SKJ Education

LC HL MATHS

FOUNDATION PROGRAM:
WEEK 2

FUNCTIONS, GRAPHS, SEQUENCES & SERIES

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LC HL MATHEMATICS – FOUNDATION PROGRAM

Week 2: Functions, Graphs, Sequences & Series

Learning Objectives

- **2.1:** To analyse and sketch polynomial and exponential functions, identifying key features such as domain and range.
- **2.2:** To define, expand, and find the general term of Arithmetic and Geometric Progressions.
- **2.3:** To calculate the sum of a given number of terms of an Arithmetic Progression (AP) and a Geometric Progression (GP), including the sum to infinity for a GP.
- **2.4:** To apply knowledge of sequences to solve financial maths problems involving loans, savings, and investments.

Key Terms - Week 2

- **Polynomial Function:** A function consisting of variables and coefficients combined using only addition, subtraction, and multiplication, and non-negative integer exponents (e.g., $f(x) = 3x^3 - 2x + 1$).
- **Exponential Function:** A function of the form $f(x) = a^x$, where a is a positive constant (e.g., $f(x) = 2^x$).
- **Arithmetic Progression (AP):** A sequence of numbers where each term after the first is obtained by adding a fixed constant to the previous term (e.g., $2, 5, 8, 11, \ldots$).
- **Geometric Progression (GP):** A sequence of numbers where each term after the first is obtained by multiplying the previous term by a fixed, non-zero number (e.g., $2, 6, 18, 54, \ldots$).
- **Sum to Infinity:** The limit of the sum of the first n terms of a sequence as n approaches infinity, applicable to convergent Geometric Progressions (e.g., for $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, the sum to infinity is 2).
- **Financial Mathematics:** The application of mathematical techniques to solve problems related to financial transactions, such as calculating the future value of investments or the repayment terms of loans.

Weekly *Challenge*: Investigate how the concept of compound interest is related to Geometric Progressions. Derive a formula to calculate the future value of a savings account that earns a fixed annual interest rate, compounded annually. ***Explain the variables used in your formula*** and discuss how this model can be applied to real-life savings plans.

WEEK 2 STUDY PLAN

Day	Activities & Time Commitment	✓	Rating (1-10)
Monday	- Review Learning Objectives (5 min) - Rank your current ability (5 min) - Review Key Terms (10 min) - Watch video (10-20 min) *Focus: PREPARATION*		
Tuesday	- Complete Exercises A1 & A2 (60 min) *Focus: EXPLORING*		
Wednesday	- Complete Exercise A3 (30 min) - Correct Exercises A1-3 and reattempt difficult questions (45 min) *Focus: PROCESSING*		
Thursday	- 1-hour online lesson (60 min) *Focus: QUESTIONING*		
Friday	- Complete Exercise B (40 min) *Focus: ERROR ANALYSIS*		
Saturday	- Complete Exam Question Assessment (C) (60 min) *Focus: EXECUTION*		
Sunday	- Correct assessment (30 min) - Complete self-reflection (15 min) - Plan next week (15 min) *Focus: REFLECTION & RECHARGING*		

Study Tips for Success

- **Active Recall:** After studying, close your notes and write down **everything you remember**. Force your brain to grow.
- **Spaced Repetition:** Review concepts **multiple times** over several days.
- **Mathematics in Action:** Look for **real-world examples** of the concepts you're learning.
- **Ask Questions:** Don't hesitate to ask for help when concepts are unclear. Reach out via *Google Classroom* or email; *steven@skjeducation.com*.
- **Celebrate Progress:** **Acknowledge your improvements**, no matter how small.

A1. Proficiency Drills (Week 2 - Revised)

Learning Focus: Foundational concepts of **functions and graphs**, **sequences and series**, and **financial mathematics**.

Part 1: Functions and Graphs - Analysis and Sketching

Key Concepts

Function Analysis: A full analysis involves identifying key characteristics to understand a function's behavior.

- **Domain:** The set of all possible input (x) values.
- **Range:** The set of all possible output ($f(x)$) values.
- **Intercepts:** Where the graph crosses the x-axis (x-intercepts) and y-axis (y-intercept).
- **Asymptotes:** Lines that the graph approaches but never touches.
- **End Behavior:** The behavior of the function as $x \to \infty$ and $x \to -\infty$.

Task #1: Complete the table below for each function.

Function	Domain	Range
$f(x) = x^3 - 2x^2 + x + 1$		
$f(x) = 2^x$		
$f(x) = e^{-x}$		
$f(x) = -x^2 + 4x - 3$		

Bonus Task: Sketch the graph of each function. Use this graph to **find the intercepts of these functions**.

Part 2: Sequences and Series - AP, GP, and Quadratic

Essential Knowledge & Formulas

Arithmetic Progression (AP): Constant first difference (d).

- n-th term: $u_n = a + (n-1)d$
- Sum of first n terms: $S_n = \frac{n}{2}(2a + (n-1)d)$

Geometric Progression (GP): Constant ratio (r).

- n-th term: $u_n = ar^{n-1}$
- Sum of first n terms: $S_n = \frac{a(r^n-1)}{r-1}$

Quadratic Sequence: Constant second difference.

- The general form is $u_n = An^2 + Bn + C$.
- To find the coefficients:
  - $2A =$ the constant second difference
  - $3A + B =$ the first term of the first differences
  - $A + B + C =$ the first term of the sequence

Task #2: Sequence Identification and Properties For each sequence below, determine if it is an Arithmetic Progression (AP), a Geometric Progression (GP), a Quadratic Sequence, or none of these. State the common difference (d), common ratio (r), or the fsecond difference (d_2) if it is quadratic.

1. $2, 5, 8, 11, ...$
2. $3, 6, 12, 24, ...$
3. $100, 50, 25, 12.5, ...$
4. $10, 6, 2, -2, ...$
5. $5, 11, 21, 35, 53, ...$
6. $2, -6, 18, -54, ...$
7. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, ...$
8. $1, 1, 2, 3, 5, 8, ...$

1. The third term of a GP is 45 and the seventh term is 3645. Find the first term (a) and the common ratio (r).
2. An arithmetic sequence has $u_5 = 14$ and $u_{12} = 35$. Find the sum of the first 50 terms.
3. Which term in the sequence $4, 9, 14, 19, ...$ is equal to 124?
4. For the quadratic sequence $5, 11, 21, 35, 53, ...$, find the formula for the n-th term, u_n, and use it to calculate the 20th term.

Part 3: Financial Mathematics - Growth, Decay, and Savings

Financial Insights & Applications

Many financial situations are direct applications of sequences and series.

- **Compound Growth:** An investment that grows by a fixed percentage each year is a **geometric progression**. The principal is multiplied by a common ratio $(1 + r)$ each compounding period. The formula is $A = P(1 + \frac{r}{n})^{nt}$.
- **Regular Savings (Annuities):** Saving a fixed amount of money at regular intervals (e.g., €100 per month) creates a series. The total future value is the **sum of a geometric series**, where each deposit has had a different amount of time to grow.
- **Depreciation:** An asset that loses value by a fixed percentage each year (e.g., a car) also follows a **geometric progression**, but with a common ratio between 0 and 1.

Task #3: Financial Applications

1. **(Compound Growth)** A person invests €1000 at an annual interest rate of 5%, compounded annually. Find the amount after 5 years.
2. **(Regular Savings)** Liam starts a savings plan. He deposits €150 at the **end of every month** into an account that pays 6% annual interest, compounded monthly. How much will he have in the account after exactly 3 years (36 deposits)? (Hint: The interest rate per month is $0.06/12 = 0.005$. This is the sum of a GP.)
3. **(Depreciation)** A new machine is purchased by a factory for €120,000. It depreciates in value by 20% each year. What is the machine's value after 5 years?
4. **(Investment Comparison)** Alex has €5,000 to invest for 10 years.
   - **Option A:** Simple interest at a rate of 7% per year.
   - **Option B:** Compound interest at a rate of 5.5% per year, compounded annually.

   Which option yields more money at the end of the 10-year period?

Answers:

- **Task #1:**

Function	Domain	Range
$f(x) = x^3 - 2x^2 + x + 1$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = 2^x$	$\mathbb{R}$	$(0, \infty)$
$f(x) = e^{-x}$	$\mathbb{R}$	$(0, \infty)$
$f(x) = -x^2 + 4x - 3$	$\mathbb{R}$	$(-\infty, 1]$

- **Bonus Task #1:** 1) y-int:(0,1); 2) y-int:(0,1); 3) y-int:(0,1); 4) y-int:(0,-3), x-ints:(1,0) & (3,0).
- **Task #2a:**
  1. AP, $d = 3$
  2. GP, $r = 2$
  3. GP, $r = 0.5$
  4. AP, $d = -4$
  5. Quadratic (constant second difference of 4)
  6. GP, $r = -3$
  7. AP, $d = 1/4$
  8. None (This is the Fibonacci sequence)
- **Task #2b:**
  1. $ar^2 = 45$, $ar^6 = 3645$. Dividing gives $r^4 = 81 \implies r = 3$. Then $a(3^2) = 45 \implies 9a = 45 \implies a = 5$.
  2. $a + 4d = 14$, $a + 11d = 35$. Subtracting gives $7d = 21 \implies d = 3$. Then $a+4(3) = 14 \implies a = 2$. Sum $S_{50} = \frac{50}{2}(2(2)+49(3)) = 25(4+147) = 3775$.
  3. AP with $a = 4, d = 5$. Solve $4 + (n-1)5 = 124 \implies 5(n-1) = 120 \implies n - 1 = 24 \implies n = 25$. It is the 25th term.
  4. 2nd diff $= 4$, so $2A = 4 \implies A = 2$. 1st diff starts with 6, so $3A + B = 6 \implies 3(2) + B = 6 \implies B = 0$. 1st term is 5, so $A + B + C = 5 \implies 2 + 0 + C = 5 \implies C = 3$. Formula is $u_n = 2n^2 + 3$. The 20th term is $u_{20} = 2(20^2) + 3 = 2(400) + 3 = 803$.
- **Task #3:**
  1. $A = 1000(1.05)^5 \approx$ £1276.28
  2. Sum of a GP with $n = 36$, first term $a = 150$, ratio $r = 1.005$. $S_{36} = \frac{150((1.005)^{36}-1)}{1.005-1} \approx$ £5903.35.
  3. Value $= 120000(1 - 0.20)^5 = 120000(0.8)^5 \approx$ £39321.60.
  4. Option A: $A = 5000 + (5000 \times 0.07 \times 10) = 5000 + 3500 =$ £8500. Option B: $A = 5000(1.055)^{10} \approx$ £8540.75. Option B is better.

C. Weekend Assessment – Past Exam Questions

Learning Focus: Applying learning to past exam questions under exam conditions.

Practical Advice: To tackle the questions in this assessment, consider the following tips:

- For questions involving algebraic manipulations (Q1, Q2, Q4), make sure to review the relevant laws and formulas, such as the laws of logarithms and the formula for the sum of a geometric series.
- When working with geometric series (Q2), pay attention to the common ratio and ensure that it is within the valid range for convergence.
- For questions involving financial mathematics (Q3), carefully read the problem statement and identify the relevant formula, such as the amortisation formula.
- When using mathematical induction (Q5), clearly state the base case and the inductive step, and ensure that your proof is logical and well-structured.
- For optimization problems (Q6), identify the objective function and the constraints, and use calculus to find the maximum or minimum value.
- When modeling real-world phenomena (Q7), carefully read the problem statement and identify the relevant mathematical model, such as exponential decay or geometric series.

General Tips:

- **Show your work** - partial credit is available for correct methods
- **Check domains/restrictions** - especially in rational equations/logarithms
- **Label answers clearly** - especially when questions have multiple parts
- **Use appropriate precision** - note where decimal places are required
- **Verify your answers** - do they make sense in the context of the problem?

Question 1 (25 marks)
(a) The function $g(x)$ is defined for $x \in \mathbb{R}$ by:

$$g(x) = 5x^2 + 20x - 12$$

Write $g(x)$ in the following form, where $a, h, k \in \mathbb{Z}$ are constants:

$$g(x) = a(x + h)^2 + k$$

(b) p is a positive constant. Use the laws of logs to write the expression:

$$ln[(e^3p)^5$$

(c) Below is a pair of simultaneous equations in x and y, where $n \in \mathbb{R}$ is a constant. One of the solutions to this pair of equations is on the y-axis. Use this information to find the value of n.

$$2x - y = 7$$

$$x^2 + y + 2y^2 = n$$

Question 2 (25 marks)
(a) The first three terms of a geometric series are x^2, $5x - 8$, and $x + 8$, where $x \in \mathbb{R}$. Use the common ratio to show that $x^3 - 17x^2 + 80x - 64 = 0$.
(b) If $f(x) = x^3 - 17x^2 + 80x - 64$, $x \in \mathbb{R}$, show that $f(1) = 0$, and find another value of x for which $f(x) = 0$.
(c) In the case of one of the values of x from part (b), the terms in part (a) will generate a geometric series with a finite sum to infinity. Find this value of x and hence find the sum to infinity.

$$x = \qquad S_\infty =$$

Question 3 (25 marks)
(a) A couple agree to take out a €250000 mortgage in order to purchase a new home. The loan is to be paid back monthly over 25 years with the repayments due at the end of each month. The bank charges an annual percentage rate (APR) which is equivalent to a monthly rate of 0-287%. Using the amortisation formula, or otherwise, find the couples' monthly repayment on the mortgage. Give your answer in euro correct to the nearest cent.
(b) Another couple agree to take out a mortgage of €350,000, at a rate of 0.3% per month, in order to purchase a new home. This loan is also to be paid back monthly over 25 years with the repayments due at the end of each month. The amount of each repayment is €1771.
After exactly 11 years of repayments, the couple receive a financial windfall. They decide to repay the remaining balance on the mortgage.
Write down a series (including the first two and last two terms) which shows the total of the present values of all the remaining monthly repayments due over the remaining 14 years of the mortgage (after the last monthly repayment at the end of year 11).
Hence, find how much the couple will need to repay in order to clear their mortgage entirely. Give your answer correct to the nearest cent.

Question 4 (30 marks)
(a) Prove that $\sqrt{2}$ is **not** a rational number.
(b) t is a positive real number, with:

$$\log_3 t + \log_9 t + \log_{27} t + \log_{81} t = 10$$

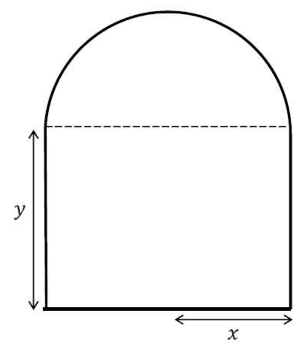
Find the value of t. Give your answer in the form 3^r, where $r \in \mathbb{Q}$.
Hint: use the formula

$$\log_a b = \frac{\log_c b}{\log_c a}$$

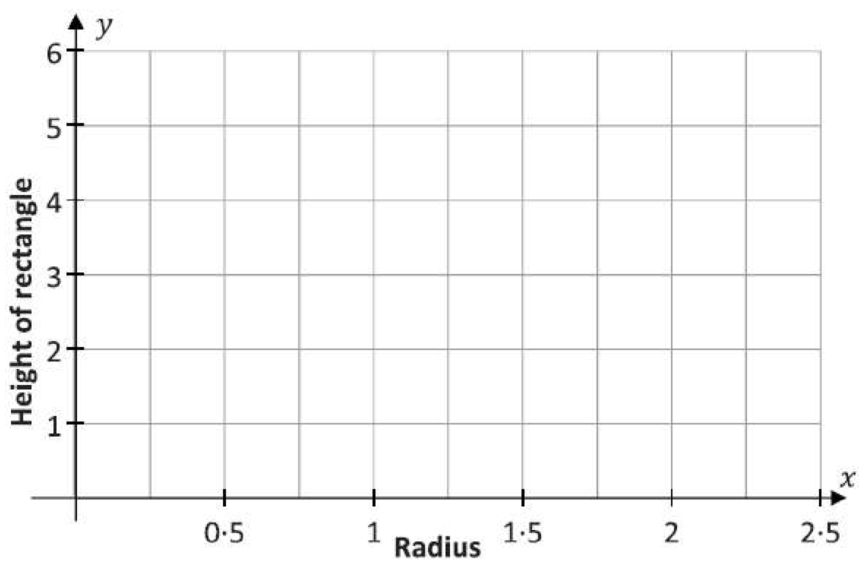
(c) (i) Explain what $\log_6 m$ means, where m is a positive real number.
(ii) m is a real number, and $m > 6$.
What information does this give about the value of $\log_6 m$?

Question 5 (30 marks)
(a) Prove using induction that $2^{3n-1} + 3$ is divisible by 7 for all $n \in \mathbb{N}$.
(b) p, $p + 7$, $p + 14$, $p + 21$, ... is an arithmetic sequence, where $p \in \mathbb{N}$.
(i) Find the n^{th} term, T_n, in terms of n and p, where $n \in \mathbb{N}$.
(ii) Find the smallest value of p for which 2021 is a term in the sequence.

Question 6 (55 marks)
In a particular Norman window the perimeter $P = 12$ metres.



(a) (i) Write P in terms of x, y, and π.
(ii) Show that $y = \frac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.
(b) (i) Draw the graph of the linear function $y = \frac{12-(2+\pi)x}{2}$ below.



(b) (ii) Find the slope of the graph y, correct to 2 decimal places. Explain the meaning of this answer.
(c) (i) The Norman window shown below has a perimeter of 12 metres and $y = \frac{12-(2+\pi)x}{2}$.
Show that the function $a(x) = \frac{24x-(\pi+4)x^2}{2}$ represents the area of the window, in terms of x and π.
(ii) Find $a'(x)$.
(iii) Find the relationship between x and y when the area of the window in part (c)(i) is at its maximum.

Question 7 (50 marks)
(a) Find the amount of the drug left in Alex's body 2-5 days after a single 15 mg injection. Give your answer in mg, correct to 2 decimal places.
(b) How long after a single 15 mg injection will there be exactly 1 mg of the drug left in Alex's body? Give your answer in days, correct to 1 decimal place.
(c) Explain why the total amount of the drug, in mg, in Alex's body immediately after the 4th injection is given by:

$$15 + 15(0-6) + 15(0-6)^2 + 15(0-6)^3$$

(d) Find the total amount of the drug in Alex's body immediately after the 10th injection. Give your answer in mg, correct to 2 decimal places.
(e) Use the formula for the sum to infinity of a geometric series to estimate the amount of the drug (in mg) in Alex's body, after a long period of time during which he gets daily injections.
(f) (i) Use the sum of a geometric series to show that the total amount of the drug (in mg) in Jessica's body immediately after the nth injection, where $n \in \mathbb{N}$, is:

$$\frac{20d(1-0.85^n)}{3}$$

(ii) Immediately after the 7th injection, there are 50 mg of the drug in Jessica's body. Find the amount of the drug in one of Jessica's daily injections. Give your answer correct to the nearest mg.

Self-Assessment

After completing the assessment:

- Grade your work honestly
- Identify areas needing improvement
- Scan and submit via Google Classroom
- Reflect on your performance in your weekly reflection

Another excellent week of work completed - ***well done!*** You are another step closer to *smashing your exams*, and another week closer to your summer holdiays!

Weekly Reflection Zone

What worked well this week?

What challenges did I face?

What surprised me the most this week?

Key mathematics concepts I want to review:

Goals for next week: