Vanderbilt University – College of Engineering
CS 2212, FALL 2021 - Sections 01/02/03/04/05/06/07
Homework Assignment #3 (maximum 55 points)

Name: (YOUR NAME)

Section: (YOUR SECTION/INSTRUCTOR HERE)

Honor Statement:
By submitting this homework under your personal Gradescope account, you are attesting that you have neither given nor received unauthorized aid concerning this homework.

HW Philosophy:
The objective of HW in CS2212 is to expand on lecture topics and examine them in more detail and more depth. For this reason, we typically give a couple of weeks to complete the HW. Keep in mind that this is not supposed to be an easy task. Expect HWs to take time, require thinking, and challenge you. The TAs and instructors are not here to provide you with an answer or tell you whether your answer is correct while you are working on the HW. We are here to help you think more critically and at a higher level than when you started this class.

Important Instructions:
You must type your answers (or you will lose 50% on the exercise). Each new exercise should begin on a new page (you can put multiple parts to the same exercises on the same page). Save your file as a PDF and upload the PDF document in an electronic format to Gradescope (https://www.gradescope.com/).

You can use different colors if you want to markup your solution but **avoid using a red font.** When submitting your work, be sure to designate the corresponding page(s) of your submission for the appropriate question (or risk losing 5 points). See the following video at the 0:46 mark:
https://www.gradescope.com/get_started#student-submission

Questions:
Post any questions to Piazza or ask your instructor or TAs during office hours.

Due Date:
Your completed homework to include all problems must be uploaded to Gradescope no later than **Thursday, 11/4/2021, 7:00 PM CT (Nashville Time).**

Exercise 1: IEP! The Return of The Inclusion-Exclusion Principle [12 points]

Skills: sets, sequences, inductive thinking

There are (4) parts to this exercise (A, B, C, D)

The Inclusion/Exclusion Principle (IEP) is often referred to as one of the most valuable enumeration principles and dates back to the early 1700s. In this exercise, we'll look at it from a different perspective to see if we can determine the sequence used in the recurring generalized formula.

A. **[2 points - 1, 1]** Given a set of 30 random integers, a set of exactly fifteen of them (call this set A) is divisible by 2, while another set of exactly ten of them (call this set B) is divisible by 3. There's also a set of exactly ten numbers that are not divisible by either 2 or 3. How many of the 30 random integers are divisible by both 2 and 3? Justify your answer using the IEP formula.

PIE:
$|A \cup B| = |A| + |B| + |A \cup B|^c - |A \cap B|$
$30 = 15 + 10 + 10 - (A \cap B)$
$30 = 35 - (A \cap B)$
$(A \cap B) = 5$

Let's also track how we arrive at our answer. Doing so may help us recognize a pattern as we proceed to add more sets. Complete the table below with the appropriate answers. You may not need all the rows.

	IEP with Two Sets	
	Plus (add these)	Minus (get rid of these)
0.	Two Individual Sets	The intersection of two sets
1.	$\lvert A\rvert + \lvert B\rvert$	$\lvert A \cap B\rvert$
2.	$\lvert A \cup B\rvert^c$	$\lvert A \cup B\rvert$

B. **[3 points – 1, 2]** Suppose we take a look closer at this set of 30 random integers and discover another set of exactly six of these numbers (call it C) are divisible by 5. In addition, exactly three of them are divisible by both 2 and 5, exactly two of them are divisible by both 3 and 5, and a set of eight of the numbers are not divisible by any of the numbers 2, 3, 5. How many of the integers are divisible by all three numbers 2, 3, and 5? Justify your answer using the IEP formula.

PIE:
$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (C \cap A) + (A \cap B \cap C)$
$30 = 15 + 10 + 6 - 5 - 3 - 2 + (A \cap B \cap C)$
$30 - 21 = (A \cap B \cap C)$
$9 = (A \cap B \cap C)$

Exercise 1 (con't)

Let's continue to track how we arrive at the answer by populating a table. Complete the table below with the appropriate answers. In row zero, write a description ala what was provided in part A's table in row zero. You may not need all the rows.

	IEP with Three Sets		
	Plus (add these)	Minus (get rid of these)	Plus (add back in these)
0.	Three individual sets	Intersection of 2 sets	Intersection of 3 sets
1.	|A||B||C|	|A ∩ B||B ∩ C||C ∩ A|	|A ∩ B ∩ C|
2.		| A ∪ B ∪ C |	
3.			

C. [4 points - 2, 2] Now, complete the table below to indicate what has to be added/subtracted when working with four and five sets using the IEP. As you did with part B, in row zero, write a description similar to what was provided in part A's table in the row zero. You may not need all the rows.

	IEP with Four Sets			
	Plus (add these)	Minus (get rid of these)	Plus (add back in these)	Minus (get rid of these)
0.	4 individual sets	Intersection of 2 sets	Intersection of 3 sets	Intersection of 4 sets
1.	|A|	|A ∩ B|	|A ∩ B ∩ C|	|A ∩ B ∩ C ∩ D|
2.	|B|	|B ∩ C|	|A ∩ B ∩ D|	
3.	|C|	|C ∩ D|	|A ∩ C ∩ D|	
4.	|D|	|D ∩ A|	|B ∩ C ∩ D|	
5.		|A ∩ C|		
6.		|B ∩ D|		
7.		|C ∩ A|		

Exercise 1 (con't)

	IEP with Five Sets				
	Plus (add these)	Minus (get rid of)	Plus (add back)	Minus (get rid of)	Plus (add back)
0.	5 individual sets	Intersection of 2 sets	Intersection of 3 sets	Intersection of 4 sets	Intersection of 5 sets
1.	|A|	|A ∩ B|	|A ∩ B ∩ C|	|A ∩ B ∩ C ∩ D|	|A∩B∩C∩D∩E|
2.	|B|	|B ∩ C|	|A ∩ B ∩ D|	|A ∩ B ∩ C ∩ E|	
3.	|C|	|C ∩ D|	|A ∩ B ∩ E|	|A ∩ B ∩ D ∩ E|	
4.	|D|	|D ∩ E|	|A ∩ C ∩ E|	|A ∩ C ∩ D ∩ E|	
5.	|E|	|E ∩ A|	|A ∩ C ∩ D|	|B ∩ C ∩ D ∩ E|	
6.		|A ∩ C|	|A ∩ D ∩ E|		
7.		|A ∩ D|	|B ∩ C ∩ D|		
8.		|B ∩ D|	|B ∩ C ∩ E|		
9.		|B ∩ E|	|B ∩ D ∩ E|		
10.		|C ∩ E|	|C ∩ D ∩ E|		
11.					

D. [3 points] The IEP formula can be generalized to any number of n sets. You can probably start to see pattern based on what the work you've completed. The trick is observing the "sign" for the cardinality of the summed combinations in each column has to flip back and forth. Derive the formula for the general form of the IEP over n sets $A_1 \ldots A_n$. **Tip:** When performing a union of a collection of sets, you can use the symbol ∪, much in the way you would utilize the Σ notation to perform a summation of integers. The formula has been started for you below:

$| \cup_{i=1}^{n} A_i |=$

Consider an element a that belongs to the union of sets from A1 to An.

Let a be present in r where 1 <= r <= n.

Conidering an example with 3 sets involving IEP and using indicator functions:

$A_i = X_i$

$A_i^c = 1-X_i$

$A_i \cap A_j = X_iX_j$

$A_i^c \cap A_j^c = (1-X_i)(1-X_j)$

$A_i \cup A_j = 1 - (1-X_i)(1-X_j)$

Now,

$P(A_1 \cup A_2 \cup A_3) = \sum \{A_1 \cup A_2 \cup A_3\}$

$= \sum \{1 - (1 - x_1)(1 - x_2)(1 - x_3)\}$

$$= \sum \{1 - 1 + X1 + x2 + x3 - x1x2 - x1x3 - x2x3 + x1x2x3\}$$

This conclusion leads us to the fact that we would have to sum up the probabilities and then subtract the two way probabilities as shown above and then add the three way probabilities and it goes on and on with the alternate sign until we reach xn.

Hence, we arrive at the general formula:

$$= \sum_{i} P(A_{\,i}) - \sum_{i1 < i2} P(Ai_{\,1} \cap \mathrm{Ai}_{\,2})$$
$$+ \sum_{i1 < i2 < i3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \ldots + (-1)^{n-1} P(A_{i_{\,1}} \cap \ldots \cap A_{i_n})$$

Where as n-1 ensures that the last value has the correct sign.

Exercise 2: NadhaSort! [12 points]

Skills: algorithm and asymptotic analysis, sequences, proof techniques

There are (4) parts to this problem A, B, C, D.

Our favorite computer programmer, Nadha Skolar, picked up a new hobby during the pandemic – making mixed drinks. While watching a TikTok video about making the best Cosmopolitan, Nadha has a clever idea for a new sort he dubbed, **NadhaSort**. The sort is similar to bubble sort, but "bubbles" in both directions of the array. Look at the example below. NadhaSort first bubbles left to right below. Then it reverses direction and bubbles right to left. The sorting continues in this fashion until no swaps are made during a traversal (either left or right). Here's an illustration of what a single pass of NadhaSort idea looks like:

Original list:

43	18	10	23	7

Pass #1:

After traversing through the list left-to-right, 43 is in the correct spot:

18	10	23	7	43

After traversing back through the list (right-to-left), 7 in the correct spot:

7	18	10	23	43

A. [4 points] Prove the worst-case running time of NadhaSort is $\theta(n^2)$ by demonstrating the big-oh and the omega for the worst-case of NadhaSort is n^2.

Consider an array (worst case) in which the whole array is reversely sorted. To traverse each element from left to right and then from right to left, the total number of elements n in an array, the traversal would each time recur to n-1. After an element n is successfully sorted at the end of the array in the first pass, the iteration from right to left will start, sorting the smallest element at index 0. Hence, the traversal would amount to n-2. This will keep going until the array is completely sorted. At the end, the traversal would lead to n, n-1, n-2, n-3,…, 3, 2, 1.

S = 1 2 3 4 ….. n-2 n-1

S = n-1 n-2 ….. 4 3 2 1

Adding both leaves us with,

2S = n n ……… n n n n

The above sequence of statements leaves us with $\frac{n(n-1)}{2}$ that is $\frac{n^2-n}{2}$ that implies the time complexity is **O(n^2)** since n^2 is a quadratic quantity making it larger than the linear quantity n.

In the worst case, the minimum time still required in Omega notation is n^2. **Ω(n^2)** = { f(n): there exist positive constants c and n_0 such that $0 \leq c(n^2) \leq \frac{n^2-n}{2}$ for all $n \geq n_0$ }.

Applying limits to abstain from abusing notations gives us 1. So, we are taking c=1.

The above expression can be described as a function $\frac{n^2-n}{2}$ belongs to the set Ω(n^2) if there exists a positive constant c such that it lies above c(n^2), for sufficiently large n.

Big theta refers to the intersection of both big omega and big oh notations.

Both of the above conclusions that leads to the running time in the worst case being n^2 inevitably points that the big theta(n^2).

B. **[2 points]** In an attempt to annoy his younger brother, the older Skolar, Bedha Skolar, tells Nadha that he could prove his silly NadhaSort is a horrible O(n!) in the worst-case. Is Bedha's analysis of the algorithm correct? Should Bedha's statement annoy Nadha? Explain.

The Bedha analysis of the algorithm is wrong. The worst case refers to the array being reversly sorted. As we have already proved that NadhaSort is O(n^2) in the worst case. Although it offers minimal upgrade as compared to bubble sort but that also in the best case only. Bedha's statement should not annoy Nadha.

C. **[2 points]** Nadha claps back at his brother and says he will have the last laugh because he's about to publish a paper proving that NadhaSort is Ω(1) in both the best and the worst-case scenarios. Nadha says this realization will make him as famous as Sir Isaac Newton. Is Nadha correct about the analysis of his algorithm? Will Nadha be as famous as Sir Isaac Newton based on this claim? Explain.

All the algorithms virtually have complexity Ω(1) i.e., algorithms that take atleast one step.

However, in the best case scenario, the array is already sorted and the algorithm will not be taking any steps and in the worst case scenario, the array is reversely sorted and will take maximum number of steps to re-arrange it. Nadha is correct about his realization and it will make him as+ famous as Sir Isaac.

D. [4 points] Nadha decides to ask his little six-year-old sister, Stellar Skolar, to prove or disprove that the set of even integers, E, is countable. If Stellar is successful, Nadha plans to submit her proof to the CS Toddler Journal of the ACM. As it turns out, Stellar has nap time and needs your help. Prove or disprove that the sets N+ = {1, 2, 3, 4, …} and E = {…-6, -4, -2, 0, 2, 4, 6, …} are the same size. If you prove the claim, clearly define a mapping function f: N+ → E and demonstrate this mapping of N+ to E.

Proof: Set of Positive Natural Numbers have the same size as set of all even integers.

Using Direct Proof:

One of the reasons that can help us in proving the same size of the given sets is if we could prove the bijection between the two given sets. Bijection helps us define that each element from either set is related or projecting to only on element of the other set. Hence, defining the same size.

Now, to prove that N+ and E have same size:

One representation of the mapping function could be:

1→0

2→2

3→-2

4→4

5→-4

6→6

7→-6

… and so on. Now, this demonstration shows us that f(n) = n if n is even or n>=0 and f(n)= 1-n.

Every element of N corresponds to one element of E. Every element of E corresponds to one element of N.

Exercise 3: Inductively Speaking [9 points]

Skills: logical reasoning, critical thinking, inductive proof techniques

There are (2) parts to this exercise (A, B).

A. **[3 points]** An often ignored fact about France is that a single one-way road connects any two cities. Utilizing this fact, use induction to prove an itinerary passes through every city at least once. As an example, assuming we only have 5 cities in France {Brest, Lyon, Marseille, Paris, Strasbourg}, one such itinerary could be Paris → Lyon → Marseille → Brest → Paris → Strasbourg. **Tip:** It's ok to include the use of cases within an inductive step if needed.

By induction, lets prove the base case first.
Suppose, n = 2, then there are only 2 cities n and the path between them exists.
Now, consider the same case for n cities.
To prove for n + 1, using induction hypothesis:
Suppose, the starting city C is Paris i.e., C_p and an itinerary passes through all the cities n to the Brest which is the last city ala n+1.
If all cities are going towards the end and there is no restriction for at most passing once, through any city.
There exists a path from C_{i-1} to C_{n+1} and from C_{n+1} to C_i.

B. **[6 points]** Let S be a set whose elements are non-zero natural numbers (N+). Let f be a function such that f(S) = square of the product of all elements in S. For example, if S = {2, 4, 5} then f(S) = $(2*4*5)^2 = 40^2$ = 1600. Now consider S_n = {1, 2, 3, 4…, n} the set of the first n natural numbers greater than 0. For each subset T of S_n that is not empty and does not contain two consecutive integers, we calculate f(T), and determine its sum Σ. As an example, consider S_3 = {1, 2, 3}. The subsets of S_3 which are not empty and do not contain consecutive numbers are: {{1}, {2}, {3}, {1, 3}}. Plugging into the previously described function gives us f({1}) = 1, f({2}) = 4, f({3}) = 9, f({1, 3}) = 9. The summation Σ = 1+4+9+9=23.

Prove by mathematical induction that the summation Σ = (n+1)! -1.

We can write it as:

$$\sum_{i=0}^{0} i.i! = (n + 1)! - 1$$

To prove by mathematical induction, We will first prove the basic step i.e., for n.

Take n = 0.

$$\sum_{i=0}^{0} \text{i.i!} = (0+1)! - 1$$

The LHS is 0 and RHS is also 0. Hence, the equation holds as both sides are zero.
Now, for any integer k such that k >= 0. We need to prove that S_n implies S_{n+1}.
So,
By using Direct Proof, we have to prove

$$\sum_{i=0}^{k+1} \text{i.i!} = ((\text{k}+1)+1)! - 1$$

Suppose,

$$\sum_{i=0}^{k+1} \text{i.i!} = (\text{k}+1)! - 1$$

Now,

$$\sum_{i=0}^{k+1} \text{i.i!} = \{\sum_{i=0}^{k} \text{i.i!}\} + (\text{k}+1)(\text{k}+1)!$$

$= \{(\text{k}+1)! - 1\} + (\text{k}+1)(\text{k}+1)!$
$= (\text{k}+1)! + (\text{k}+1)(\text{k}+1)! - 1$
$= \{1 + (\text{k}+1)\}(\text{k}+1)! - 1$
$= (\text{k}+2)(\text{k}+1)! - 1$
$= (\text{k}+2)! - 1$
$= ((\text{k}+1)+1)! - 1$

Hence, proved LHS =RHS.
We have now proved that
$\sum_{i=0}^{0} i.i! = (n+1)! - 1$ for every integer in S_n.

Exercise 4: It's All Related [12 points]

Skills: relations, specialty relations, proof techniques

There are (5) parts to this exercise (A, B, C, D, E)

A. **[1 point]** Consider an equivalence relation defined on the set A = {0, 1, 2, 3, 4, 5, 6, 7, 11, 12} by: a ∼ b as meaning 3|(a – b). List the elements of the equivalence classes of ∼.

> [0] = {0,2,4,6,12} = set of odd numbers in A.
> [1] = {1,3,5,7,11} = set of prime numbers in A.

B. **[2 points]** Given the relation R = {(1,3), (3,1), (1,1)} over the set A = {1, 2, 3, 4, 5}, indicate whether the relation R is reflexive? Symmetric? Transitive? Justify each of your answers.

> It is symmetric because for (3,1) we have (1,3).
> It is not reflexive because we have (1,1) but we don't have (2,2), (3,3), (4,4) and (5,5).
> It is transitive because for xRy and yRx it implies that xRx. Take x = 1, y = 3 and it holds the transitive property due to (1,1).

C. **[3 points]** If a minus b is an integer multiple of 5 (i.e., a - b = 5k) and aRb, prove or disprove that R is an equivalence relation. You may assume a, b, k ∈ Z.

> a – b = 5k can also be written as 5k|a-b.
> Since, a, b, k ∈ Z.
> Reflexive: a − b = 5k ; a − a = a(5k). Since, a belongs to Z and is an integer multiple of 5, so is 5k.
> Symmetric: a – b = 5k ; a = 5k – b
> Now, b – a:
> b – 5k + b
> 2b – 5k where 2b ∈ Z but is not an integer multiple of 5. Hnece, the ssymmetric property doesn't hold.
> aRb is not an equilance relation.

D. **[4 points]** Let a, b ∈ Z, where Z is the set of integers. Define a relation S on Z as aSb iff an + bn is even for some number n. Prove that S is an equivalence relation.

> Reflexive: aSa iff an+an = 2an which is even since we can define even for any number of n with 2a for any integer a.

Symmmetric: If aSb is even, then bSa is even by upholding the commutative property as a+b =b+a for a,b ∈ Z.
Transitive: If aSb and bSc then aSc i.e.,
an+bn=bn+cn
an+bn-bn-cn=0
an-cn=0
-(an+cn) = 0 where a,b ∈ Z and Z is closed under addition and multiplication, for some number n
There are two cases to be discussed as well:
Case 1: a/b are both even.
bSc means b+c is even, so b and c both must be even. Since, a and c are also even that means an+cn is also even for n > 0.
Case 2: a/b are both odd.
bSc is odd, so a and b both must be odd too. Sum of two odds is even. Hence, a+c is even too which means an+cn is also even for n>0

E. **Extra Credit:** Let a, b ∈ Z, where Z is the set of integers. Define a relation S on Z as aSb iff $a^n + b^n$ is even for some number n. Prove that S is an equivalence relation.

Reflexive: aSb if $a^n + a^n = 2a^n$ and for even, we can write 2a where a is some integer for any number n.
Symmetric: $a^n + b^n = b^n + a^n$. Since, an is even that means than means bn is also even and by upholding the comuutative law, we prove that it is symmetric.
Transitive: $a^n + b^n = b^n + c^n = a^n + c^n$ where a,b,c ∈ Z.
So there exist k,l ∈ Z such that $2k = a^n + b^n$ and $2l = b^n + c^n$.
Now,
$a^n = 2k - b^n$
$c^n = 2l - b^n$
$a^n + c^n = 2k - b^n + 2l - b^n$
$a^n + c^n = 2k + 2l - 2b^n$
$a^n + c^n = 2(k + l - b^n)$ where k,l ∈ Z and Z is closed under multiplication and addition.

F. **[2 points]** If S is an equivalence relation in the above question in part (D), determine its distinct equivalent classes where n=2.

The equation an+bn where n = 2 is:
2a+2b = 2(a+b) where a,b ∈ Z.
Hence,
[2] = {x: xR1} = {x: 2(a+b) are even} = All even numbers.

Exercise 5: A Recurring Theme [10 points]

Skills: sequences, summations, recurrence relations, closed form

There are (2) parts to this exercise (A, B)

A. [3 points] Find the closed formula for 10 + 16 + 22 + … + (6n-2)

To find the arithemetic sequence:
6n-2 is the last term as given.
To get 10 with 6n-2, we get n = 2.
So, terms range from 2, 3, …, n. Hence, it is n – 1.
Now, reverse and add
S = 10 + 16 + … + 6n – 8 + 6n – 2
S = 6n – 2 + 6n – 8 + … + 16 + 10
2S = 6n + 8 + 6n + 8 + … + 6n + 8 + 6n + 8
2S = (n – 1)(6n + 8)
S = ((n-1)(6n + 8))/2

B. [7 points – 1, 2, 4] For Homecoming week, the Vanderbilt Parent Organization decides it would be fun to temporarily install a "Magical Mystery Chocolate Machine" outside the football stadium. This special vending machine behaves rather strangely. The first time someone inserts a dollar bill into the machine, one chocolate is dispensed. The second time, however, a quantity of seven chocolates is dispensed. The third time someone inserts a dollar bill into the machine, 31 chocolates are dispensed. The fourth time 127 chocolates, etc. This sequence continues forever because the magical mystery chocolate machine is constantly and rapidly making its own chocolate:

(i) [1 point] Being from France, it is not surprising that Prof. Piot is an active chocolatier during his summer months. After studying the machine carefully for a while, he discovers that the number of chocolates that are dispensed the "nth time" is $2^{2n-1} – 1$. Based on this revelation, indicate how many chocolates the 10th student receives after inserting their money into the machine. Show your work by filling in the table below.

nth student	Chocolates Dispensed
1	1
2	7
3	31
4	127
5	511
6	2047
7	8191

8	32767
9	131071
10	524287

ii. **[2 points]** Prof. Hasan spent some time in Switzerland at a conference and learned about chocolates during his free weekend. After studying the machine, he disputes Prof. Piot's claim about the rate at which the chocolate is dispensed. Instead, he insists the dispensing of chocolates follows a linear recurrence relation. If the production of the machine is a_n for the n^{th} student, then we actually have: $a_n = k_{n-1}*a_{n-1} + k_{n-2}*a_{n-2} + ... + k_1*a_1 + k_0*a_0.$ By studying the values of k_{n-1}, k_{n-2}, $k_{n-3}...k_0$ from part (i), devise the recurrence relation that Prof. Hasan is hinting at.

Taking k from part 1 leaves us with the following sequence:
1, 7, 31, 127, 511, 2047, …
Difference between each term is growing by a factor of 4.
To check: 1.4 = 4, 4.7 = 28, 31.4 = 124
It seems that the every answer we get by multiplying with a factor of 4 is 3 times less than the original sequence.
Hence,
$a_n= 4a_{n-1} + 3$.

iii. **[4 points]** We now need to determine whether both Prof. Piot and Prof. Hasan are actually talking about the same formula (spoiler alert: They are!). Solve the recurrence relation you derived in part (ii) to arrive at a closed form for the recurrence relation.

We have the relation as: $a_n= 4a_{n-1} + 3$.
Using Iterative Method:
Consider $a_0 = 1$.
$a_1 = 4a_0 + 3$
$a_2 = 4(a_1) + 3 = 4(4a_0 + 3) + 3 = 4^2a_0 + 3 . 4+3$

$a_3 = 4(a_2) + 3 = 4(4^2a_0 + 3 . 4+3) + 3 = 4^3a_0 + 3.4^2 + 3.4 + 3$

…

$a_n = 4(a_{n-1}) + 3 = 4(4^{n-1}a_0 + 3.4^{n-2} + 3.4^{n-3}+ … + 3) + 3 = 4^na_0 + 3.4^{n-1} + 3.4^{n-2}+ … + 4.3 + 3$

We can simply assume that 4^n along with the multiple factor of 3. This shows us the geometric term with first term 3 and common ratio 4.
By simplifying the equation of $3 + 3.4 + 3.4^2 + 3.4^3 + … + 3.4^{n-1}$, we get $\frac{3 - 3.4^n}{-3}$ which can also be simplified to $4^n – 1$.
Putting this together with the factor of 3, we get:
$a_n= 3.4^n – 1$ or $3.2^{2n} – 1$.