SKJ Education

LC HL MATHS

FOUNDATION PROGRAM: WEEK 6

TRIGONOMETRY

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LC HL MATHEMATICS – FOUNDATION PROGRAM

Week 6: Trigonometry

Learning Objectives

- **6.1:** To apply trigonometric ratios (sin, cos, tan) to solve problems involving right-angled triangles.
- **6.2:** To use formulae like the Sine and Cosine Rules to find unknown sides, angles, and the area of a triangle in non-right triangles.
- **6.3:** To prove and manipulate key trigonometric identities.
- **6.4:** To solve complex trigonometric equations for a general solution.

Key Terms - Week 6

- **Trigonometric Ratios:** The ratios of the sides of a right-angled triangle, defined as $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, and $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.
- **Sine Rule:** A formula for finding unknown sides or angles in a non-right triangle: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- **Cosine Rule:** A formula for finding unknown sides or angles in a non-right triangle: $c^2 = a^2 + b^2 - 2ab\cos C$.
- **Trigonometric Identities:** Equations that involve trigonometric functions, which are true for all values of the variable(s) (e.g., $\sin^2\theta + \cos^2\theta = 1$).
- **General Solution:** A solution to a trigonometric equation that includes all possible values of the variable, often expressed in terms of a parameter (e.g., n).

Weekly *Challenge*: A surveyor is measuring the height of a building using trigonometry. The angle of elevation to the top of the building is measured from two different points on the ground. Using the Sine Rule and trigonometric identities, derive a formula to calculate the height of the building. ***Explain the steps taken to arrive at the solution*** and discuss the assumptions made.

WEEK 2 STUDY PLAN

Day	**Activities & Time Commitment**	✓	**Rating (1-10)**
Monday	- Review Learning Objectives (5 min) - Rank your current ability (5 min) - Review Key Terms (10 min) - Watch video (10-20 min) *Focus: PREPARATION*		
Tuesday	- Complete Exercises A1 & A2 (60 min) *Focus: EXPLORING*		
Wednesday	- Complete Exercise A3 (30 min) - Correct Exercises A1-3 and reattempt difficult questions (45 min) *Focus: PROCESSING*		
Thursday	- 1-hour online lesson (60 min) *Focus: QUESTIONING*		
Friday	- Complete Exercise B (40 min) *Focus: ERROR ANALYSIS*		
Saturday	- Complete Exam Question Assessment (C) (60 min) *Focus: EXECUTION*		
Sunday	- Correct assessment (30 min) - Complete self-reflection (15 min) - Plan next week (15 min) *Focus: REFLECTION & RECHARGING*		

Study Tips for Success

- **Active Recall:** After studying, close your notes and write down **everything you remember**. Force your brain to grow.
- **Spaced Repetition:** Review concepts **multiple times** over several days.
- **Mathematics in Action:** Look for **real-world examples** of the concepts you're learning.
- **Ask Questions:** Don't hesitate to ask for help when concepts are unclear. Reach out via *Google Classroom* or email; *steven@skjeducation.com*.
- **Celebrate Progress:** **Acknowledge your improvements**, no matter how small.

A1. Proficiency Drills (Week 6)

Learning Focus: Mastering key concepts in **trigonometry**, from solving right-angled and non-right triangles to proving identities and solving complex trigonometric equations.

Part 1: Right-Angled Triangle Trigonometry

Key Concepts

Trigonometric Ratios (SOH CAH TOA): For an angle θ in a right-angled triangle:
- $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$
- $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
- $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$

Pythagoras' Theorem: $a^2 + b^2 = c^2$, where c is the hypotenuse.

Task #1: Solve the following problems.
1. In a right-angled triangle, one angle is 35° and the side adjacent to it is 10 cm. Find the length of the opposite side and the hypotenuse.
2. From a point on the ground 50 meters away from the base of a building, the angle of elevation to the top of the building is 60°. What is the height of the building?

Part 2: The Sine and Cosine Rules

Essential Formulae for Non-Right Triangles

For any triangle with angles A, B, C and opposite sides a, b, c:
- **Sine Rule:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (Used when you have a side-angle pair).
- **Cosine Rule:** $c^2 = a^2 + b^2 - 2ab\cos C$ (Used for Side-Angle-Side or Side-Side-Side cases).
- **Area of a Triangle:** Area $= \frac{1}{2}ab\sin C$.

Task #2: In triangle PQR, side $p = 12$ cm, side $q = 15$ cm, and angle $R = 50°$.
1. Find the length of side r.
2. Find the measure of angle P.
3. Calculate the area of triangle PQR.

