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Preliminary Draft

Sceptical Notes on the Establishment of Rational Expectations Equilibria Without Knowledge Being Widespread

An effort spanning more than a decade has begun to clarify the conditions for the existence and stability of rational expectations equilibria where agents' information and/or initial beliefs about the model are not homogenous.[1] Most results, depending as they usually do on specific learning mechanisms, show that at best a rational expectations equilibrium is approached only as a long-run limit.

Accepting the plausibility of a long-run rational expectations equilibrium is, thus, not inconsistent with a scepticism toward the usefulness of such a concept in the short run. On the other hand, however, Mishkin (1983) and others have argued that alternative conditions are present that, at least in the context of financial markets, dominate slow-acting learning mechanisms and are likely to assure a rapid convergence to a rational expectations equilibrium. In particular, Mishkin suggests that the presence of a few rational and well-informed speculators can substitute for slower learning mechanisms:

> **"First, although the efficient markets or rational expectations condition in (4) may not hold exactly, it is an extremely useful approximation for macroeconomic analysis. Second, this condition should be a useful approximation even if not all market participants have expectations that are rational. Indeed, even if most market participants were irrational, we would still expect the market to be rational as long as some market participants stand ready to eliminate unexploited profit opportunities."**[2]

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1. See e.g., Cyert and DeGroot (1974), Bray (1983), Radner (1983), and Marcet and Sargent (1988).
2. Mishkin (1983), p. 11.

It is the purpose of this note to examine and, ultimately, to argue against this proposition. The position reached below is that under all, or at least most plausible assumptions, the actions of one or a few rational speculators cannot alone be sufficient to produce the rational expectations equilibrium. This conclusion will not rest on borrowing constraints, the existence of which would of course be sufficient to produce it; even with the rather unrealistic assumption of unlimited borrowing and lending opportunities at a constant rate of interest, the conclusion does not change. It turns out that a command of unlimited resources can only rarely be a perfect substitute for the universal availability of information. This conclusion is reached, for different reasons, whether speculators are risk neutral or risk averse.

Below we will consider three cases: two under risk neutrality of speculators' perferences and one under risk aversion. In each of the following examples the assumption will be that information pertaining to an increase in an asset's return is obtained by a single or, at most, a few rational speculators. Further, the informed speculator is assumed to have unlimited borrowing and lending opportunities at a constant riskless interest rate, r.

I. An Informed, Risk-Neutral Speculator Facing a Market of Uninformed, Risk Neutral Investors

A. The Case of Identical Expectations

Although I think the most telling objections to Miskin's hypothesis occur under the assumption of risk aversion, the subject of the next section, there are also problems in a world of risk neutrality. In this first case assume that we have an informed, but risk neutral speculator operating in a market of uninformed, risk-neutral investors, this latter group sharing common beliefs about the expected value and other moments of the distribution of each risky asset. In the next section we will examine the effects of allowing the uninformed investors to have differing beliefs.

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In this and subsequent sections we will examine a particular case where information becomes available to the speculator alone about the return on a particular risky asset with a one-period horizon. The return during the period, y, will in fact be made up of two components: a random return x, whose mean μ and other moments are known to all; and a component, a, known with certainty by the speculator and completely unknown to or assumed to have a mean of zero by the uninformed investors. This second component could be, for example, a new subsidy to be paid in every state of the world (in which case "a" would be positive). The return, a, is assumed to become known and simultaneously to be paid at the end of the period -- at the same time x is realized.

At time zero, prior to the speculator obtaining knowledge of the subsidy, given the universal risk neutrality and homogenous expectations concerning the random part of the return x, the market price (P) of the asset will be:

$$P_0 = E(x)/(1+r) = \mu/(1+r), \tag{1}$$

or THINK of Perpetual return $E(x)/r = E(x)\left[\sum^{\infty}\left(\frac{1}{1+r}\right)\right] = \frac{\mu}{r}$ repeated drawing

where, in addition to the symbols already defined, E(x) is the expected value of x and r is the riskless rate of interest.

The speculator knows that after the subsidy becomes widely known, the price will jump to:

$$P_1 = (\mu+a)/(1+r) = P_0 + a/(1+r). \tag{2}$$

If the knowledge of the new subsidy were universal, the equilibrium price of the asset would immediately jump to P_1, as it will as soon as the subsidy is announced. The question of interest here is: As hypothesized by Mishkin and others, can the speculator, by using his information, force the market to move to the full information, rational expectations price P_1?

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Given the (rather artificial) assumption of unlimited borrowing opportunities, it will indeed be in the interest of the speculator to

1) purchase all existing units of the asset at any price less than P_1, given that each unit so purchased will lead to a rate of return greater than the common market return r. Despite this, P_1 is only one of an infinity of

2) possible equilibrium prices between P_0 and P_1. Given the risk neutrality and, especially, the homogenous expectations of the uninformed investors, the speculator ought to be able to purchase all outstanding shares of the asset at a small premium, ϵ, over P_0. Any small premium will increase the expected return of selling the asset for the uninformed investors above the market equilibrium of r. As long as the total supply of the asset is not large enough to affect the overall market rate of return r, the speculator should not force the market price to the full information equilibrium.

Of course, the above solution depends on the assumption of a **single** knowledgable speculator. With two or more speculators, there is the possibility of any number of equilibrium prices between P_0 and P_1, depending, among other things, on the degree of cooperation among them. The solutions can vary from a Bertrand-type solution, where the price would be bid up to the rational expectations price, P_1, to a price as low as $P_0+\epsilon$ under complete cooperation.[3]

Case IB

B. The Case of Diverse Expectations

The above indeterminancy resulted because of a constant supply price, independent of the amount supplied, and a consequential **constant**, supernormal, rate of profit for each unit purchased by the single, informed speculator. One can change the result in the single-speculator case and still maintain the risk-neutrality of the speculator, by changing the behavior of the **uninformed** investors so as to make their supply price an increasing function of the amount sold. For example, assume that the uninformed do not hold homogenous beliefs about the mean return μ, and that because of risk aversion or imperfect borrowing opportunities, such disparate beliefs are consistent with the original equilibrium price P_0.

IB:1

Assuming that the disparate beliefs of the uninformed investors lead to a continuously differentiable, upward sloping supply curve for the asset, the speculator will find himself in a typical monopsony situation.[4] Let us assume a supply price, $P(Q)$, a function of the number of units, Q, sold to the speculator; the first derivative of the supply curve, $P'(Q)$, is everywhere positive. The new price forced by the speculator, P^*, will be determined by equalizing the present value of the expected marginal return

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3. I am indebted to Michael Gavin for suggesting the plausibility of the Bertrand solution. For a discussion of the Bertrand solution, see J.W. Friedman (1977), pp. 38-39, and Martin Shubik (1959), pp. 80-82. For a discussion of a non-cooperative environment that leads to a solution that has many of the properties of the fully cooperative solution, see Friedman (1977), pp. 180-189.

4. This of course assumes that the speculator cannot extract all the seller's surplus by paying a different price to each investor.

$\Pi = TR - TC = \frac{\mu+a}{1+r} \cdot Q - P(Q)Q = Q\left[\frac{\mu+a}{1+r} - P(Q)\right]$

$\frac{\partial \Pi}{\partial Q} = 0 = \frac{\mu+a}{1+r} - P_{paid} - Q P'(Q)$

MR = MC

earned on the asset $[(\mu+a)/(1+r)]$ to its marginal cost (which, because of the monopsony is greater than the purchase price, P(Q)):

$$(\mu+a)/(1+r) = P^* + P'(Q)Q. \qquad P^* = \frac{\mu+a}{1+r} - P'(Q)Q \quad (3) \quad < P_1$$

Once again the equilibrium forced by the single speculator will not be the rational expectations price under universal information. The equilibrium price P* will be less than P_1(which equals the lefthand side of (3)). When the information about the subsidy becomes public knowledge, the market price will jump from P* to P_1 .

As noted in the previous section, in a market with more than one informed speculator, depending on the degree of cooperation, the equilibrium price can vary from P* up to the rational expectations price, P_1.

II. Making the Speculator Risk Averse

Although I think risk neutrality on the part of speculators was what Mishkin and others had in mind when suggesting the original hypothesis, the truly realistic case, in my view, is risk aversion. After all, even the Ivan Boeskys of this world are not known to put all their eggs in one basket.

The analysis of risk aversion on the part of speculators provides a telling rebuttal to the view that the command of large resources can be a substitute for widely disseminated information. Under risk aversion, even with unlimited borrowing possibilities, there is no necessary tendency for the speculator to absorb all of the underpriced shares of the asset. Because of the buildup of portfolio risk as the speculator accumulates risky assets, eventually the (constant) increased expected return on the marginal unit of the asset is counterbalanced by increased portfolio risk. Depending on the speculator's inital wealth, his degree of risk aversion, & the systematic risk of the asset and the number of shares of the underpriced asset, a new equilibrium will be reached, possibly with little or no effect of the speculator's purchases on the initial market price of the asset, P_0. In any case, once again there will be no necessary tendency for the speculator to force the market price to $\bar{P}_1$.

These points can be demonstrated by considering a simple specific example. We will use a numerical example developed by Mossin (1973) of the portfolio decision of a risk averse speculator.[5]

Generally, any speculator is assumed to maximize the expected utility, U, of end-of-period wealth, Y. For a given utility function, u(Y), one obtains the general marginal condition:

$Y = rW + z(X - rp)$ single risky asset

excess return over putting (price) into riskless asset

$$\partial U/\partial z_j = E\{\underbrace{u'(Y)}_{\text{Marg U of income for particular realization of X}}[\overbrace{x_j - (1+r)P_j}]\} = 0, \quad (j = 1,\ldots,n), \tag{4}$$

expected Marg U = 0

where: z_j is the number of shares bought of asset j, and P_j is its price; E is the expectation operator; u' is the first derivative of u, the marginal utility; x_j is the random return from a share of asset j; and r is the riskless rate of interest.

Mossin solves explicitly a form of the problem incorporating two risky assets and a quadratic utility function, thus limiting attention to the first two moments of the relevant probability distributions. Other utility functions may be preferred on theoretical grounds, but using the quadratic leads to tractable results without distorting the

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5. See Mossin (1973), pp. 49-52.

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general conclusion that holds for virtually all utility functions. For the case of two risky assets, Mossin shows that the system of marginal conditions becomes:[6]

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\begin{aligned}
z_1\{\sigma_{11} + [\mu_1-(1+r)P_1]^2\} + z_2\{\sigma_{12} + [\mu_1-(1+r)P_1][\mu_2-(1+r)P_2]\} &= [\mu_1-(1+r)P_1][c/2-rW] \\
z_1\{\sigma_{12} + [\mu_1-(1+r)P_1][\mu_2-(1+r)P_2]\} + z_2\{\sigma_{22} + [\mu_2-(1+r)P_2]^2\} &= [\mu_2-(1+r)P_2][c/2-rW] \quad .
\end{aligned}
\tag{5}
$$

The dependent variables are of course the number of shares of each risky asset purchased, z_1 and z_2. The definitions of those variables not defined above and the values assigned to them by Mossin are as follows:

μ_1, μ_2 = expected values of the two assets = 0.5 and 1.25;
σ_{11}, σ_{22} = variances of the assets = 1/4 and 3/16;
σ_{12} = covariance between the returns = 1/8;
r = the riskless rate of interest = 0;
P_1, P_2 = prices of the two assets = 0.3 and 1.1;
c = risk aversion parameter in the utility function = 1/2180;
W = initial wealth = 500.

New paragraph

The optimal number of shares purchased by the speculator for Mossin's original problem are z_1 = 300 and z_2 = 200.

Suppose that our speculator learns, as he did for previous examples, that the return on asset 1 will increase in all states of the world because of the awarding of a subsidy at rate a per share. Only he possesses this information and, to weight the situation in favor of the Mishkin hypothesis, we assume he can borrow unlimited amounts at the riskless rate of interest. We will set the value of a at 0.5, in order to

6. See equation (12), p. 51.

double the expected return of asset 1, μ_1, to 1.0. Because the subsidy affects all states of the world identically, neither the covariance nor the variances of the informed speculator are changed. All other values imposed by Mossin are assumed to be unchanged, the speculator being too small to affect the riskless rate of interest or the prices of the two assets. Since μ_1 appears in all but one term of the equations (5), the numerical values of most coefficients change from Mossin's initial example. It can be verified that the new solution is $z_1 = 647.56$ and $z_2 = -287.8$. The speculator's holding of asset 1 approximately doubles. The negative holding of the other asset means that optimally he now sells asset 2 short. The major point remains, however: despite doubling the expected return of asset 1, known only by this single speculator, he does not attempt to corner the market, but optimally increases his holdings by only 347.56 more shares. The increased risk of this larger portfolio of risky assets deters him from going further in buying more of the asset with a supernormal return.

New Sol.

In fact one can calculate the increased risk of his new optimal portfolio and compare it to the risk of the portfolio that would have been optimal had the knowledge of the subsidy been available to all investors. Using the formula for the variance of a sum, the overall risk of the initial portfolio of 300 shares of asset 1 and 200 shares asset 2 can be calculated to be 45,000:

New Risk

$$\mathrm{Var}(300,200) = (300)^2\sigma_{11} + (200)^2\sigma_{22} + 2(300\cdot 200)\sigma_{12} = 45000. \tag{6}$$

The change in the variance of the portfolio held by the knowledgeable speculator is attributable solely to the change in the number of shares held -- since the variances and covariance per share are by assumption unaffected by the speculator's action. The variance of the new portfolio increases by 64 percent, up to 73,772:

$$\mathrm{Var}(648,-288) = (648)^2(1/4) + (288)^2(3/16) - 2(648\cdot 288)(1/8) = 73772. \qquad (7)$$

Thus, the rapidly increasing risk the speculator must assume in order to take advantage of his superior knowledge eventually forces him to stop acquiring further shares of asset 1. This is the case despite the fact that he also adjusts his holdings of asset 2 optimally; he even goes short in asset 2, but the overall portfolio risk rises nevertheless.

The above situation is in contrast to the rational expectations equilibrium that eventually would be established when all investors gain knowledge of the subsidy. To calculate the new equilibrium one must now use the formula for the pricing of risky assets in this mean-variance world. As discussed by Mossin (1973), and derived in the classic articles by Lintner (1965), Mossin (1966), and Sharpe (1964), the market price of asset j, P_j, is equal to the following expression:

$$P_j = 1/(1+r)[\mu_j - R\cdot\sum_j \sigma_{jk}], \qquad (8)$$

where the only new term is R, the so-called market price of risk. By assumption, μ_j increases from 0.5 to 1.0, and none of the variances or convariances, the σ_{jk}, change. Hence, if asset j is small enough relative to the market, implying that r and R will not change, then the price of the newly subsidized asset, P_1, changes by $0.5/(1+r)$. The resulting change in P_1 is identical in this case to what would have happened under risk neutrality. It can also be shown that the price of the second asset in Mossin's example, P_2, does not change at all, since none of the terms in the equation for P_2 change. Returning to equations (5), which determine each investors's optimal portfolio, it is easy to show that any optimal portfolio remains unchanged after the information on the subsidy is distributed universally.[7] Thus, in the final rational expectations equilibrium, the price of the subsidized asset jumps to eliminate any excess profit and, once this occurs, the former asset levels, z_1=200 and z_2=300, remain optimal.

It might be useful to reflect further on the nature of the speculator's problem under risk aversion. In this world, which I would argue is a realistic one, the speculator cannot avoid facing a dilemma. He possesses, in this polar case, valuable and certain knowledge, but because of the rules of the game, he can profit from this knowledge only by purchasing the whole of a **risky** probability distribution, i.e. a share of the risky asset on which the subsidy will be paid. And because of this required assumption of more risk, the scale of his purchases is curtailed. What the speculator would really like to do would be to find a way to exploit his knowledge without having to combine it with the assumption of more risk. One way, if allowed and if feasible, would be to

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7. In equations (5) the only variables or parameters that are changed as a result of the release of the subsidy information are μ_1 and, of course, P_1. However, the terms in which these variables appear **are not changed in value**. This is true because the change in P_1 times (1+r) equals exactly the change in μ_1. Hence the term in the equations, μ_1-(1+r)P_1, remains unchanged. Thus the optimal holdings, z_1 and z_2, also remain unchanged: once the information of the subsidy becomes common knowledge, the price of the asset fully reflects this new information, and the old portfolio continues to be optimal.

sell his information separately -- that is, become a consultant. The bottom line seems to be that if a "market participant" (to use Miskin's words) is to be able to use his information to achieve the rational expectations equilibrium, he cannot do it through the purchase of what he knows to be undervalued assets, but only through the efficient dissemination of that information.

IV. Conclusions

The preceding analysis indicates that it is difficult to suggest an adjustment mechanism that can substitute for the widespread dissemination of new information, a process that, depending on market and other conditions, may lead only slowly or not at all to a rational expectations equilibrium. Miskin's hypothesis that a few well-informed speculators could quickly overcome the rest of the market's lack of information and/or rationality and, by eliminating "unexploited profit opportunities," establish a rational-expectations equilibrium, rarely, if ever, holds. Of the three cases examined, the actions of a single speculator, even with the unlikely advantage of unlimited borrowing opportunities, never led to the desired equilibrium. When the number of informed speculators was increased to two or more, it was the case, under risk neutrality, that a Bertrand-type game could lead to the rational expectations solution; however, this was but one of many possible outcomes. Under risk aversion, my preferred assumption, the number of informed speculators would have little effect on the single-speculator outcome until they control a significant share of the market's wealth -- thus implying a significant sharing of the information.

My conclusion is, therefore, that if one wants to argue for the relevance of a rational expectations equilibrium in the short run, one must focus on the conditions that lead to a rapid and widespread dissemination of information.

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