedding of each node i in $\mathbb{R}^{N-1}$ is defined as:

$$r_i := \left(\frac{u_{i2}}{\sqrt{\lambda_2}}, \frac{u_{i3}}{\sqrt{\lambda_3}}, \ldots, \frac{u_{iN}}{\sqrt{\lambda_N}} \right) \in \mathbb{R}^{N-1}$$

Here, u_{ik} is the i-th component of the eigenvector u_k, and the components associated with the zero eigenvalue have been discarded.

Then:

$$\|r_i - r_j\|^2 = \sum_{k=2}^{N} \left(\frac{u_{ik} - u_{jk}}{\sqrt{\lambda_k}} \right)^2 = R_{ij}$$

We define explicitly:

$$r_{ij} := \|r_i - r_j\|, \quad R_{ij} = r_{ij}^2$$

thus distinguishing between: - R_{ij}: the resistance distance (negative type metric), - r_{ij}: the Euclidean distance (positive type metric).

Role of the Change-of-Basis Matrix

We define:

$$G := \Lambda^{+1/2} U^\top \quad \Rightarrow \quad L^+ = G^\top G$$

Since:

$$R_{ij} = (e_i - e_j)^\top L^+ (e_i - e_j) = \|G(e_i - e_j)\|^2$$

the images of the vectors e_i under G generate the embedding:

$$r_i = Ge_i \in \mathbb{R}^{N-1}$$

That is, the matrix G acts as an active change-of-basis matrix: it transforms the canonical basis vectors $\{e_i\}$ into their corresponding geometric representations r_i in the Euclidean space where the metric is the square root of the resistance distance.

Appendix O. Hilbert Space and Position Operator: Between the Graph and Physical Space

O.1 Introduction

In this theory, the Hilbert space associated with the graph of observers is not assumed a priori but emerges from the structure of interaction between souls. The metric on the graph induces a distance between nodes—the resistance distance—whose square root defines a Euclidean metric in a space of dimension $N-1$. This structure allows the construction of a Hilbert space with canonical inner product and complex wavefunctions.

The goal of this appendix is to rigorously define the **position operator** in this Hilbert space, establishing its relationship with the geometric coordinates of the spectral embedding and clearly distinguishing between physical space and state space.

O.2 Two Distinct Spaces

1. Geometric space: we denote it by $\mathbb{R}^N$, and its canonical basis is formed by the vectors $|n\rangle$, with $n = 1, \ldots, N$. Each vector $|n\rangle$ represents observer n as a point-like entity. Functions defined over the nodes of the graph are expressed as linear combinations of these vectors.

2. Emergent Hilbert space: it has dimension $N-1$, and is denoted with basis $\{|e_i\rangle\}_{i=1}^{N-1}$. These vectors are the nontrivial orthonormal eigenvectors of the system's symmetric Laplacian. It is in this space that the wavefunctions $|\psi\rangle$ live, and where the quantum evolution of the system is expressed.

These two bases—$\{|n\rangle\}$ and $\{|e_i\rangle\}$—must not be confused: one represents concrete observers, the other principal directions in state space.

O.3 Spectral Embedding and Emergent Coordinates

The spectral embedding of the graph is built from the principal eigenvectors of the Laplacian. For each node n, we define its spectral position as:

$$x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(d)}) \in \mathbb{R}^d$$

where $x_n^{(k)}$ is the k-th component of node n in the principal direction k, extracted from the eigenvector $|e_k\rangle$.

These spectral positions define the emergent geometric space where:

$$r_{nm}^2 = \sum_{k=1}^{d} \left(x_n^{(k)} - x_m^{(k)} \right)^2 \approx R_{nm}$$

that is, Euclidean distances approximate resistance distances.

O.4 Definition of the Position Operator

In the emergent Hilbert space, we can define a position operator $\widehat{X}_k$ associated with the k-th spectral coordinate as:

$$\widehat{X}_k = \sum_{i=1}^{N-1} x_i^{(k)} |e_i\rangle\langle e_i|$$

This operator acts on wavefunctions $|\psi\rangle \in \mathbb{C}^{N-1}$, and extracts the position component in direction k. In this formulation:

- $|e_i\rangle$ represents the i-th principal direction of the graph (not observer i), - $x_i^{(k)}$ is the spectral coordinate of node i in direction k.

O.5 Physical Interpretation

This operator should not be interpreted as a classical spatial position. It represents a projection within Hilbert space, whose internal structure contains the system's geometric information. The transition to physical space—i.e., the assignment of real spatial coordinates—occurs via the metric derived from the spectral embedding.

In particular, if the spectral dimension of the system is $d_s = 2$, one may work with coordinates x, y in the embedding, and recover a third coordinate z by considering the full distance $R = r^2$, so that physical space emerges as $\mathbb{R}^3$.

O.6 Conclusion

The position operator in this theory is defined on the emergent Hilbert space and reflects the metric structure of the graph. This operator is not postulated but derived from the interaction between souls. The distinction between geometric space and state space is fundamental, and it enables us to understand how the physical universe can emerge from a purely relational network.

Appendix G. Laplacian Spectrum and Energy Levels

In this appendix, we show how the proposed theory reproduces, in the case of regular networks, the characteristic spectral behavior of known quantum systems. Specifically, we observe how the spectrum of the Laplacian operator of the graph generates an energy structure identical to that appearing in the Schrödinger equation for free particles.

G.1 Regular Networks and Fourier Modes

Consider a graph with N nodes arranged in a regular network, for example a ring (circular topology), an open line (segment), or a grid with periodic boundary conditions.

In these cases, the eigenvectors of the Laplacian are discrete harmonic functions: the Fourier modes. For the ring, for instance, the eigenvectors are:

$$v_n(j) = \frac{1}{\sqrt{N}} e^{2\pi i n j/N}$$

and they form an orthonormal basis of the discrete Hilbert space.

G.2 Laplacian Eigenvalues

The eigenvalues of the Laplacian for different graphs are well known:

- For the ring (circular connection):

$$\lambda_n = 1 - \cos\left(\frac{2\pi n}{N}\right)$$

- For the open line (segment with free ends):

$$\lambda_n = 1 - \cos\left(\frac{n\pi}{N+1}\right)$$

In both cases, when $n \ll N$, the eigenvalues behave as:

$$\lambda_n \approx \left(\frac{2\pi n}{N}\right)^2 \quad \text{or} \quad \lambda_n \approx \left(\frac{n\pi}{N+1}\right)^2$$

G.3 Identification with Energy

If we identify the evolution operator as:

$$\mathbf{H} := c \cdot \mathbf{L}$$

where c is a constant (for example $c = \hbar^2/2m$), then the Laplacian eigenvalues are directly interpreted as energy levels:

$$E_n = c \cdot \lambda_n \sim c \cdot n^2$$

This behavior matches what is observed in quantum mechanics for particles in a box or in a square potential well.

G.4 Wavefunction Evolution

The wavefunction can be decomposed in the Fourier basis:

$$|\psi(t)\rangle = \sum_n \tilde{\psi}_n(0) e^{-iE_n t} |v_n\rangle$$

where $\tilde{\psi}_n(0)$ are the initial components of ψ in the graph's eigenmodes.

This expression is exactly the general solution of the free Schrödinger equation in the discrete Hilbert space that emerges from the network.

G.5 Conclusion

The energy structure of quantum mechanics does not need to be postulated: in this theory, it **emerges directly from the spectrum of the interaction graph between souls**. Quantum energies are nothing more than the natural frequencies of oscillation of light traveling through the network.

A Discrete Synge Function from Resistance Distances

All modern physics begins with the idea that space is a continuous manifold endowed with a metric that gives it shape. This assumption is so fundamental that it is rarely questioned: points, distances, tangents... everything stems from it.

But in this theory, we do not assume that such a space exists. We only assume that there are observers —or souls— and that there are relations between them, probabilities of interaction. From those relations, we construct a graph, and its dynamics give rise to an emergent metric. Nothing continuous, nothing absolute.

And it turns out that this metric is not just similar to that of relativity: it is formally identical. The resistance distance between nodes satisfies exactly the same properties as the Synge function. And if we densify the graph, we recover classical spacetime as a special case.

That is why this is not a metaphor nor an approximation: it is a generalization. What used to be an assumption (space as a background) is now a consequence.

In this view, space does not exist on its own: it emerges from interaction. Differential geometry is not the starting point but the continuous limit of a deeper relational theory.

Summary

We present a rigorous and self-contained derivation of a discrete analog of Synge's world function, constructed from resistance distances on a directed, weighted graph. We demonstrate that the pseudoinverse of the Laplacian operator plays a mathematically equivalent role to that of the metric tensor on a differentiable manifold, and that the resistance-based Synge function structurally coincides with its continuous counterpart. This establishes a concrete link between combinatorial Laplacians and differential geometry.

Synge Function in Smooth Geometry

Let (M, g) be a pseudo-Riemannian manifold, and let $x, x' \in M$ be two points connected by a unique geodesic $\gamma(s)$, with $\gamma(0) = x$, $\gamma(1) = x'$. The Synge function is defined as:

$$\sigma(x, x') := \frac{1}{2} \left(\int_0^1 \sqrt{g_{\mu\nu}(\gamma(s))\dot{\gamma}^\mu(s)\dot{\gamma}^\nu(s)}\, ds \right)^2$$

In normal coordinates around x, this function expands locally as:

$$\sigma(x, x') = \frac{1}{2} g_{\mu\nu}(x)(x' - x)^\mu (x' - x)^\nu + o(\|x - x'\|^3)$$

Discrete Geometry and Resistance Distance

Let $G = (V, E)$ be a directed, weighted graph with $|V| = N$ nodes. Let $\mathbf{T} \in \mathbb{R}^{N \times N}$ be a column-stochastic transition matrix, and π a stationary distribution satisfying $\mathbf{T}\pi = \pi$. Define $\mathbf{\Pi} := \text{diag}(\pi)$ and the left Laplacian:

$$\mathbf{L} := (\mathbf{I} - \mathbf{T})\mathbf{\Pi}$$

Let $\mathbf{L}^+$ denote the Moore–Penrose pseudoinverse of $\mathbf{L}$.

We define the resistance distance between nodes i, j as:

$$R_{ij} := (e_i - e_j)^\top \mathbf{L}^+ (e_i - e_j)$$

and the discrete Synge function as:

$$\sigma_{ij} := \frac{1}{2} R_{ij}$$

Structural Equivalence

The function σ_{ij} satisfies:

- Symmetry: $\sigma_{ij} = \sigma_{ji}$
- Positivity: $\sigma_{ij} \geq 0$
- Diagonal nullity: $\sigma_{ii} = 0$
- Quadratic form: $\sigma_{ij} = \frac{1}{2}(e_i - e_j)^\top \mathbf{L}^+ (e_i - e_j)$

Let $\phi_i := \mathbf{L}^+ e_i$. Then:

$$\sigma_{ij} = \frac{1}{2} \|\phi_i - \phi_j\|^2$$

The Metric Tensor as Operator

In smooth geometry, the Laplace–Beltrami operator is:

$$\partial_g f = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu f \right)$$

Its inverse defines a Green's function $G(x, x')$ satisfying:

$$\|x - x'\|^2 \propto G(x, x) + G(x', x') - 2G(x, x')$$

Analogously, for resistance distances:

$$R_{ij} = L^+_{ii} + L^+_{jj} - 2L^+_{ij}$$

Therefore, $\mathbf{L}^+$ is structurally the discrete analog of ∂_g^{-1}, and encodes geometry in the same way that $g_{\mu\nu}$ does in the continuous case.

Conclusion

We conclude that:

1. The resistance distance defines a discrete Synge function $\sigma_{ij} = \frac{1}{2}R_{ij}$;
2. The pseudoinverse of the Laplacian $\mathbf{L}^+$ plays the structural role of the inverse of the Laplace–Beltrami operator;
3. Therefore, $\mathbf{L}^+$ acts as a discrete metric tensor, and σ_{ij} as a discrete Synge function.

This establishes a rigorous bridge between the metric notions of graph theory and those of differential geometry.

Beyond Geometry: Space as a Network of Relations

Classical differential geometry begins from a fundamental assumption: that the universe is a differentiable manifold endowed with a metric tensor, which defines distances, angles, and volumes. Physics is built upon this geometric scaffold: points, curves, tangents, and geodesics.

In this work, we take a radical step. We show that such a geometric structure can emerge naturally and rigorously from a purely discrete model, without assuming continuity or coordinates. The central object is not a field on a background, but a network of interactions among elementary entities. These relationships define a transition matrix, whose Laplacian gives rise to an emergent metric.

In this framework, the pseudoinverse of the Laplacian operator plays exactly the role of the metric tensor. The resistance distance, derived from this pseudoinverse, satisfies all the structural properties of Synge's world function in general relativity. In the limit of dense graphs, classical geometry is recovered as a special case.

This is not an analogy. It is a generalization. In this new perspective, differential geometry is not a fundamental axiom: it is the limiting case of a more basic relational theory. Space ceases to be a stage: it is an emergent property of the interaction pattern between observers.

This discrete, relational, and algebraically rigorous view of geometry not only reproduces classical relativity—it contains it.

Appendix N. Natural Units and the Emergence of c and h

N.1 Introduction

Throughout the article, we have worked in natural units, where the speed of light c and Planck's constant h have been set to 1. This choice simplifies the formulas and allows focus on the mathematical structure of the model. However, to connect this theory with the physical world and human measurement units, it is necessary to explicitly reintroduce these fundamental constants.

In this appendix, we show how the constants c and h naturally appear in the model's equations when arbitrary physical units are considered.

N.2 The Constant c: Speed of Light

In this theory, light represents the fundamental particle that travels between souls. We have defined the absolute distance R_{ij} as the average number of instants it takes for light to propagate from soul i to soul j. In natural units, one "instant" lasts $\tau = 1$, and thus the distance $r_{ij} = \sqrt{R_{ij}}$ is also measured in units of "natural space."

To generalize this to physical units, we need to introduce:

- τ: duration of an instant in seconds (s), - c: speed of light in meters per second (m/s).

Then, the physical distance in meters between two souls is:

$$r_{ij}^{(\text{physical})} = c \cdot \tau \cdot \sqrt{R_{ij}}$$

And the absolute resistance distance is expressed as:

$$R_{ij}^{(\text{physical})} = \tau \cdot R_{ij}$$

This equation shows us that the geometry of physical space emerges directly from counting instants, once the units of time and velocity are fixed.

N.3 The Constant h: Quantization of Action

In quantum physics, Planck's constant h relates energy to frequency:

$$E = h\nu$$

In our model, the Hamiltonian can be derived from the Laplacian (or its symmetric version) of the graph:

$$H = \alpha \cdot \mathbf{L}$$

where the eigenvalues of $\mathbf{L}$ have units of inverse fundamental time. To translate this to physical energy, we set:

$$\alpha = \frac{h}{\tau}$$

Then:

$$H = \frac{h}{\tau} \cdot \mathbf{L}$$

And the time evolution of the wavefunction becomes:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = e^{-i\frac{h}{\hbar\tau}\mathbf{L}t}|\psi(0)\rangle$$

If working with $\hbar = h/2\pi$, then the eigenvalues of $\mathbf{L}$ are interpreted as angular frequencies, and H directly acquires dimensions of energy.

N.4 Conceptual Unification

The explicit appearance of c, h, and τ allows us to reinterpret these fundamental parameters:

- τ: minimum duration of a subjective instant (related to the temporal resolution of the soul's consciousness),
- c: speed of light propagation in the space generated by the network of observers,
- h: minimal action, associated with the elementary process of interaction between souls.

In this model, these values are not arbitrary but emerge from the causal weave of fundamental actions. They are the constants that translate the internal language of the universe of souls into the measurable language of human instruments.

N.5 Conclusion

The model developed in this article can be expressed both in natural units and in conventional physical units. The explicit appearance of c and h when translating the equations into human measurement systems reveals the explanatory power of the model: it not only reconstructs time, space, and mass from first principles, but also contains within its structure the fundamental constants that govern the physical world.