Part 3: Trigonometric Identities

Fundamental Identities

An identity is an equation that is true for all possible values of the variable.

- **Quotient Identity:** $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- **Pythagorean Identity:** $\sin^2\theta + \cos^2\theta = 1$

Proof Strategy: To prove an identity, start with one side (usually the more complex one) and use algebraic manipulation and known identities to transform it until it is identical to the other side.

Task #3: Prove the following trigonometric identities.

1. Prove that $(\sin x + \cos x)^2 \equiv 1 + 2\sin x\cos x$.
2. Prove that $\frac{1}{\cos^2\theta} - 1 \equiv \tan^2\theta$.

Part 4: Trigonometric Equations

Solving Trigonometric Equations

The goal is to find all angles that satisfy the equation.

1. Isolate the trigonometric function (e.g., $\sin x = k$).
2. Find the **principal value** or reference angle using the inverse function (e.g., $x = \arcsin(k)$).
3. Use the properties of the unit circle (or ASTC diagram) to find all solutions within a specific range.
4. For the **general solution** (in radians):
   - If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$
   - If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$
   - If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$ (where n is any integer)

Task #4: Find the general solution for the following equations (in radians).

1. $2\sin\theta - 1 = 0$
2. $2\cos^2\theta - 3\cos\theta + 1 = 0$

Answers:

- **Task #1:**
  1. Opposite side: $\tan(35°) = \frac{\text{Opp}}{10} \implies \text{Opp} = 10\tan(35°) \approx 7.00$ cm. Hypotenuse: $\cos(35°) = \frac{10}{\text{Hyp}} \implies \text{Hyp} = \frac{10}{\cos(35°)} \approx 12.21$ cm.
  2. Let h be the height. $\tan(60°) = \frac{h}{50}$. Since $\tan(60°) = \sqrt{3}$, $h = 50\sqrt{3} \approx 86.6$ meters.
- **Task #2:**
  1. Using Cosine Rule: $r^2 = 12^2 + 15^2 - 2(12)(15)\cos(50°) \approx 144 + 225 - 360(0.6428) \approx 137.59$. So, $r \approx \sqrt{137.59} \approx 11.73$ cm.
  2. Using Sine Rule: $\frac{\sin P}{12} = \frac{\sin 50°}{11.73} \implies \sin P = \frac{12 \sin 50°}{11.73} \approx 0.783$. So, $P = \arcsin(0.783) \approx 51.53°$.
  3. Area $= \frac{1}{2}(12)(15)\sin(50°) = 90(0.766) \approx 68.94$ cm^2.
- **Task #3:**
  1. LHS $= (\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$. Since $\sin^2 x + \cos^2 x = 1$, this simplifies to $1 + 2\sin x \cos x =$ RHS.
  2. LHS $= \frac{1}{\cos^2\theta} - 1 = \frac{1-\cos^2\theta}{\cos^2\theta}$. Using $\sin^2\theta + \cos^2\theta = 1 \implies \sin^2\theta = 1 - \cos^2\theta$. So, LHS $= \frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta =$ RHS.
- **Task #4:**
  1. $2\sin\theta = 1 \implies \sin\theta = 1/2$. The principal value is $\alpha = \arcsin(1/2) = \pi/6$. General solution: $\theta = n\pi + (-1)^n(\frac{\pi}{6})$.
  2. Let $y = \cos\theta$. The equation is $2y^2 - 3y + 1 = 0$, which factors to $(2y-1)(y-1) = 0$. So, $y = 1/2$ or $y = 1$. Case 1: $\cos\theta = 1/2$. Principal value $\alpha = \arccos(1/2) = \pi/3$. General solution: $\theta = 2n\pi \pm \frac{\pi}{3}$. Case 2: $\cos\theta = 1$. Principal value $\alpha = \arccos(1) = 0$. General solution: $\theta = 2n\pi \pm 0 = 2n\pi$.

A2. Worked Example & Questions

Learning Focus: Understand and apply trigonometric functions to model real-world periodic phenomena.

Scenario

The height of the tide, h meters, at a certain beach is modeled by $h(t) = 2 + 1.5\sin(\frac{\pi t}{6})$, where t is the time in hours after midnight. Find $h(3)$ and interpret the result.

Solution Framework:

1. **Decode & Define:** "The given function models the height of the tide. We need to evaluate $h(t)$ at $t = 3$."
2. **Plan:**
   - Substitute $t = 3$ into the function $h(t)$.
   - Calculate $h(3)$.
   - Interpret the result in the context of the problem.
3. **Execute:**
   - $h(3) = 2 + 1.5\sin(\frac{\pi \cdot 3}{6}) = 2 + 1.5\sin(\frac{\pi}{2})$.
   - Since $\sin(\frac{\pi}{2}) = 1$, $h(3) = 2 + 1.5 \cdot 1 = 3.5$ meters.
4. **Evaluate:** "At $t = 3$ hours after midnight, the height of the tide is 3.5 meters. This demonstrates how trigonometric functions can model periodic phenomena like tides."

