SKJ Education

LC HL MATHS

FOUNDATION PROGRAM: WEEK 3

BASIC CALCULUS

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LC HL MATHEMATICS – FOUNDATION PROGRAM

Week 3: Basic Calculus

Learning Objectives

- **3.1:** To define the derivative from first principles and apply differentiation rules to polynomials.
- **3.2:** To differentiate trigonometric, exponential, and logarithmic functions.
- **3.3:** To use differentiation to find slopes, equations of tangents/normals, and to identify maxima, minima, and points of inflection.
- **3.4:** To define integration as the inverse of differentiation and integrate polynomial, trigonometric, and exponential functions.

Key Terms - Week 3

- **Derivative:** The rate of change of a function with respect to one of its variables (e.g., $\frac{d}{dx}f(x)$ represents the derivative of $f(x)$ with respect to x).
- **Differentiation from First Principles:** The process of finding the derivative of a function using the limit definition: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
- **Differentiation Rules:** Rules for differentiating various types of functions, including the power rule, product rule, and quotient rule.
- **Integration:** The process of finding the antiderivative of a function, i.e., finding a function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$.
- **Tangent and Normal Lines:** The tangent line to a curve at a point is the line that just touches the curve at that point, while the normal line is perpendicular to the tangent line.
- **Maxima and Minima:** The maximum and minimum values of a function, which can be found using differentiation.
- **Point of Inflection:** A point on a curve where the curvature changes sign, often identified using the second derivative.

Weekly *Challenge*: Investigate a real-world scenario where optimisation using calculus is applied, such as minimising the cost of materials for a container or maximising the area of a fenced region. Use differentiation to find the optimal solution and ***explain the steps taken to arrive at the solution***.

WEEK 2 STUDY PLAN

Day	Activities & Time Commitment	✓	Rating (1-10)
Monday	- Review Learning Objectives (5 min) - Rank your current ability (5 min) - Review Key Terms (10 min) - Watch video (10-20 min) *Focus: PREPARATION*		
Tuesday	- Complete Exercises A1 & A2 (60 min) *Focus: EXPLORING*		
Wednesday	- Complete Exercise A3 (30 min) - Correct Exercises A1-3 and reattempt difficult questions (45 min) *Focus: PROCESSING*		
Thursday	- 1-hour online lesson (60 min) *Focus: QUESTIONING*		
Friday	- Complete Exercise B (40 min) *Focus: ERROR ANALYSIS*		
Saturday	- Complete Exam Question Assessment (C) (60 min) *Focus: EXECUTION*		
Sunday	- Correct assessment (30 min) - Complete self-reflection (15 min) - Plan next week (15 min) *Focus: REFLECTION & RECHARGING*		

Study Tips for Success

- **Active Recall:** After studying, close your notes and write down **everything you remember**. Force your brain to grow.
- **Spaced Repetition:** Review concepts **multiple times** over several days.
- **Mathematics in Action:** Look for **real-world examples** of the concepts you're learning.
- **Ask Questions:** Don't hesitate to ask for help when concepts are unclear. Reach out via *Google Classroom* or email; *steven@skjeducation.com*.
- **Celebrate Progress:** **Acknowledge your improvements**, no matter how small.

A1. Proficiency Drills

Learning Focus: An introduction to **differential and integral calculus**, covering derivatives from first principles, differentiation of standard functions, applications of derivatives, and basic integration.

Part 1: The Derivative - First Principles and Basic Rules

Key Concepts

The Derivative from First Principles: The derivative of a function $f(x)$, denoted $f'(x)$, represents the instantaneous rate of change. It is formally defined as the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The Power Rule: For any real number n, the derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1}$. This is the fundamental rule for differentiating polynomials. The derivative of a sum of terms is the sum of their derivatives.

Task #1: Complete the table below for each function.

Function $f(x)$	**Derivative** $f'(x)$ **using the Power Rule**
$f(x) = x^3 - 5x^2 + 2x + 8$	
$f(x) = 4x^5 - 3x^{-2}$	
$f(x) = \frac{1}{2}x^4 + 7x - 1$	
$f(x) = 4x^{\frac{1}{2}} + 7\frac{1}{x^2} - 2\sqrt{x}$	
$f(x) = (x+2)(x-3)$	
$f(x) = \frac{x^2-4}{x+2}$	

Bonus Task: Using first principles, prove that the derivative of $f(x) = 3x^2$ is $f'(x) = 6x$.

Part 2: Differentiating Trigonometric, Exponential, & Logarithmic Functions

Essential Derivatives

Beyond polynomials, several standard functions have known derivatives:

- **Trigonometric:** $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$
- **Exponential:** $\frac{d}{dx}(e^x) = e^x$
- **Logarithmic:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

The Chain Rule (Simple Form): To differentiate a function of a function, such as $f(ax+b)$, we use the chain rule: $\frac{d}{dx}[f(ax+b)] = a \cdot f'(ax+b)$.

Task #2: Find the derivative of the following functions.

1. $f(x) = 4\sin(x) - 3\cos(x)$
2. $f(x) = 5e^x + 2\ln(x)$
3. $f(x) = \sin(3x)$
4. $f(x) = e^{5x+1}$

Part 3: Applications of Differentiation

Geometric Interpretation of the Derivative

The derivative gives us powerful tools to analyze the geometry of curves.

- **Slope and Tangents:** The value of the derivative at a point, $f'(a)$, is the slope of the tangent line to the curve at $x = a$.
- **Normals:** The normal line is perpendicular to the tangent. Its slope is the negative reciprocal of the tangent's slope: $m_{\text{normal}} = -\frac{1}{f'(a)}$.
- **Stationary Points:** These occur where the slope is zero, i.e., where $f'(x) = 0$. They can be local maxima, local minima, or points of inflection.
- **The Second Derivative Test:** We can classify stationary points using the second derivative, $f''(x)$. If $f'(a) = 0$:
  - If $f''(a) > 0$, the point is a local minimum.
  - If $f''(a) < 0$, the point is a local maximum.

Task #3: Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$.

1. Find the equation of the tangent to the curve at the point where $x = 1$.
2. Find the coordinates of the stationary points of the curve.
3. Use the second derivative test to determine the nature of these stationary points.

Part 4: Introduction to Integration

Integration as Antidifferentiation

Integration is the reverse process of differentiation. If the derivative of $F(x)$ is $f(x)$, then the integral of $f(x)$ is $F(x)$.

- **Indefinite Integral:** $\int f(x)\,dx = F(x) + C$. The "$+C$" is the constant of integration, representing an unknown constant lost during differentiation.
- **Power Rule for Integration:** For $n \neq -1$, $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C$.
- **Standard Integrals:** $\int \cos x\,dx = \sin x + C$, $\int \sin x\,dx = -\cos x + C$, $\int e^x\,dx = e^x + C$.

Task #4: Find the following indefinite integrals.

1. $\int (6x^2 - 8x + 3)\,dx$
2. $\int (4\sin(x) - 2e^x)\,dx$
3. Given $f'(x) = 3x^2 + 4x - 1$ and the point $(2, 10)$ lies on the curve $f(x)$, find the function $f(x)$.

Answers:

- **Task #1:**

Function $f(x)$	**Derivative** $f'(x)$
$f(x) = x^3 - 5x^2 + 2x + 8$	$f'(x) = 3x^2 - 10x + 2$
$f(x) = 4x^5 - 3x^{-2}$	$f'(x) = 20x^4 + 6x^{-3}$
$f(x) = \frac{1}{2}x^4 + 7x - 1$	$f'(x) = 2x^3 + 7$
$2(x^{\frac{1}{2}}) + 7(x^-2)$	$f'(x) = x^{\frac{-1}{2}} - 14(x^-3)$
$f(x) = x^2 - x - 6$	$f'(x) = 2x - 1$
$f(x) = \frac{x^2-4}{x+2}$	$f'(x) = 1$

- **Bonus Task #1:** $f'(x) = \lim_{h\to 0} \frac{3(x+h)^2-3x^2}{h} = \lim_{h\to 0} \frac{3(x^2+2xh+h^2)-3x^2}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0}(6x + 3h) = 6x$.
- **Task #2:**
  1. $f'(x) = 4\cos(x) + 3\sin(x)$
  2. $f'(x) = 5e^x + \frac{2}{x}$
  3. $f'(x) = 3\cos(3x)$ (Chain Rule)
  4. $f'(x) = 5e^{5x+1}$ (Chain Rule)
- **Task #3:**
  1. $f'(x) = 3x^2 - 6x - 9$. At $x = 1$, slope $m = f'(1) = 3 - 6 - 9 = -12$. Point is $(1, f(1)) = (1, -6)$. Equation: $y - (-6) = -12(x - 1) \implies y = -12x + 6$.
  2. Set $f'(x) = 0 \implies 3(x^2 - x - 3) = 0 \implies 3(x - 3)(x + 1) = 0$. Stationary points at $x = 3$ and $x = -1$. Coords: $(3, -22)$ and $(-1, 10)$.
  3. $f''(x) = 6x - 6$. At $x = 3$, $f''(3) = 12 > 0 \implies$ local minimum. At $x = -1$, $f''(-1) = -12 < 0 \implies$ local maximum.
- **Task #4:**
  1. $2x^3 - 4x^2 + 3x + C$
  2. $-4\cos(x) - 2e^x + C$
  3. $f(x) = \int(3x^2 + 4x - 1)\,dx = x^3 + 2x^2 - x + C$. Use $(2, 10)$: $10 = (2)^3 + 2(2)^2 - 2 + C \implies 10 = 8 + 8 - 2 + C \implies 10 = 14 + C \implies C = -4$. So, $f(x) = x^3 + 2x^2 - x - 4$.

B. Calculation Error Analysis: Forensic Mathematics

Learning Focus: Developing **critical analytical skills** by identifying and correcting common mathematical **misconceptions** and **procedural errors** in algebra.

Analysis Protocol

1. **Locate the Error:** Pinpoint the specific step or statement that is incorrect.
2. **Diagnose the Error:** Classify the error type: **Procedural Error** (miscalculation, incorrect formula application), **Conceptual Error** (misunderstanding a definition or principle), or **Omission Error** (incomplete conditions or overlooked restrictions).
3. **Explain the Misconception:** Articulate the underlying flawed reasoning or missing knowledge demonstrated by the error.
4. **Correct the Solution:** Provide the complete, accurate, and step-by-step mathematical solution.
5. **Metacognitive Reflection:** "This error (e.g., misinterpreting discriminant conditions) is subtle because the algebra might seem correct initially. What is one personal strategy I can adopt to ensure I never overlook a crucial detail like this under exam pressure (e.g., always double-checking discriminant inequalities)?"

Forensic Mathematics Task

Your job is to find the **flaw in the thinking**, not just the right answer. Explain ***why*** each statement/calculation is wrong and **correct them**.

Error Analysis Exercises

Exercise 1: Derivative from First Principles
Question: Find the derivative of $f(x) = x^2 + 3x$ from first principles.
Incorrect Calculation: $f'(x) = \lim_{h\to 0} \frac{(x+h)^2+3(x+h)-(x^2+3x)}{h}$
$= \lim_{h\to 0} \frac{x^2+2xh+h^2+3x+3h-x^2-3x}{h}$
$= \lim_{h\to 0} \frac{2xh+h^2+3h}{h} = \lim_{h\to 0}(2x+h) = 2x$.

Flawed Thinking

Error Analysis:

Correct Approach

Correction:

C. Weekend Assessment – Past Exam Questions

Learning Focus: Applying learning to past exam questions under exam conditions.

Practical Advice: To tackle the questions in this assessment, consider the following tips:

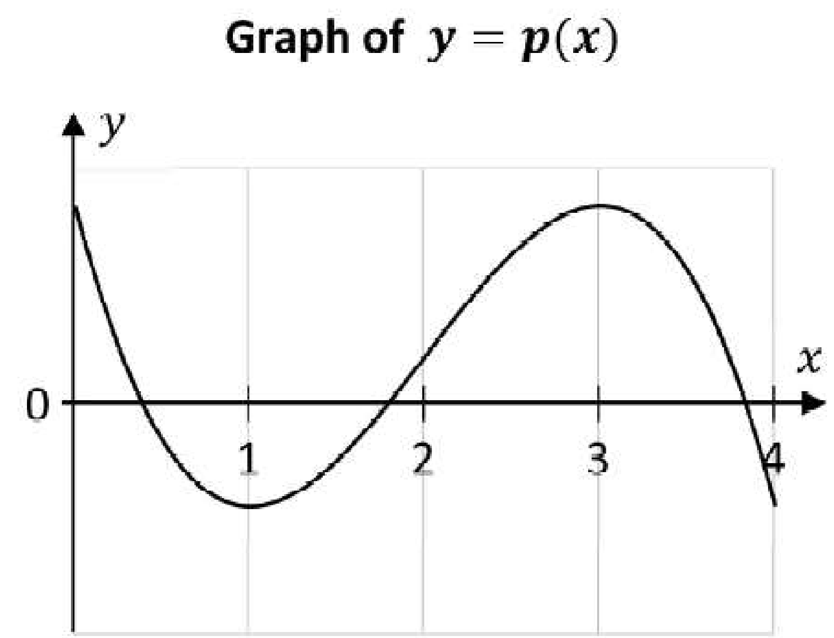
- For differentiation from first principles (Q1(a)), carefully apply the limit definition of a derivative.
- When solving related rates problems (Q1(b)), identify the relevant variables and their relationships, and then differentiate with respect to the appropriate variable.
- When analyzing graphs of functions and their derivatives (Q1(c), Q2), pay attention to key features such as maxima, minima, and inflection points.
- For optimization problems (Q6(a)), identify the objective function and the constraints, and use calculus to find the maximum or minimum value.
- When working with definite integrals (Q4, Q6(b)(iii), Q7(c)), ensure that you are using the correct limits of integration and that your answer is in the required form.
- For problems involving logarithmic and exponential functions (Q3, Q5, Q7), be sure to apply the relevant properties and rules, such as the chain rule and the product rule.

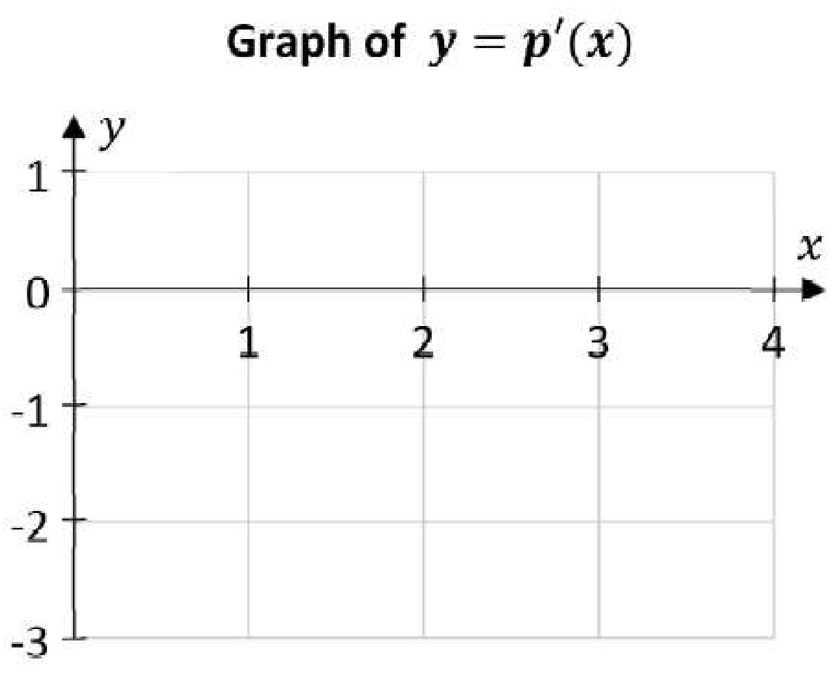
General Tips:

- **Show your work** - partial credit is available for correct methods
- **Check domains/restrictions** - especially in rational equations/logarithms
- **Label answers clearly** - especially when questions have multiple parts
- **Use appropriate precision** - note where decimal places are required
- **Verify your answers** - do they make sense in the context of the problem?

Question 1 (30 marks)
(a) Differentiate $f(x) = 2x^2 + 4x$ with respect to x, from first principles.
(b) A rectangle is expanding in area. Its width is x cm, where $x \in \mathbb{R}$ and $x > 0$. Its length is always four times its width.
Find the rate of change of the area of the rectangle with respect to its width, x, when the area of the rectangle is 225 cm^2.
(c) The graph of a cubic function $p(x)$ is shown in the first diagram below, for $0 \leq x \leq 4$, $x \in \mathbb{R}$. The maximum value of $p'(x)$ in this domain is 1, and $p'(0) = -3$, where $p'(x)$ is the derivative of $p(x)$.
Use this information to draw the graph of $p'(x)$ on the second set of axes below, for

$0 < x < 4$, $x \in \mathbb{R}$.
Graph of $y = p(x)$

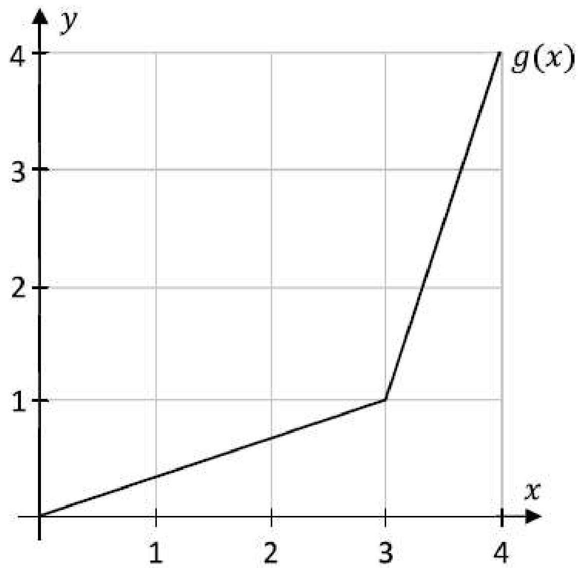




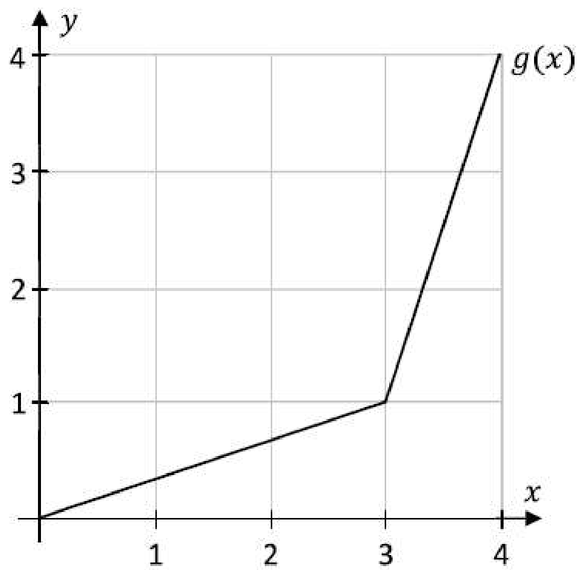
Question 2 (30 marks)
(a) A function is defined for $x \in \mathbb{R}$ by:

$$f(x) = 6 + x^2 + \sin 4x$$

(i) Find $f'(x)$, the derivative of f with respect to x.
(ii) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 0$. Give your answer in the form $ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$.
(b) The function $g(x)$ is defined for $0 \leq x \leq 4$, $x \in \mathbb{R}$. Its graph is shown in the diagram below, and is made up of two line segments.

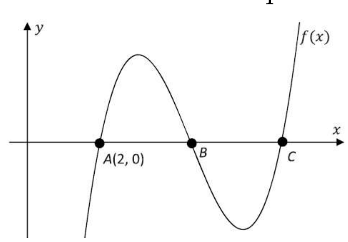


(i) State the range of values of x for which $g'(x) > 2$.
(ii) Find the value of $g(g(3))$. Give your answer in the form $\frac{a}{b}$ where a, $b \in \mathbb{N}$.
(iii) The graph of $y = g(x)$ is shown again on the diagram below. Draw and label the graph of $y = g^{-1}(x)$ on the same diagram.



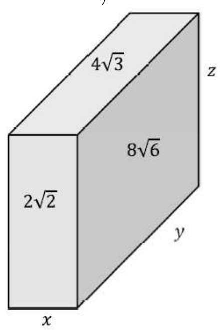
Question 3 (25 marks)
(a) Factorise fully: $3xy - 9x + 4y - 12$.
(b) $g(x) = 3x \ln x - 9x + 4 \ln x - 12$. Using your answer to part (a) or otherwise, solve $g(x) = 0$.
(c) Evaluate $g'(e)$ correct to 2 decimal places.

Question 4 (25 marks)
(a) Find $\int (4x^3 - 6x + 10)\, dx$.
(b) Part of the graph of a cubic function $f(x)$ is shown below. The graph cuts the x-axis at the three points $A(2, 0)$, B, and C.



(i) Given that $f'(x) = 6x^2 - 54x + 109$, show that $f(x) = 2x^3 - 27x^2 + 109x - 126$.
(ii) Find the co-ordinates of the point B and the point C.

Question 5 (30 marks)
(a) The diagram shows a cuboid with dimensions x, y and z cm. The areas, in cm^2, of three of its faces are also shown. Find the volume of the cuboid in the form $a\sqrt{b}$cm^3, where $a, b \in \mathbb{N}$.



(b) (i) Given that $f(x) = 3x^2 + 8x - 35$, where $x \in \mathbb{R}$, find the two roots of $f(x) = 0$.
(ii) Hence or otherwise, solve the equation $3^{2m+1} = 35 - 8(3^m)$, where $m \in \mathbb{R}$. Give your answer in the form $m = \log_3 p - q$, where $p, q \in \mathbb{N}$.

Question 6 (50 marks)
Dani drives a car.
(a) The fuel consumption, F, of Dani's car depends on the speed of the car, c. For one particular journey, F is given by:

$$F(c) = 0.05c^2 - 8.5c + 800$$

where F is in litres per 10 000 km, and c is in km/hour, with $40 \leq c \leq 120$.
(i) Show that there is no difference between the fuel consumption (F) when Dani's car is travelling at 60 km/hour and when it is travelling at 110 km/hour.
(ii) Find an expression for $\frac{dF}{dc}$, the rate of change of fuel consumption with respect to speed.
During part of the journey, the speed, c, of Dani's car at time t is given by:

$$c = 78 + 9\ln(t^2)$$

where t is the time in minutes, $1 \leq t \leq 10$, and c is in km/hour.
(iii) Use this, and your answer to part (a)(ii), to find the value of $\frac{dF}{dc}$ when $t = 7$. Give your answer correct to 1 decimal place.
(iv) Show that the rate of change of the car's speed with respect to time is given by:

$$\frac{dc}{dt} = \frac{18}{t}$$

(v) Use your answers to parts (a)(iii) and (a)(iv) to find the rate of change of the car's fuel consumption, F, with respect to time, at the instant when $t = 7$ minutes. Give your answer in (litres per 10 000 km) per minute.

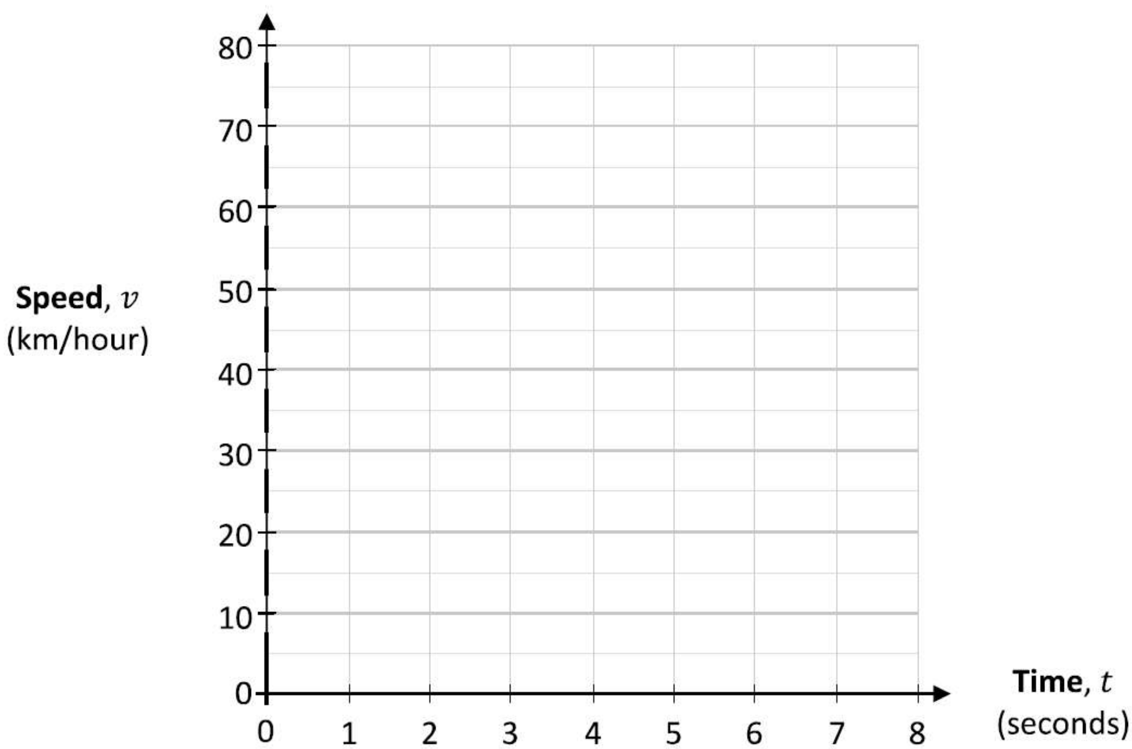
(b) Over the first 8 seconds that Dani is driving her car, the car's speed, in km/hour, can be approximated using:

$$v(t) = \begin{cases} 8e^{0.4t} - 8, & 0 \leq t \leq 4 \\ -t^2 + 24t - 48.4, & 4 < t \leq 8 \end{cases}$$

(i) Fill in the table below to show the values of $v(t)$ for the given values of t, up to $t = 8$. Give each value correct to 1 decimal place.

Time, t (seconds)	0	1	2	3	4	5	6	7	8
Speed, v (km/hour)	0		9.8		31.6			70.6	79.6

(ii) Hence, draw the graph of the function $y = v(t)$ on the axes below.



(iii) Use integration, and $v(t) = -t^2 + 24t - 48.4$, to find the average speed of Dani's car for $4 < t \leq 8$. Give your answer in km/hour, correct to 1 decimal place.

Question 7 (50 marks)
Aidan is a hydrogeologist who has drilled a cylindrical bore with a diameter of 16 centimetres and a depth of 35 metres.
(a) If 1 litre of water has a volume of 1,000 cubic centimetres, find the volume of the bore in litres, correct to 1 decimal place.
Underground water flows into the bore. Aidan pumps out all the water from the bore so that at time $t = 0$ the bore contains no water. He then turns off the pump and the bore starts to fill up with water. The volume, in litres, of water in the bore t minutes after the pump is turned on can be modelled by the function $V(t)$:

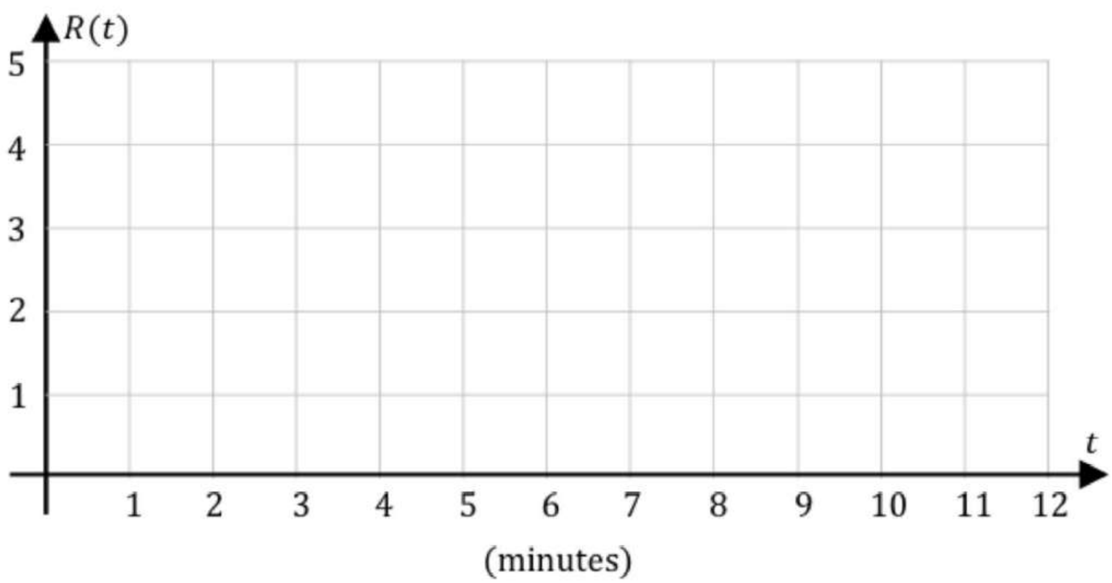
$$V(t) = t\ln(10t+1) + \frac{\ln(10t+1)}{10} - t$$

(b) (i) By differentiation show that the net rate of water flow (in litres per minute), $R(t)$ can be written as:

$$R(t) = \ln(10t+1)$$

(ii) Complete the table below and draw the graph of $y = R(t)$ on the diagram provided.

t (minutes)	0	2	4	6	8	10	12
$R(t) = \ln(10t+1)$				4.1			



(c) Find the average rate of flow of water for the first 12 minutes after the time the pump is turned off, in litres per minute, correct to 2 decimal places.

Self-Assessment

After completing the assessment:

- Grade your work honestly
- Identify areas needing improvement
- Scan and submit via Google Classroom
- Reflect on your performance in your weekly reflection

Another excellent week of work completed - ***well done!*** You are another step closer to *smashing your exams*, and another week closer to your summer holdiays!

Weekly Reflection Zone

What worked well this week?

What challenges did I face?

What surprised me the most this week?

Key mathematics concepts I want to review:

Goals for next week: