Exercise 1: It's A Small World [10 points]

Skills: sets, cardinality, justifications, and proofs.

There are (3) parts to this exercise (A, B, C).

Surveying 1000 students who speak English and at least Chinese or Spanish or French, we have the following:

- 750 speak at least the Chinese language (and possibly other languages)
- 800 speak at least the Spanish language (and possibly other languages)
- 550 speak at least the French language (and possibly other languages)

Let's call **MAX** the maximum number of students who speak these 3 languages and **MIN** the minimum number of students who speak these 3 languages.

A. [2 points] Calculate MAX. Briefly justify your answer.

> Number of people speaking Chinese = n(C)= 750
>
> Number of people speaking French = n(F)= 800
>
> Number of people speaking Spanish = n(S)= 550
>
> n(CUFUS) = 750+800+550-1000
>
> Max = 1100 (To maximize, we need to maximize the intersection of all 3. Adding 1 to the intersection of all 3 takes care of a surplus of 2. To take care of 1100:
>
> Max = 1100/2=550
>
> There are 550 people who speak these 3 languages.

B. [5 points] Calculate MIN. Justify your answer with solid step-by-step reasoning (you may include a handwritten drawing of sets ala Venn diagram if that helps with your explanation).

> To minimize who speak all three, we need to minimize the intersection.
>
> To take care of surplus of 1100, we can add that to the intersection of any two or all three.
>
> Adding one to the intersection of two sets take care of a surplus of one.

So,

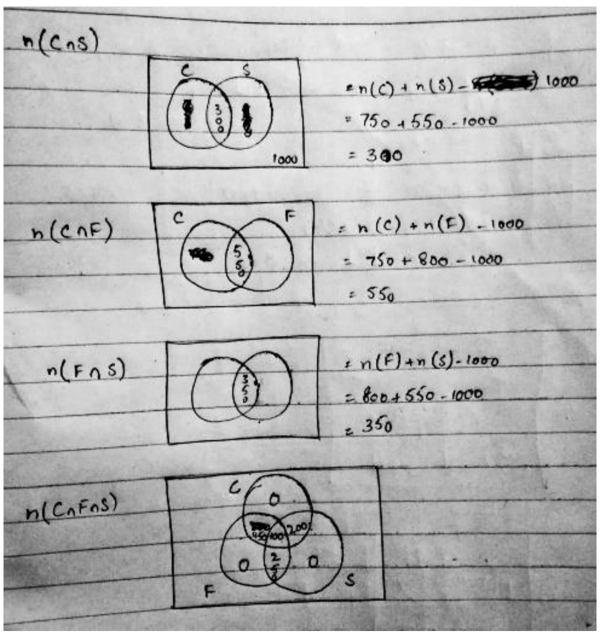
1000 = n(S)+n(C)+n(F)-n(C∩S)-n(C∩F)-n(F∩S)+n(C∩F∩S)

1000 = 800+750+550-300-500-350+n(C∩F∩S)

1000 = 900+ n(C∩F∩S)

1000-900 = n(C∩F∩S)

n(C∩F∩S) = 100



c. **[3 points]** The pandemic strikes and a subset of the 1000 students begin exhibiting symptoms of COVID, including **fever, cough, and loss of smell**. When the medical center screened the 200 symptomatic students, they found 150 students had a fever, 25 students had a cough, 25 students had loss of smell. In addition, 20 students appeared with fever and cough, 10 students showed cough and loss of smell, 5 students had fever and loss of smell. Calculate the number of students exhibiting all three symptoms: fever, cough, and loss of smell.

Following is the representation of the statements along with their values:
Number of people having flu = n(F)= 150
Number of people having loss of smell = n(L)= 25
Number of people having cough = n(C)=25
Number of people having flu and cough = n (F ∩ C) =20
Number of people having flu and loss of smell= n (F ∩ L) =5
Number of people having cough and loss of smell = n (C ∩ L) =10
Number of people screened = n(CUFUL)
Number of people having all three symptoms = n (F∩C∩L) = 200-25-150-25+20+5+10

= 35
35 students exhibit all three of the given symptoms.
Extra Credit Response:
Applying the Inclusion/Exclusion Principle, Subtracting the cardinality of F, L and C from the main set gives us a zero set. Hence, the answer we found is basically the addition of the intersection of all three sets indicating that everyone who had 2 symptoms, had all three of them which appears to be a counterintuitive result.

Exercise 2: Fun with Sets [15 points]

Skills: logical reasoning, critical thinking, set properties.

There are (4) parts to this exercise (A, B, C, D). Note: $N^{>0}$ indicates the set of Natural numbers > 0.

Assume we have $n \in N^{>0}$ distinct sets: $S_1, S_2, S_3, \ldots, S_n$ with the following property:

$\forall x \forall y((1 \le x \le n) \land (1 \le y \le n) \land (x \ne y) \rightarrow S_x \ne S_y)$

A. **[2 points]** Consider the proposition P: $\exists k \forall z((1 \le k \le n) \land ((1 \le z \le n) \land (z \ne k) \rightarrow S_z \not\subset S_k))$. Write an English sentence that accurately and clearly reflects this proposition.

> There exists a natural number k which is greater than or equal to 1 and less than or equal to n **AND** all-natural number z are greater than or equal to 1 and less than or equal to n **AND** z is not equal to k implies that Set Z is not a proper subset of Set K.

B. **[2 points]** Now, consider the negation of the above statement in part A. Express the negation clearly in English.

> For all-natural number k which are less than 1 and greater than n **OR** there are some natural number z which are less than 1 and greater than n **OR** z is equals to k implies that Set Z is a proper subset of Set K.

C. **[5 points]** Prove using BWOC the following: One of the sets from the domain does not have any of the other sets as one of its subsets.

Premise – One of the sets from Domain doesn't have any of the other sets as one of its subsets.
Re-Premise – Every set has another set as one of its subsets within the domain.
Proof by **Contradiction:**
To prove: All the sets have another set as their subsets.

1	All the sets have another set as their subsets	Premise
2	Let set S1 be the smallest set i.e. having minimum cardinality	Domain element
3	S1 have a subset in the domain, consider it S2	1, Universal Insanitiation
4	To S1 be a subset to S2, S1 = S2	Definition of subset
5	S1 ≠ S2 since $\forall x \forall y((1 \le x \le n) \land (1 \le y \le n) \land (x \ne y) \rightarrow Sx \ne Sy)$	Conclusion from Premise
6	Therefore, S1 have no other set in the domain as a subset	3, Contradiction
7	$\exists k \forall z((1 \le k \le n) \land ((1 \le z \le n) \land (z \ne k) \rightarrow S_z \not\subset S_k))$	6, Existential Generalization
8	QED	

Exercise 2 (con't)

D. **[6 points]** In general when trying to solve a problem, you need the following two steps:

1. You prove the problem has at least one solution.
2. You provide an algorithm to find the solution(s).

You have proven in part C above that one of the sets from the domain does not have any of the other sets from the domain as subsets. Now, suppose we want algorithm to find it. Consider one arbitrary set from the domain: S_1 and answer the following:

i. **[2 point]** What is the maximum number of subsets that S_1 may have?

Consider S1 with n elements, then a single element x has two choices, either $x \in S_1$ or $x \notin S_1$. This gives 2 possibilities for every x given 2^n possibilities. Once those choices are fixed, the set is completely determined. There are precisely 2^n subsets.

ii. **[2 point]** What can you say if S_1 does not have any sets from the domain as subsets?

If S1 doesn't have any other set from the domain as its subset, we can safely assume that there are no common elements between S_1 and all the other sets or S1 has the minimum cardinality within domain.

iii. **[2 points]** If S_1 has some sets from the domain as subsets, consider an arbitrary one – call it S2 and indicate an algorithm to find a set which does not have any subset.

A recursive algorithm can be used to find out the sets as subsets of a set from a domain and if the answer is Null or Empty Set, then it indicates that there are no subsets of that set from the domain. A recursive algorithm will keep checking the subset of S2 (if found) to have a same set as a subset of the domain. If there is one, keep repeating until a new subset is found.

Exercise 3: WTF - What the Function?! [10 points]

Skills: functions, injective, surjective, bijective, proof techniques

There are (2) parts to this exercise (A, B).

For the following parts A and B, determine if the functions are injective, surjective, or bijective. You must prove your answer. For example, if you decide a function is only injective, you must prove that it is injective and prove that it is not surjective and that it is not bijective. Similarly, if you claim a function is only surjective, you must prove it is surjective and then prove it is not injective and not bijective.

A. [5 points] The function f: Z → Z where f(x) = 3x.

SURJECTIVE:
f: Z → Z where f(x) = 3x
We will disprove the statement using **Counterexample Proof**.
F is **not SURJECTIVE.**
Proof: No x belongs to number 1. Range of the function is 3Z (integer multiples of 3) which is not equal to Z.

Suppose, x ∈ Z so, f(x)=3x	Premise
For f(1) =1/3	Contradiction within Range

But 1/3 is not in the domain.
Hence, F is not surjective.
QED

INJECTIVE:
f: Z → Z where f(x) = 3x
We will prove the statement using **Direct Proof**.
F is **INJECTIVE.**
Proof: Each element in range is assigned to at most one element of the x. If x is a multiple of 3, only x/3 is mapped to x. If x is not a multiple of 3, there is no input to output x.
Suppose,

Let x, y ∈ Z so, f (x) = f (y).	Definition of Injective
3x = 3y	Math, Division
divide both sides by 3.	
Hence, x = y	Proved

QED

BIJECTIVE:
f: Z → Z where f(x) = 3x
F is not **BIJECTIVE**.
Proof: The corresponding function is missing elements in the range, the function is not Bijective and Surjective but only Injective.

B. [5 points] Define the function g: $N^{>0} \rightarrow N^{>0} \cup \{0\}$ such that g(x) = floor(x/2). You may use the fact that if a = z + r where z ∈ Z and 0 <= r < 1, then floor(a) = z.

SURJECTIVE:
g: $N^{>0} \rightarrow N^{>0}$ where g(x) = floor(x/2)
g is **SURJECTIVE**.
We will prove the statement by **contradiction**.
Proof: Each output gets mapped to at least one input. To prove f is onto, you want to show for any number y in the codomain, there exists a preimage x in the domain mapping to it.

Assume g is not surjective.	Negation of Premise
Note: a = z + r where z ∈ Z and 0 <= r < 1, then floor(a) = z.	Premise
Therefore, there is no a such that g(b) = c where b and c ∈ $N^{>0}$	Definition of Surjective
But there will always be b such that b = c + r	Math, Addition
Therefore, there always exists a b for all c such that g(b) = c.	Contradiction

QED

INJECTIVE:
We will prove the statement by a **counterexample** in a step-by-step reasoning proof style.
g: $N^{>0} \rightarrow N^{>0}$ where g(x) = floor (x/2)
g is **NOT INJECTIVE**.
Proof: Examining the floor function, we see that we don't always get the number we input as an output. A single output can be mapped to different inputs

Suppose, g is Injective.	Premise
Let x, y ∈ N as x = y so, floor (x/2) = floor (y/2).	Definition of Injective
floor(x/2) = floor(y/2)	Definition of Injective
but x ≠ y	Contradiction

Consider an arbitrary example:
floor (2/2) = floor (3/2)
1=1
Since, the value of both of these functions is 1, the function is not injective.
QED

BIJECTIVE:
g: $N^{>0} \rightarrow N^{>0}$ where g(x) = floor (x/2)
g is **not BIJECTIVE**.
Proof: The corresponding function is mapping more than one element to a same output in the range, the function is not Bijective and Injective but only Surjective.
It is not injective since g(3) = g(2). It is surjective since g(1) = 0 and g(2n) = n for n ∈ N.

Exercise 4: Prove It! [15 points]

Skills: sets, cartesian product, critical thinking, proof techniques

There are (5) parts to this exercise (A, B, C, D, E)

A. [4 points] Let X and Y be sets with P(X) and P(Y) being the power sets of X and Y respectively. Prove or disprove that X ⊆ Y if and only if P(X) ⊆ P(Y). Use a numbered line-by-line proof style.

1	X ⊆ Y	Premise 1
2	x ∈ P(X)	x is an element of P(X)
3	x ⊆ X	Since, a set is also a subset of its own set
4	x ⊆ X so x ⊆ Y	3, Transitivity Property
5	x ∈ P(Y)	4,2
6	P(X) ⊆ P(Y)	Proven, **QED**

Conversely,

1	P(X) ⊆ P(Y)	Premise 1
2	x ∈ P(Y)	x is an element of P(Y)
3	x ⊆ Y	Since, a set is also a subset of its own set
4	x ⊆ Y so x ⊆ X	3, Transitivity Property
5	x ∈ X	4,3
6	X ⊆ Y	Proven, **QED**

The statements are proved using **Direct Proof.**

B. [3 points] Let's say M, N, and S be sets, where S is a non-null set. Prove that if M × S = N × S, then M = N. Use a written proof style.

Using **Direct Proof:**
Suppose M x S = N x S. We must now show M = N. First, we will show M is a subset of N. Suppose that m belongs to M. Since, S is a non-null set, so s belongs to S where s is an arbitrary element. By definition of Cartesian Product, (m,s) belongs to M x S. Since, M = N, then (m,s) ∈ N x S. Now, m belongs to N. Hence, M is a subset of N. Now, we will show N is a subset of M. Suppose that m belongs to N. Since, S is a non-null set so s belongs to S, just like said before. By definition of Cartesian Product, (m,s) ∈ N x S. Then, (m,s) ∈ M x S which makes m ∈ M. Hence, N is a subset of M. Above proofs show that M ⊆ Y and Y ⊆ M. Hence, M = N. **QED**

C. **[2 points]** Let M, N, S be finite sets. Use a written style to prove or disprove that if M ∈ N and N ∈ S then M ∈ S.

Using **Counterexample proof:**
Let M be any set, N be the set containing only M, S be the set only containing N. By the set logic, M ∈ N such that N = {M}. By the set logic, N ∈ S such that S = {N} = {{M}}
Then, (M ∈ N) Λ (N ∈ S) -> (M ∉ S). According to the statements, M ∉ S since M ≠ {M}
The condition doesn't match in the considered sets.
Hence, the given condition is **disproved** to be TRUE.
QED

D. **[2 points]** Let X and Y be sets. Use a written style to prove or disprove that if X ∈ Y then P(X) ∈ P(Y).

By using Counterexample **Proof**:
Let X and Y be sets. We will disprove the following statement by using counterexample proof. Let X={a} And Y={{a}}. Now, according to definition, a belongs to X and Y. The P(X)= {{}, a} and P(Y)= {{}, {a}}. This example shows us that while a belongs to both X and Y, the P(X) doesn't belong to P(Y). Hence, the statement is disproved to be true.
QED

E. **[4 points]** Let A = {x ∈ Z | 2 divides x}, B = {x ∈ Z | 9 divides x} and C = {x ∈ Z | 6 divides x}, where Z represents the set of Integers. Use a numbered line-by-line style to prove that A ∩ B ⊆ C.

Using **Direct Proof:**

1	A ∩ B ⊆ C	Premise
2	a ∈ A ∩ B	a is an element belongs to A ∩ B
3	a ∈ A, a ∈ B	By definition of Intersection
4	a ∈ A, 2 divides x	Definition of set A
5	c=2|x, so a=2c	Definition of even
6	a ∈ B, 9 divides x	Definition of set B
7	d=9|x, so a=9d	Definition of even
8	a=even, 9d=even	7, Definition of equality
9	d=2k	Definition of even
10	a=9d, a=9(2k), a=6(3k)	7, Substitution
11	a=6(3k), a=6|x	Definition of set C
12	a ∈ C	11, **QED**
13	We have shown that a ∈ A ∩ B and a ∈ C hence, a ∈ A ∩ B ⊆ C.	