Test Yourself

1. **(Knowledge)** What are the key characteristics of the sine and cosine functions that make them suitable for modeling periodic phenomena?
2. **(Application)** A company's seasonal sales are modeled by $S(t) = 100 + 20\cos(\frac{\pi t}{3})$, where t is the time in months. Find $S(2)$ and interpret the result.
3. **(Analysis)** Consider the function $h(t) = 1 + 2\sin(3t)$. Discuss how the amplitude, period, and vertical shift affect the graph of $h(t)$.
4. **(Synthesis)** The temperature in a city is modeled by $T(t) = 15 + 5\sin(\frac{\pi t}{12})$, where t is the time in hours after midnight. Find the maximum and minimum temperatures and the times at which they occur.

C. Weekend Assessment – Past Exam Questions

Learning Focus: Applying learning to past exam questions under exam conditions.

Practical Advice: To tackle the questions in this assessment, consider the following tips:

- For trigonometric identities (Q1(a), Q2(a), Q4(a)), use similar formulae and simple algebraic manipulation.
- When working with 3D geometry (Q1(b)), visualise the cube and identify relevant vectors.
- For compound angle formulas (Q2), break down angles into sums of standard angles.
- When solving trigonometric equations (Q2(c), Q3(b), Q4(b)), consider all possible solutions in the given domain (graphs really help!).
- For triangle problems (Q3, Q6, Q7), apply sine rule, cosine rule, and area formulas appropriately.
- When working with vectors and relative motion (Q5), set up position vectors carefully (coordinates can help!).
- For optimisation problems (Q5(c)), differentiate the distance squared function.
- When dealing with applied trigonometry (Q6, Q7), draw clear diagrams and identify relevant triangles.

General Tips:

- **Show your work** - partial credit is available for correct methods
- **Check domains/restrictions** - especially in trigonometric equations
- **Label answers clearly** - especially when questions have multiple parts
- **Use appropriate precision** - note where decimal places are required
- **Verify your answers** - do they make sense in the context of the problem?

Question 1 (25 marks)
(a) Show that $\cos 2\theta = 1 - 2\sin^2\theta$.
(b) Find the cosine of the acute angle between two diagonals of a cube.

Question 2 (30 marks)
(a) Prove that $\sin(A+B) = \sin A \cos B + \cos A \sin B$.
(b) Using the formula in part (a), and without using a calculator, find the value of $\sin 75°$. Give your answer in surd form.
(c) Find all solutions of the following equation in t, for $0° \leq t \leq 360°$:

$$\sin t = \sin(2t)$$

Question 3 (30 marks)
(a) $ABCD$ is a parallelogram. $|AB| = 10\text{cm}, |BC| = 13\text{cm}, \text{and} |\angle ABC| = 110°$. Find the area of $ABCD$, correct to the nearest cm^2.
(b) X is an angle, with $0° \leq X \leq 360°$, and

$$\cos(2X) = \frac{\sqrt{3}}{2}$$

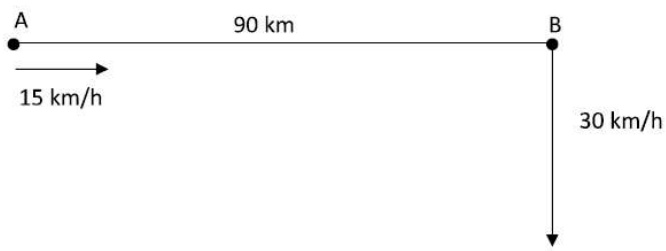
Find all the possible values of X.
(c) KLM is a triangle where $|MK| = 15\sqrt{3}\text{cm}$, $|ML| = 45\text{cm}$, and $|\angle KLM| = 25°$. θ is the angle $\angle LKM$.
Work out the two possible values of θ, for $0° < \theta < 180°$. Give each answer correct to the nearest degree.

Question 4 (30 marks)
(a) (i) Prove that $\cos 2A = \cos^2 A - \sin^2 A$.
(ii) $\sin \frac{\theta}{2} = \frac{1}{\sqrt{5}}$, where $0 \leq \theta \leq \pi$. Use the formula $\cos 2A = \cos^2 A - \sin^2 A$ to find the value of $\cos \theta$.
(b) Solve the equation:

$$\tan(B + 150°) = -\sqrt{3},$$

for $0° \leq B \leq 360°$.

Question 5 (25 marks)
Two ships, P and Q, are moving with constant velocities. At 12:00, ship P is at the point O, and ship Q is 20 km due north of O. Ship P is moving at a constant velocity of $10\mathbf{i} + 24\mathbf{j}$ km/h, and ship Q is moving at a constant velocity of $-10\mathbf{i} + 10\mathbf{j}$ km/h, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the east and north directions respectively.

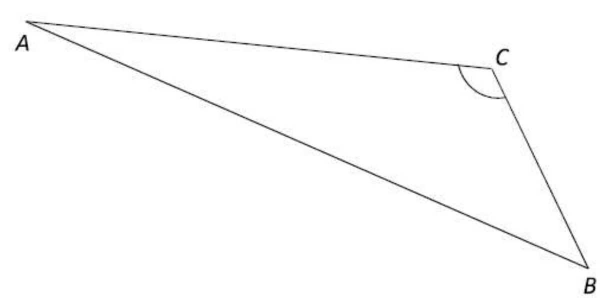


(a) Write down the position vectors of ships P and Q at time t hours after 12:00.
(b) Show that the distance d km between the two ships at time t is given by:

$$d^2 = 500t^2 - 400t + 400$$

(c) Use calculus to find the value of t when the ships are closest to each other, and find the distance between the ships at your value of t. Give the distance in km, correct to 1 decimal place.

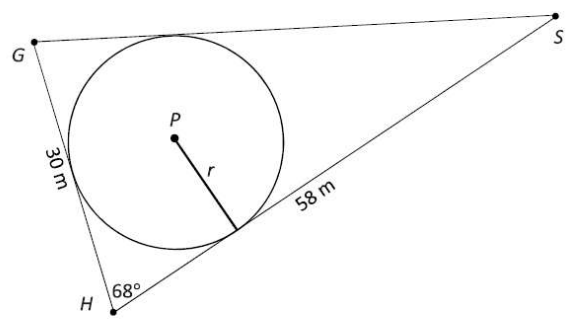
Question 6 (50 marks)
The diagram (Triangle ABC) shows the 3 sections of a level triathlon course. In order to complete the triathlon, each contestant must swim 4 km from C to B, cycle from B to A, and then run 28 km from A to C. Mary can cycle at an average speed of 25 km/hour. It takes her 1 hour and 12 minutes to cycle from B to A.



(a) Show that the total length of the course is 62 km.
(b) On average, Mary can run 5.6 times as fast as she can swim. It takes her 4.8 hours to complete the course. Find her average swimming speed in km/h.

(c) Show that $|\angle ACB| = 116.5°$, correct to 1 decimal place.
(d) To comply with safety regulations, the region inside the triangular course must be kept clear of people. Find the area of this region. Give your answer, in km^2, correct to 1 decimal place.
(e) Find the shortest distance from the point C to the side AB. Give your answer in km, correct to 1 decimal place.
(f) The course is viewed from a camera tower which rises vertically from point A. The top of the tower is point T. The angle of elevation of T from B is 0.05°. Find $|AT|$, the vertical height of the tower. Give your answer correct to the nearest metre.

Question 7 (50 marks)
The diagram below shows a triangular patch of ground ΔSGH, with $|SH| = 58$m, $|GH| = 30$m, and $|\angle GHS| = 68°$. The circle is a helicopter pad. It is the incircle of ΔSGH and has centre P.



(a) Find $|SG|$. Give your answer in metres, correct to 1 decimal place.
(b) Find $|\angle HSG|$. Give your answer in degrees, correct to 2 decimal places.
(c) Find the area of ΔSGH. Give your answer in m^2, correct to 2 decimal places.
(d) (i) Find the area of ΔHSP, in terms of r, where r is the radius of the helicopter pad.
(ii) Show that the area of ΔSGH, in terms of r, can be written as $71.2r\ m^2$.
(iii) Find the value of r. Give your answer in metres, correct to 1 decimal place.
(e) There is a vertical pole at the point S. The angle of elevation of the top of the pole from the point P is 14°. Find the height of the pole. Give your answer, in metres, correct to 1 decimal place.

Self-Assessment

After completing the assessment:

- Grade your work honestly
- Identify areas needing improvement
- Scan and submit via Google Classroom
- Reflect on your performance in your weekly reflection

Another excellent week of work completed - ***well done!*** You are another step closer to *smashing your exams*, and another week closer to your summer holdiays!

Weekly Reflection Zone

What worked well this week?

What challenges did I face?

What surprised me the most this week?

Key mathematics concepts I want to review:

Goals for next week: