Around an idea of Bombieri on the Selberg sieve

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Introduction

In these notes, we are interested in the number of prime factors of two sequences of integers and more precisely, we aim at giving an upper bound of this number by using the weighted sieve. In the first example, we will consider the integers $p+2$ when p is a prime number; as of today, the problem of prime twins ("there is infinitely many primes p such that $p+2$ is prime") remains a mystery. Nevertheless, we have some ideas about the quantity of prime twins: they are, in some sense, rare: Viggo Brun (1882-1978) proved that the sum of the reciprocals of twin primes is convergent or finite which is in contrast with the fact that the sum of the reciprocals over all prime numbers is divergent.

Let $\{a_n\}_{n\geq 1}$ be a sequence of (positive) integers. We look at the set $E(x) := \{n \leq x | \omega(a_n) \leq r_0\}$ (x positive real number) of integers a_n such that a_n has at most r_0 prime factors for some positive integer r_0 (and $n \leq x$). The general problem is to determine an admissible value of r_0 such that $\#\{n \geq 2 | \omega(a_n) \leq r_0\} = +\infty$. The best result in this direction for the sequence $\{p+2\}_{p\geq 2}$ is due to J.-R. Chen (Sci. Sinica, 1973) with $r_0 = 2$. For the sequence $\{n(n+2)\}_{n\geq 1}$ (which is our second application of the weighted sieve) the best result is attributed to Rényi with the integer $r_0 = 5$. We will further prove a lower bound of the cardinality of $E(x)$; the boundedness of the weights in the weighted sieve will allow us to obtain such a result. Moreover, we refine our results by showing that we can assume $p+2$ (respectively $n(n+2)$) to have no prime factor less than a small power of p (respectively n), which we explicitly determine.

Contents

Chapter 1

Sieve method/principle of the proof

1.1 Method

We define first the object that we are going to study:

Definition 1. *A sieve is by definition given by the datas* ($\mathcal{N}$,$\mathcal{P}$,$\{\Omega_p\}_p$), *where:*
(a) $\mathcal{N}$ *is a set of integers,*
(b) $\mathcal{P}$ *is a set of prime numbers,*
(c) for all $p \in \mathcal{P}$, Ω_p *is a set of residue modulo* p.

The set we want to study is

$$\mathcal{N}_0 = \{n \in \mathcal{N} | n \mod p \notin \Omega_p, \forall p \in \mathcal{P}\}.$$

In many cases, we take $\mathcal{N}$ to be $\{M < n \leq M + N\}$ an interval of length N, or $\{p \leq N\}$ the set of prime numbers less than N, or $\{f(n) | n \leq N\}$, where f is a polynomial. For instance, in the case $\mathcal{N} = [1, N]$, $\mathcal{P} = \{p \leq \sqrt{N+2}\}$ and $\Omega_p = \{-2, 0\}$, $\mathcal{N}_0$ is the set of prime twins in $]\sqrt{N+2}, N]$.

The fundamental problem is to obtain an estimation of $|\mathcal{N}_0|$. Let $\lambda_1 = 1$ and λ_d ($d \geq 2$) be arbitrary real numbers. The first form of the Selberg sieve, called the simplest Selberg upper bound sieve method (see for instance [2]), derives from the inequality

$$\sum_{n \in \mathcal{N}_0} 1 \leq \sum_{n \in \mathcal{N}} \Big(\sum_{\substack{\nu | n \\ \nu | P(y)}} \lambda_\nu \Big)^2,$$

where $P(y) = \prod_{\substack{p \in \mathcal{P} \\ p < y}} p$ for some parameter y (for when $n \in \mathcal{N}_0$, $\nu = 1$ is the only divisor appearing on the right and it makes a contribution 1 since $\lambda_1 = 1$; while all the other terms on the right hand side, namely whose associated with $n \in \mathcal{N}_0$, $(n, P(y)) > 1$, are non-negative, the λ_ν's being real). Using this method, we can show for instance that:

$$\sum_{\substack{p \le x \\ p+2 \text{ prime}}} 1 \ll \frac{x}{\log^2 x}.$$

To do so, we take $\mathcal{N} = \{n(n+2) | n \le x\}$ and $\Omega_p = \{0, -2\}$ for every prime number p; then, $\mathcal{N}_0 = \{p \le x | p+2 \text{ prime}\}$ and:

$$\sum_{p \in \mathcal{N}_0} 1 \le \sum_{n \le x} \Big(\sum_{\substack{d | n(n+2) \\ d < z}} \lambda_\nu \Big)^2 + z.$$

In this paper, we consider a stronger form of the Selberg sieve which allows us to get better informations about the set $\mathcal{N}_0$. In order to do so, we look at the sum

$$S = \sum_{n \in \mathcal{N}} \Big(\sum_{\substack{d | n \\ d < y}} a_d \Big) \Big(\sum_{\substack{\nu | n \\ \nu < z}} \lambda_\nu \Big)^2$$

where a_d and λ_ν are bounded real numbers, that vanish for large values of d and ν, namely when $d \ge y$ and $\nu \ge z$, where y and z are parameters which will be some powers of x. Here we will be able to get a lower bound for the cardinality of a set "close" of $\mathcal{N}_0$, say $\tilde{\mathcal{N}_0}$; for instance, we will establish that

$$\tilde{\mathcal{N}_0} \gg \frac{x}{\log^2 x},$$

where $\tilde{\mathcal{N}_0}$ is in this case the set of primes not more than x such that $p+2$ has at more four prime factors.

On expanding the sum S, a new set appears: for each positive integer d, we define the set $\mathcal{N}_d = \{n \in \mathcal{N} | n \equiv 0\ [d]\}$ of elements of $\mathcal{N}$ divisible by d. For sieve problems, we hope to find a multiplicative function f so that $1/f$ approximates the proportion of elements of $\mathcal{N}$ divisible by d. Then, we define a remainder term $\mathcal{R}_d$ by:

$$|\mathcal{N}_d| = \frac{|\mathcal{N}|}{f(d)} + \mathcal{R}_d.$$

Furthermore, we will use the multiplicative function f_1 which is equal to the convolution product $f \star \mu$. The strongest form of the Selberg sieve, which can be found in [1], is:

Theorem 1. *We have*

$$S = |\mathcal{N}| \mathfrak{S} + \mathcal{O}\Big(\sum_{m < yz^2} \big(\sum_{\substack{d | m \\ d < y}} |a_d| \big) \big(\sum_{\substack{\nu | m \\ \nu < z}} |\lambda_\nu| \big)^2 |\mathcal{R}_m| \Big)$$

where

$$\mathfrak{S} = \sum_{m<z} \sum_{\substack{d<y \\ (m,d)=1}} \frac{\mu^2(m)}{f_1(m)} \frac{a_d}{f(d)} \Big(\sum_{\substack{r|d \\ r<z/m}} \mu(r)\zeta_{rm} \Big)^2, \tag{1.1}$$

and

$$\zeta_r = \mu(r) f_1(r) \sum_{\nu<z/r} \frac{\lambda_{\nu r}}{f(\nu r)}. \tag{1.2}$$

We note here that we can also express the λ_ν in terms of the coefficients ζ_r, thanks to the Möbius inversion formula:

$$\lambda_\nu = \mu(\nu) f(\nu) \sum_{r<z/\nu} \frac{\mu^2(r\nu)}{f_1(r\nu)} \zeta_{r\nu}.$$

In our applications, we will always take the same sequence of functions (of the variable z) $\{\zeta_r\}_{r\geq 1}$: $\zeta_r = \zeta_1$ when r is a squarefree integer not more than z and $\zeta_r = 0$ otherwise. We prove in Lemma 12 that in this case the weights λ_ν are bounded in absolute value by $|\lambda_1|$.

Now, let us explain the principle of the proof of the estimation of $|\tilde{\mathcal{N}_0}|$. Firstly, we choose the sequence $\{a_d\}_{d\geq 1}$ as $a_1 = b$, where b is a positive real number, and $a_d = 0$ either. In a second step, we take $a_p = -1$ when $p \in \mathcal{P}$ is less than y, and $a_d = 0$ either. Our goal is to get:

$$\sum_{n\in\mathcal{N}} \Big[b - \sum_{\substack{p|n \\ p|P(y)}} 1 \Big] \Big(\sum_{\substack{\nu|n \\ \nu<z}} \lambda_\nu \Big)^2 \sim K(b, y, z) \tag{1.3}$$

for some function K which tends to infinity when y and z goes to infinity; the sizes of y and z (relative to the size of $\mathcal{N}$) are choosen at the end of the proof in order to make the function K positive. We will conclude that $b - \sum_{\substack{p|n \\ p|P(y)}} 1 > 0$ i.e. $\sum_{\substack{p|n \\ p|P(y)}} 1 \leq \lfloor b \rfloor$ for many integers $n \in \mathcal{N}$.

1.2 Aims

In these notes, we give two applications of the Selberg sieve. The first of them deals with the set of prime numbers and more precisely about the number of prime factors of $p + 2$.

Theorem 2 (Upper bound for $\omega(p + 2)$)**.** *There are infinitely many primes p such that $p + 2$ has at most four prime factors.*

The other one is on the set of integers; in this case, we aim at giving an upper bound of the number of prime factors of $n(n + 2)$.

Theorem 3 (Upper bound for $\omega(n(n+2))$)**.** *There are infinitely many positive integers n such that $n(n+2)$ has at most six prime factors.*

In fact, we will see that we can "forget" the smaller prime factors. For a positive real number a, we define the set $\mathcal{P}_a$ of integers n satisfying the following property: "the prime factors of $n+2$ are bigger than n^a". Then, we can state the next result:

Theorem 4. *For $a = 1/17$, we have:*

$$\#\{p \leq x | \omega(p+2) \leq 4,\ p \in \mathcal{P}_a\} \gg \frac{x}{\log^2 x}.$$

Analogously, let $\mathcal{Q}_{a'}$ be the set of integers n satisfying the following property: "the prime factors of $n(n+2)$ are bigger than $n^{a'}$". Hence, we have:

Theorem 5. *For $a' = 1/17$, we have:*

$$\#\{n \leq x | \omega(n(n+2)) \leq 6,\ n \in \mathcal{Q}_{a'}\} \gg \frac{x}{\log^2 x}.$$

In order to control the remainder term of S in Theorem 2, we need the Bombieri-Vinogradov Theorem.

Theorem 6 (Bombieri-Vinogradov (1965))**.** *For all $A > 0$, there is a constant $B = B(A) > 0$ such that*

$$\sum_{q \leq x^{1/2}(\log x)^{-B}} \max_{y \leq x} \max_{(a,q)=1} \left| \pi(y; q, a) - \frac{\mathrm{Li}(y)}{\phi(q)} \right| \ll_A x(\log x)^{-A}.$$

The proofs of these Theorems bear signifiant similarities; in particular, they are both based on a series of four very similar technical lemmas. In order to achieve these goals and to lighten the proofs, we shall establish in the following chapter some essential estimates.

Chapter 2

Lemmas

We order into two sections the two applications that we were talking about.

2.1 On the sequence $\{p+2\}_{p\geq 2}$

In our first application of the Selberg sieve, we consider the set of integers $\mathcal{N} = \{p+2|p \leq x\}$, where x is a positive real number greater than 2. Here, $\mathcal{N}_d = \{p+2|p \equiv -2\ [d],\ p \leq x\}$. The Prime Number Theorem in arithmetic progression gives us $|\mathcal{N}_d| \sim \mathrm{Li}(x)/\phi(d)$ (when x goes to infinity), when d is an odd integer. Moreover, $\mathcal{N}_2 = \{4\}$ and $\mathcal{N}_{2\delta} = \emptyset$ for every integer $\delta \geq 2$. Then, the function f (defined on odds integers) in the sieve will be the Euler totient function ϕ (if we want to define f on even integers, we ask $f(2k) = \infty$ for all $k \geq 1$).

About the function ϕ_1

We have $\phi_1(2) = \infty$ and $\phi_1(p) = p-2$ when $p > 2$. First and foremost, it is required to know an estimation of the sums

$$\sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)},$$

called $G(z)$. It is the object of the next lemma.

Lemma 1. *There are two numerical constants $\alpha_1 > 0$ and α_2 such that, for every $z > 0$, we have:*

$$G(z) = \alpha_1 \log z + \alpha_2 + \mathcal{O}(z^{-1/3}).$$

Proof. We prove this first asymptotic result by using the method of convolution. For it, we want to write the multiplicative function, say h, defined by $h(n) = 0$ when n is an even integer and $h(n) = \mu^2(n)/f_1(n)$ when n is odd, as a convolution product. Clearly, h is given by $h(2) = 0$, $h(p) = 1/(p-2)$ when p is an odd prime number, and $h(p^\nu) = 0$ when $\nu \geq 2$ and p is a prime number. The Dirichlet series of h is defined by:

$$D(h, s) = \sum_{n \geq 1} \frac{h(n)}{n^s}.$$

Since h is a multiplicative function, we can give the formal Euler product of $D(f, s)$:

$$D(h, s) = \prod_{p \geq 2} \Big(\sum_{k \geq 0} \frac{h(p^k)}{p^{ks}} \Big) = \prod_{p \geq 3} \Big(1 + \frac{1}{(p-2)p^s} \Big) \tag{2.1}$$

We want to compare this Euler product to a Zêta function in order to obtain a new product with a lower abscissa of absolute convergence. We note that the expansion (2.1) looks like $\zeta(s+1)$ as an Euler product. Then, we compare $D(h, s)$ with the function $\zeta(s+1)$; let us consider $C(s) = D(h, s)/\zeta(s+1)$. We have:

$$\begin{aligned} C(s) &= \Big(1 - \frac{1}{2^{s+1}} \Big) \prod_{p \geq 3} \Big(1 + \frac{1}{(p-2)p^s} \Big) \Big(1 - \frac{1}{p^{s+1}} \Big) \\ &= \Big(1 - \frac{1}{2^{s+1}} \Big) \prod_{p \geq 3} \Big(1 + \frac{2}{(p-2)p^{s+1}} - \frac{1}{(p-2)p^{2s+1}} \Big) \end{aligned}$$

where the product is absolutely convergent for $\Re(s) > -1/2$. Let f and g be the multiplicative functions whose Dirichlet series are respectively C and $\zeta(s+1)$ (in particular, f is defined by $f(n) = 1/n$, for every $n \geq 1$). Now, we have $h = f \star g$, and then, on expanding this convolution product, we get:

$$\begin{aligned} \sum_{\substack{m<z \\ m \equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} &= \sum_{m<z} (f \star g)(m) \\ &= \sum_{n<z} \sum_{\ell m = n} \frac{g(m)}{\ell} = \sum_{m \geq 1} g(m) \sum_{\ell < z/m} \frac{1}{\ell}. \end{aligned}$$

Now, we use the following estimate (see Lemma 10 in Appendix): for every $t > 0$,

$$\sum_{\ell < t} \frac{1}{\ell} = \log t + \gamma + \mathcal{O}(t^{-1/3})$$

where γ is the Euler's constant:

$$G(z) = \sum_{m \geq 1} g(m) \Big(\log(z/m) + \gamma + \mathcal{O}((z/m)^{-1/3}) \Big)$$

$$= \Big(\sum_{m\geq 1} g(m)\Big)(\log z + \gamma) - \sum_{m\geq 1} g(m)\log m + \mathcal{O}\Big(z^{-1/3}\sum_{m\geq 1}|g(m)|m^{1/3}\Big).$$

The first sum $\sum_{m\geq 1} g(m)$ is a (positive) constant (since the Dirichlet series of C is absolutely convergent for $\sigma = 0$ or because $\prod_{p\geq 3}\big(1+1/p(p-2)\big) < \infty$), say α_1. Furthermore, we note that the second sum is also a constant:

$$-\sum_{m\geq 1} g(m)\log m = C'(0),$$

say α_2. Lastly we use the fact that the Dirichlet series of C is absolutely convergent for $\Re(s) > -1/2$. Indeed,

$$\sum_{m\geq 1}|g(m)|m^{1/3} = \Big(1+\frac{1}{2^{2/3}}\Big)\prod_{p\geq 3}\Big(1+\frac{2}{(p-2)p^{2/3}}+\frac{1}{(p-2)p^{1/3}}\Big)$$

which is finite. Then, we can rewrite the remainder term as $\mathcal{O}(z^{-1/3})$ and the lemma is proved. □

We keep the notations α_1 and α_2 of the previous lemma until the end of these notes. This one is essential in our study; we establish now the estimations which appear naturally in the proof of Theorem 2.

Lemma 2. *We have:*

$$\sum_{\substack{m<z\\ m\equiv 1[2]}}\frac{\mu^2(m)}{\phi_1(m)}\frac{1}{1+\log(z/m)} \ll \frac{\log\log z}{\log z}G(z).$$

Proof. We prove this result by summation by parts; we start with the integration's formula:

$$\frac{1}{1+\log(z/m)} = \int_1^m \frac{du}{u(1+\log(z/u))^2} + \frac{1}{1+\log z}$$

which gives us:

$$\begin{aligned}\sum_{\substack{m<z\\ m\equiv 1[2]}}\frac{\mu^2(m)}{\phi_1(m)}\frac{1}{1+\log(z/m)} &= \sum_{\substack{m<z\\ m\equiv 1[2]}}\frac{\mu^2(m)}{\phi_1(m)}\Big[\int_1^m \frac{du}{u(1+\log(z/u))^2}+\frac{1}{1+\log z}\Big]\\ &= \sum_{\substack{m<z\\ m\equiv 1[2]}}\frac{\mu^2(m)}{\phi_1(m)}\int_1^m \frac{du}{u(1+\log(z/u))^2}+\frac{G(z)}{1+\log z}\\ &= \int_1^z (G(z)-G(u))\frac{du}{u(1+\log(z/u))^2}+\frac{G(z)}{1+\log z}.\end{aligned}$$

On appealing to the first lemma, we infer that:

$$\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \frac{1}{1+\log(z/m)} = \int_1^z (\alpha_1 \log(z/u) + \mathcal{O}(u^{-1/3})) \frac{1}{u(1+\log(z/u))^2}\, du + \frac{G(z)}{1+\log z}.$$

Now, we have:

$$\begin{aligned}\int_1^z \frac{\log(z/u)}{u(1+\log(z/u))^2} du &= -\int_1^z \frac{\log t}{(1+\log t)^2} \frac{t}{z} \frac{(-z)}{t^2} dt \\ &= \int_1^z \frac{\log t}{t(1+\log t)^2} dt \\ &= \log(1+\log z) - 1 + \frac{1}{1+\log z} \ll \log\log z.\end{aligned}$$

Moreover,

$$\int_1^z \mathcal{O}(u^{-1/3}) \frac{du}{u(1+\log(z/u))^2)} \ll \int_1^z \frac{du}{u^{4/3}(1+\log(z/u))^2} \leq \int_1^{+\infty} \frac{du}{u^{4/3}} < \infty,$$

and $G(z)/(1+\log z)$ is negligible in front of $(\log\log z/\log z)G(z)$. Finally, since $G(z) \sim \alpha_1 \log z$ when z goes to infinity, we conclude that:

$$\begin{aligned}\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \frac{1}{1+\log(z/m)} &\ll \log\log z \\ &\ll \frac{\log\log z}{\log z} G(z).\end{aligned}$$

□

Lemma 3. *We have:*

$$\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \log\Big(\frac{\log z}{1+\log(z/m)}\Big) \sim G(z)$$

when z goes to infinity.

Proof. As in the proof of Lemma 2, we use summation by parts; we start with the following integration's formula:

$$\log\Big(\frac{\log z}{1+\log(z/m)}\Big) = \int_1^m \frac{du}{u(1+\log(z/u))} + \log\Big(\frac{\log z}{1+\log z}\Big).$$

We have:

$$\begin{aligned}
&\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \log\big(\frac{\log z}{1+\log(z/m)}\big)\\
&= \int_1^z (G(z)-G(u))\frac{du}{u(1+\log(z/u))} + G(z)\log\big(\frac{\log z}{1+\log z}\big)\\
&= \int_1^z (\alpha_1 \log(z/u) + \mathcal{O}(u^{-1/3}))\frac{du}{u(1+\log(z/u))} + G(z)\log\big(\frac{\log z}{1+\log z}\big)
\end{aligned}$$

by using Lemma 1. Then,

$$\begin{aligned}
\int_1^z \frac{\log(z/u)}{u(1+\log(z/u))}du &= \int_1^z \frac{\log t}{t(1+\log t)}dt\\
&= \log z - \log(1+\log z) \sim \log z
\end{aligned}$$

when z goes to infinity. Moreover,

$$\int_1^z \mathcal{O}(u^{-1/3})\frac{du}{u(1+\log(z/u))} \ll \int_1^{+\infty} \frac{du}{u^{4/3}} < \infty,$$

and

$$G(z)\log\big(\frac{\log z}{1+\log z}\big) = \mathcal{O}(\frac{G(z)}{\log z}) = \mathcal{O}(1).$$

Actually, we obtain:

$$\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \log\big(\frac{\log z}{1+\log(z/m)}\big) \sim \alpha_1 \log z \sim G(z)$$

when z goes to infinity. □

Lemma 4. *We have:*

$$\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \sum_{\substack{p|2m\\ z/m\leq p<2z}} \frac{1}{p-1} = \epsilon(z)G(z)$$

where $\epsilon(z)$ tends to zero when z goes to infinity.

Proof. Firstly, we exchange the summation's symbol:

$$\sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \sum_{\substack{p|2m\\ z/m\leq p<2z}} \frac{1}{p-1} = \sum_{p<2z} \frac{1}{p-1} \sum_{\substack{z/p\leq m<z\\ p|2m\\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \tag{2.2}$$

and we deal with the prime number $p=2$ separately; the contribution is

$$G(z) - G(z/2) \ll 1.$$

Now, for the odd prime numbers p, by putting $m = p\ell$ in the interior sum of the right hand side term of (2.2), we get (since $\phi_1(p) = p - 2$):

$$\sum_{2<p<2z} \frac{1}{p-1} \sum_{\substack{z/p \leq m < z \\ p|m \\ m \equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \leq \sum_{2<p<2z} \frac{G(z/p) - G(z/p^2)}{(p-2)^2}$$

$$\ll \sum_{p>2} \frac{\log p}{(p-2)^2} < \infty$$

and the lemma follows readily. □

2.2 On the sequence $\{n(n+2)\}_{n\geq 1}$

In the second application, we consider the set of integers $\mathcal{N} = \{n(n+2|n \leq x\}$, where x is a positive real number. Here, $|\mathcal{N}_d| = \{n(n+2)|n(n+2) \equiv 0\ [d],\ n \leq x\}$. The equation $n(n+2) \equiv 0 \mod p$ has two solutions when p is an odd prime number, and only one solution when $p = 2$. Then $\mathcal{N}_d = 2^{\omega_2(d)} \lfloor x/d \rfloor$ where the (additive) function w_2 counts the number of odd prime divisors of integers. So, the function f will be defined by $f(d) = d/2^{\omega_2(d)}$.

About the function f_1

We have $f_1(2) = 1$ and $f_1(p) = (p-2)/2$ when $p > 2$. Then,

$$f_1(d) = \frac{\prod_{2<p|d}(p-2)}{2^{w_2(d)}}.$$

As in the last subsection, we need an estimation of the sums

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)},$$

called $G'(z)$. This is given by the following lemma.

Lemma 5. *There are three numerical constants $\beta_1 > 0$, β_2 and β_3 such that, for every $z > 0$, we have:*

$$G'(z) = \beta_1 \log^2 z + \beta_2 \log z + \beta_3 + \mathcal{O}(z^{-1/3}).$$

Proof. We denote by h the multiplicative function μ^2/f_1. We use the convolution method like in Lemma 1 but we will need here an estimate of the sums $\sum_{n\leq t} d(n)/n$ (instead of estimate of the sums $\sum_{n\leq t} 1/n$) where the proof of this can be found in [3]: there is a numerical constant β such that for every $t > 0$, we have:

$$\sum_{n\leq t} \frac{d(n)}{n} = \frac{\log^2 t}{2} + 2\gamma \log t + \beta + \mathcal{O}(t^{-1/3}). \qquad (2.3)$$

The Dirichlet series (as a formal Euler product) of the multiplicative function h is:

$$D(h,s) = \prod_{p\geq 2}\Big(\sum_{k\geq 0} g(p^k)p^{-ks}\Big) = \Big(1+\frac{1}{2^s}\Big)\prod_{p\geq 3}\Big(1+\frac{2}{p^s(p-2)}\Big) \qquad (2.4)$$

We note that the expansion (2.4) looks like the $\zeta(s+1)^2$'s expansion as an Euler product. Then, we compare $D(h,s)$ with the function $\zeta(s+1)^2$; let us consider the function $C(s) = D(h,s)/\zeta(s+1)^2$. We have:

$$C(s) = \Big(1+\frac{1}{2^s}\Big)\Big(1-\frac{1}{2^{s+1}}\Big)^2\prod_{p\geq 3}\Big(1+\frac{2}{p^s(p-2)}\Big)\Big(1-\frac{1}{p^{s+1}}\Big)^2$$

$$= \Big(1+\frac{1}{2^s}\Big)\Big(1-\frac{1}{2^{s+1}}\Big)^2\prod_{p\geq 3}\Big(1+\frac{2}{p^s(p-2)}\Big)\Big(1-\frac{2}{p^{s+1}}+\frac{1}{p^{2(s+1)}}\Big)$$

$$= \Big(1+\frac{1}{2^s}\Big)\Big(1-\frac{1}{2^{s+1}}\Big)^2\prod_{p\geq 3}\Big(1+\frac{2}{p^{s+1}(p-2)}+\frac{1}{p^{2(s+1)}}-\frac{4}{p^{2s+1}(p-2)}+\frac{2}{p^{3s+2}(p-2)}\Big)$$

where this product is absolutely convergent for $\Re(s) > -1/2$. But we know that

$$\zeta(s)^2 = \sum_{n\geq 1}\frac{d(n)}{n^s}.$$

As a result, the function $\zeta(s+1)$ is the Dirichlet series associated to the multiplicative function, say ψ, defined by $\psi(n) = d(n)/n$. Let g be the multiplicative function whose Dirichlet series is C; now, we have $h = \psi\star g$ and on expanding this convolution product, we obtain:

$$G'(z) = \sum_{n<z}(\psi\star g)(n) = \sum_{m\geq 1} g(m)\sum_{\ell<z/m}\frac{d(\ell)}{\ell}.$$

By using estimation (2.3), we get:

$$G'(z) = \sum_{m\geq 1} g(m)\Big[\frac{1}{2}\log^2(z/m) + 2\gamma\log(z/m) + \beta + \mathcal{O}\big((z/m)^{-1/3}\big)\Big],$$

and thus:

$$\begin{aligned} G'(z) = {} & \Big(\sum_{m\geq 1} g(m)\Big)\big(\frac{1}{2}\log^2 z + 2\gamma\log z + \beta\big) \\ & + \sum_{m\geq 1} g(m)(-2\gamma\log m + \frac{1}{2}\log^2 m - (\log m)\log z) \\ & + \mathcal{O}\big(z^{-1/3}\sum_{m\geq 1}|g(m)|m^{1/3}\big). \end{aligned}$$

The sums:

$$\sum_{m\geq 1} g(m), \ \sum_{m\geq 1} g(m)\log m = -C'(0),$$
$$\sum_{m\geq 1} g(m)\log^2 m = C''(0) \text{ and } \sum_{m\geq 1} |g(m)|m^{1/3}$$

are finite since the Dirichlet series of C is convergent for $\Re(s) > -1/2$. In particular, we can rewrite the remainder term as $\mathcal{O}(t^{-1/3})$. Finally, we define $\beta_1 = 1/2C(0)$, $\beta_2 = 2\gamma C(0) + C'(0)$, and $\beta_3 = \beta C(0) + 1/2C''(0) - 2\gamma C'(0)$ and the lemma is proved. □

We keep the notations β_1, β_2 and β_3 which appeared in Lemma 5 until the end of these notes. This lemma allows us to obtain three needed estimates.

Lemma 6. *We have:*

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \frac{1}{1+\log(z/m)} \ll \frac{\log\log z}{\log z} \sum_{m<z} \frac{\mu^2(m)}{f_1(m)}.$$

Proof. We prove this result by summation by parts by starting with the integration's formula:

$$\frac{1}{1+\log(z/m)} = \int_1^m \frac{du}{u(1+\log(z/u))^2} + \frac{1}{1+\log z},$$

as in Lemma 3. Then,

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \frac{1}{1+\log(z/m)} = \int_1^z (G'(z) - G'(u)) \frac{du}{u(1+\log(z/u))^2} + \frac{G'(z)}{1+\log z} \tag{2.5}$$

and we call the integral term in the right hand side of (2.5) by $I(z)$. Thanks to Lemma 5, we have:

$$I(z) = \int_1^z (\beta_1(\log^2 z - \log^2 u) + \beta_2 \log(z/u) + \mathcal{O}(u^{-1/3})) \frac{du}{u(1+\log(z/u))^2}.$$

Now, we have ever seen in Lemma 3 that

$$\int_1^z \frac{\log(z/u)}{u(1+\log(z/u))^2} \ll \log\log z$$

and

$$\int_1^z \frac{du}{u^{4/3}(1+\log(z/u))^2} \ll 1.$$

Furthermore,

$$\begin{aligned}\int_1^z \frac{du}{u(1+\log(z/u))^2} &= \int_1^z \frac{dt}{t(1+\log t)^2}\\ &= 1 - \frac{1}{1+\log z}\\ &= 1 + \mathcal{O}(1/\log z).\end{aligned}$$

Lastly,

$$\begin{aligned}\int_1^z \frac{\log^2 u}{u(1+\log(z/u))^2}du &= \int_1^z \frac{\log^2(z/t)}{t(1+\log t)^2}dt\\ &= \log^2 z\int_1^z \frac{dt}{t(1+\log t)^2} - 2\log z\int_1^z \frac{\log t}{t(1+\log t)^2}dt\\ &+ \int_1^z \frac{\log^2 t}{t(1+\log t)^2}dt\end{aligned}$$

Now,

$$\int_1^z \frac{dt}{t(1+\log t)^2} = 1 + \mathcal{O}(1/\log z), \tag{2.6}$$

$$\int_1^z \frac{\log t}{t(1+\log t)^2}dt \ll \log\log z \tag{2.7}$$

and

$$\int_1^z \frac{\log^2 t}{t(1+\log t)^2}dt \leq \log z. \tag{2.8}$$

Then, by putting the results (2.6), (2.7) and (2.8) together, we get:

$$\int_1^z \frac{\log^2 u}{u(1+\log(z/u))^2}du = \log^2 z + \mathcal{O}((\log z)\log\log z)$$

Finally, since $G'(z)/(1+\log z)$ is negligible in front of $(\log\log z/\log z)G'(z)$, we conclude that

$$\sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\frac{1}{1+\log(z/m)} = \log^2 z - \log^2 z + \mathcal{O}((\log z)\log\log z)$$

which is that we wanted since $G'(z) \sim \beta_1 \log^2 z$ at infinity. □

Lemma 7. *We have:*

$$\sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\log\big(\frac{\log z}{1+\log(z/m)}\big) \sim \frac{3}{2}G'(z)$$

when z goes to infinity.

Proof. As usual, we use summation by parts and we begin with the integration's formula:

$$\log\big(\frac{\log z}{1+\log(z/m)}\big) = \int_1^m \frac{du}{u(1+\log(z/u))} + \log\big(\frac{\log z}{1+\log z}\big).$$

Then,

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \log\big(\frac{\log z}{1+\log(z/m)}\big) = \int_1^m (G'(z) - G'(u)) \frac{du}{u(1+\log(z/u))} + G'(z)\log\big(\frac{\log z}{1+\log z}\big). \quad (2.9)$$

Let $I(z)$ be the integral term which appears in the right hand side of (2.9). We apply Lemma 5 and we have:

$$I(z) = \int_1^z \big[\beta_1(\log^2 z - \log^2 u) + \beta_2 \log(z/u) + \mathcal{O}(u^{-1/3})\big] \frac{du}{u(1+\log(z/u))}.$$

We know that (see the proof of Lemma 3)

$$\int_1^z \frac{\log(z/u)}{u(1+\log(z/u))} du \sim \log z$$

when z goes to infinity, so this term is negligible in front of $G'(z)$, and

$$\int_1^z \frac{du}{u(1+\log(z/u))} = \log(1+\log z). \quad (2.10)$$

Moreover,

$$\int_1^z \frac{\log^2 u}{u(1+\log(z/u))} du = (\log z)^2 \log(1+\log z) - 2\log z(\log z - \log(1+\log z)) + \int_1^z \frac{\log^2 t}{t(1+\log t)} dt \quad (2.11)$$

with

$$\int_1^z \frac{\log^2 t}{t(1+\log t)} dt = \int_1^z \frac{\log t + 1}{t} dt - 2\int_1^z \frac{\log t}{t(1+\log t)} dt - \int_1^z \frac{dt}{t(1+\log t)} = \frac{\log^2 z}{2} + \mathcal{O}(\log z). \quad (2.12)$$

Thus, by putting the previous estimates (2.10), (2.11) and (2.12) together, we conclude that:

$$\beta_1^{-1} \sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \log\big(\frac{\log z}{1+\log(z/m)}\big) = (\log z)^2 \log(1+\log z) - (\log z)^2 \log(1+\log z) + (2 - \frac{1}{2}) \log^2 z + \mathcal{O}((\log z)\log\log z),$$

i.e.

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \log\Big(\frac{\log z}{1+\log(z/m)}\Big) \sim \frac{3}{2}\beta_1 \log^2 z$$

when z goes to infinity, which conclude the proof of our lemma. □

Lemma 8. *We have:*

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \sum_{\substack{z/m\leq p<2z\\ p|2m}} \frac{1}{p} = \epsilon(z) \sum_{m<z} \frac{\mu^2(m)}{f_1(m)},$$

where $\epsilon(z)$ tends to zero when z goes to infinity.

Proof. Firstly, we exchange the summation's symbol:

$$\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \sum_{\substack{z/m\leq p<2z\\ p|2m}} \frac{1}{p} = \sum_{p<2z} \frac{1}{p} \sum_{\substack{z/p\leq m<z\\ p|2m}} \frac{\mu^2(m)}{f_1(m)} \tag{2.13}$$

and we deal with the prime number $p=2$ separately; the contribution is

$$G'(z) - G'(z/2) \ll \log z.$$

Now, for the odd prime numbers p, by putting $m = p\ell$ in the interior sum of the right hand side of (2.13), we get, since $f_1(p) = (p-2)/2$,

$$\begin{aligned}
\sum_{2<p<2z} \frac{1}{p} \sum_{\substack{z/p\leq m<z\\ p|2m}} \frac{\mu^2(m)}{f_1(m)} \leq 2 \sum_{2<p<2z} & \frac{G'(z/p)-G'(z/p^2)}{(p-2)^2} \\
= & \ (4\beta_1 \log z + \mathcal{O}(1)) \sum_{2<p<2z} \frac{\log p}{(p-2)^2} \\
\ll & \ \log z.
\end{aligned}$$

and the lemma follows readily since the size of $G'(z)$ is $\log^2 z$. □

Chapter 3

Proof of Theorem 2

We recall that we want to prove that there are infinitely many primes p such that $p+2$ has at most four prime factors. We do that in two steps: first we control the remainder term and afterwards we deal with the main term.

3.1 Control of the remainder term

Here, $\mathcal{N} = \{p+2 | p \leq x\}$ and

$$S = \sum_{p \leq x} \Big(\sum_{\substack{d|p+2 \\ d<y}} a_d \Big) \Big(\sum_{\substack{\nu|p+2 \\ \nu<z}} \lambda_\nu \Big)^2$$

with bounded coefficients a_d and λ_ν that we are going to choose later. Then, Theorem 1 gives us:

$$S = \mathrm{Li}(x)\mathfrak{S} + \mathcal{O}\Big(\sum_{\substack{m<yz^2 \\ m \equiv 1[2]}} d^3(m) |\mathcal{R}_m| \Big)$$

where

$$\mathfrak{S} = \sum_{\substack{m<z \\ m\equiv 1[2]}} \sum_{\substack{d<y \\ d\equiv 1[2] \\ (m,d)=1}} \frac{\mu^2(m)}{\phi_1(m)} \frac{a_d}{\phi(d)} \Big[\sum_{\substack{r|d \\ r<z/m}} \mu(r) \zeta_{rm} \Big]^2 \tag{3.1}$$

and

$$\mathcal{R}_m = \pi(x; m, -2) - \frac{\mathrm{Li}(x)}{\phi(m)}.$$

Proof. The expression of $\mathfrak{S}$ is given by (1.1); when p is an odd prime number, then all divisors of $p+2$ are odds and the contribution of $p=2$ in S is a constant (it would be negligible in front of the remainder term) that's why the summations in (3.1) are over odd integers. Hence, we can rewrite the remainder term

$$\mathcal{O}\Big(\sum_{\substack{m<yz^2\\ m\equiv 1[2]}}\big(\sum_{d|m}|a_d|\big)\big(\sum_{\nu|m}|\lambda_\nu|\big)^2|\mathcal{R}_m|\Big)$$

as

$$\mathcal{O}\Big(\sum_{\substack{m<yz^2\\ m\equiv 1[2]}}\big(\sum_{a|m}1\big)^3|\mathcal{R}_m|\Big)$$

what is that we have announced since $\sum_{a/m}1 = d(m)$. □

Now, by using the Bombieri-Vinogradov Theorem, we shall prove that this remainder term is $\mathcal{O}_A\big(x(\log x)^{-A}\big)$ on using the fact that:

$$|\mathcal{R}_m| \leq \mathrm{R}_m := \max_{u\leq x}|\pi(u;m,-2) - \frac{\mathrm{Li}(u)}{\phi(m)}|.$$

Then, by using the Cauchy-Schwarz inequality, we have:

$$\sum_{\substack{m<yz^2\\ m\equiv 1[2]}} d^3(m)|\mathcal{R}_m| \leq \Big(\sum_{m<yz^2}\frac{d^6(m)}{m}\Big)^{1/2}\Big(\sum_{\substack{m<yz^2\\ m\equiv 1[2]}} m|\mathcal{R}_m|^2\Big)^{1/2}.$$

We admit temporarily that there is a positive integer k (equal to 2^6 for instance according to Lemma 9) such that $d^6(m) \leq d_k(m)$. Hence,

$$\begin{aligned}\sum_{\substack{m<yz^2\\ m\equiv 1[2]}} d^3(m)|\mathcal{R}_m| &\leq \Big(\sum_{m<yz^2}\frac{d_k(m)}{m}\Big)^{1/2}\Big(\sum_{m<yz^2} m|\mathcal{R}_m|^2\Big)^{1/2}\\ &\leq \Big(\sum_{m<yz^2}\frac{1}{m}\Big)^{k/2}\Big(\sum_{m<yz^2} m|\mathcal{R}_m|^2\Big)^{1/2}\\ &\ll (\log yz^2)^{32}\Big(\sum_{m<yz^2} m|\mathcal{R}_m|^2\Big)^{1/2}\end{aligned}$$

for $k = 64$ which is admissible. Since $|\mathcal{R}_m| \ll x/\phi(m)$ (indeed, $\pi(u;m,-2) \leq (x+2)/m$ and $\mathrm{Li}(x) \ll x/\log x$) and $m/\phi(m) \ll \log m$ (see Lemma 13), we obtain:

$$\begin{aligned}\sum_{\substack{m<yz^2\\ m\equiv 1[2]}} d^3(m)|\mathcal{R}_m| &\ll (\log yz^2)^{32+1/2}x^{1/2}\Big(\sum_{m<yz^2}|\mathcal{R}_m|\Big)^{1/2}\\ &\ll (\log yz^2)^{65/2}x^{1/2}\Big(\sum_{m<yz^2}\mathrm{R}_m\Big)^{1/2},\end{aligned}$$

because $|\mathcal{R}_m| \leq \mathrm{R}_m$. Let $A > 0$ and $A' := 2A + 65$. We use the Bombieri-Vinogradov Theorem which gives us, if $B = B(A') > 0$ is large enough and $yz^2 < x(\log x)^{-B}$:

$$\begin{aligned}\sum_{m<yz^2} d^3(m)|\mathcal{R}_m| &\ll (\log yz^2)^{65/2} x^{1/2} \Big(\sum_{m<x(\log x^{-B})} \mathrm{R}_m \Big)^{1/2} \\ &\ll_A (\log x)^{65/2} x^{1/2} x^{1/2} (\log x)^{65/2-A} \\ &= x(\log x)^{-A}.\end{aligned}$$

Finally, we prove that:

$$\forall A > 0, \; S = \mathrm{Li}(x)\mathfrak{S} + \mathcal{O}_A(x(\log x)^{-A}). \tag{3.2}$$

3.2 Main term

As we said in the introduction, we study the sum S with two sequences of real numbers $\{a_d\}_{d\geq 1}$ which each of them gives a different term $\mathfrak{S}$. We recall that we take ζ_r as more simple as we can: $\zeta_r = \zeta_1$ if $r < z$ is squarefree, $\zeta_r = 0$ either. First and foremost, let us consider the case $a_1 = 1$ (in fact, we will take $a_1 = 2$ later) and $a_d = 0$ when $d > 1$. Hence, we have:

$$\mathfrak{S} = \Big[\sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \Big] \zeta_1^2.$$

Then,

$$\sum_{p\leq x} \Big(\sum_{\substack{\nu | p+2 \\ \nu < z}} \lambda_\nu \Big)^2 \sim G(z)\zeta_1^2 \frac{x}{\log x} \tag{3.3}$$

when x goes to infinity (for any choice of y and z such that (3.2) is true for $A > 2$). Now, we consider the sequence $\{a_d\}_{d\geq 1}$ such that $a_{p'} = -1$ when $2 < p' < y$ is an odd prime number and $a_d = 0$ otherwise. Then,

$$-\mathfrak{S} = \sum_{\substack{m<z \\ m\equiv 1[2]}} \sum_{\substack{2<p<y \\ (m,p)=1}} \frac{\mu^2(m)}{\phi_1(m)} \frac{1}{p-1} \Big[\sum_{\substack{r|d \\ r<z/m}} \mu(r)\zeta_{rm} \Big]^2$$

We easily see that

$$\sum_{r|p} \mu(r)\zeta_{mr} = \zeta_1$$

when $p \geq z/m$, and

$$\sum_{r|p} \mu(r)\zeta_{mr} = 0$$

otherwise. Hence, we can simplify the main term:

$$-\mathfrak{S} = \sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \Big[\sum_{\substack{z/m\leq p<y \\ p\not| 2m}} \frac{1}{p-1} \Big] \zeta_1^2$$

We want to give an estimation of $-\mathfrak{S}$. For this, we use the following approximation which is the object of Lemma 11: for every $t>0$,

$$\sum_{p\leq t} \frac{1}{p-1} = \log(1+\log t) + A_0 + \mathcal{O}(\frac{1}{1+\log t}) \tag{3.4}$$

(where A_0 is a numerical constant which is not important for our subject) and our lemmas in the chapter 2. Before applying them, we write:

$$\begin{aligned} -\mathfrak{S}\zeta_1^{-2} = G(z) \sum_{p<y} \frac{1}{p-1} &- \sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \sum_{p<z/m} \frac{1}{p-1} \\ &- \sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \sum_{\substack{z/m\leq p<y \\ p|2m}} \frac{1}{p-1}. \end{aligned}$$

The estimate (3.4) gives us:

$$\begin{aligned} -\mathfrak{S}\zeta_1^{-2} = G(z)&\Big(\log(1+\log y) + A_0 + \mathcal{O}(\frac{1}{1+\log y})\Big) \\ &- \sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \Big(\log(1+\log(z/m)) + A_0 + \mathcal{O}(\frac{1}{1+\log(z/m)})\Big) \\ &- \sum_{\substack{m<z \\ m\equiv 1[2]}} \frac{\mu^2(m)}{\phi_1(m)} \sum_{\substack{z/m\leq p<y \\ p|2m}} \frac{1}{p-1} \end{aligned}$$

where we apply Lemma 2, Lemma 3 and Lemma 4:

$$-\mathfrak{S}\zeta_1^{-2} = \Big\{ \log \frac{\log y}{\log z} + 1 + \epsilon(z) \Big\} G(z)$$

where $\epsilon(z)$ tends to zero when z goes to infinity. Moreover, $\mathrm{Li}(x) \sim x/\log x$ at infinity and $\zeta_1 = \lambda_1/G(z)$ (according to (1.2)), so we get:

$$\sum_{p\leq x} \Big(\sum_{\substack{p'|p+2 \\ p'<y}} 1 \Big) \Big(\sum_{\substack{\nu|p+2 \\ \nu<z}} \lambda_\nu \Big)^2 \sim \Big\{ 1 + \log \frac{\log y}{\log z} \Big\} \frac{x}{G(z)\log x} \lambda_1^2 \tag{3.5}$$

if $yz^2 < x^{1/2}(\log x)^{-B}$, so as to make the remainder term smaller than the main term. Hence, the two expressions (3.3) and (3.5) of S give us:

$$\sum_{p\leq x}\Big[2-\sum_{\substack{p'|p+2\\ p'<y}}1\Big]\Big(\sum_{\substack{\nu|p+2\\ \nu<z}}\lambda_\nu\Big)^2 \sim \{1-\log\frac{\log y}{\log z}\}\frac{x}{G(z)\log x}\lambda_1^2$$

when x goes to infinity. Now, if we choose $y = x^{1/4+\epsilon_0}$ and $z = x^{1/8-\epsilon_0}$ (with ϵ_0 small), we have $yz^2 < x^{1/2}(\log x)^{-B}$ and

$$\sum_{p\leq x}\Big[2-\sum_{\substack{p'|p+2\\ p'<y}}1\Big]\Big(\sum_{\substack{\nu|p+2\\ \nu<z}}\lambda_\nu\Big)^2 \sim \{1-\log(\frac{2+8\epsilon_0}{1-8\epsilon_0})\}\frac{x}{G(z)\log x}\lambda_1^2. \qquad (3.6)$$

We take $\epsilon_0 > 0$ small enough in order to make the term into embraces in the right hand side of (3.6) positive. Since $x/(G(z)\log x)$ tends to infinity when x goes to infinity, it follows that

$$2-\sum_{\substack{p'|p+2\\ p'<y}}1 > 0$$

for a lot of prime numbers p. But $\sum_{\substack{p'|p+2\\ p'<y}}1$ is exactly the number of prime factors of $p+2$ smaller than y, so there is at most one prime factor of $p+2$ not more than y. Furthermore, with our choice of y, we see that there is at most $\lfloor\log(x+2)/\log y\rfloor = \lfloor 4\log(x+2)/(\log x(1+4\epsilon_0))\rfloor = 3$ prime factors of $p+2$ bigger than y. Eventually, $p+2$ has at most four prime factors, which conclude the proof of our first Theorem.

3.3 A lower bound

We proved in the previous section that

$$|\tilde{\mathcal{N}_0}| = \#\{p \leq x | \omega(p+2) \leq 4\} \to +\infty$$

when x goes to infinity and we want to quantify this fact. We start with (3.6). We note that the function of p: $2-\sum_{\substack{p'|p+2\\ p'<y}}1$ is bounded by twice the characteristic function of $\tilde{\mathcal{N}_0}$; moreover the weights λ_ν are bounded in absolute value by one (with the choice $\lambda_1 = 1$; see Lemma 12) and there is at most four squarefree divisors of $p+2$ which contribute in the sum (if ν is not squarefree, then $\lambda_\nu = 0$ and if p' and p'' are the possible prime factors of

$p+2$, the squarefree divisors of $p+2$ are 1, p', p'' and $p'p''$). Then,

$$\sum_{p\le x}\Big[2-\sum_{\substack{p'|p+2\\ p'<y}}1\Big]\Big(\sum_{\substack{\nu|p+2\\ \nu<z}}\lambda_\nu\Big)^2 \le 2(1+1+1+1)^2\sum_{\substack{p\le x\\ \omega(p+2)\le 4}}1$$
$$=32\sum_{\substack{p\le x\\ \omega(p+2)\le 4}}1.$$

So, since $G(z)\sim\alpha_1\log z$ when z goes to infinity, we have:

$$\#\{p\le x|\omega(p+2)\le 4\}\gg\frac{x}{\log^2 x}. \tag{3.7}$$

Chapter 4

Proof of Theorem 3

We recall that we want to prove that there are infinitely many integers n such that $n(n+2)$ has at most six prime factors. In order to do so, we sieve the set of integers of the form $n(n+2)$, where n is an integer not more than x. We control the remainder term first, and we deal with the main term lastly as in the previous chapter.

4.1 Control of the remainder term

Here, $\mathcal{N} = \{n(n+2) | n \leq x\}$ and

$$S = \sum_{n \leq x} \Big(\sum_{\substack{d | n(n+2) \\ d < y}} a_d \Big) \Big(\sum_{\substack{\nu | n(n+2) \\ \nu < z}} \lambda_\nu \Big)^2$$

with bounded coefficients a_d and λ_ν that we are going to choose later. Then Theorem 1 gives us:

$$S = x\mathfrak{S} + \mathcal{O}\Big(\sum_{m < yz^2} \big(\sum_{\substack{d | m \\ d < y}} |a_d| \big) \big(\sum_{\substack{\nu | m \\ \nu < z}} |\lambda_\nu| \big)^2 |\mathcal{R}_m| \Big),$$

where

$$\mathfrak{S} = \sum_{m < z} \sum_{\substack{d < y \\ (m,d)=1}} \frac{\mu^2(m)}{f_1(m)} \frac{2^{w_2(d)}}{d} a_d \Big(\sum_{\substack{r | d \\ r < z/m}} \mu(r) \zeta_{rm} \Big)^2,$$

and

$$\mathcal{R}_m = |\mathcal{N}_m| - \frac{2^{\omega_2(m)}}{m} x.$$

Under the same hypothesis that the real numbers a_d and λ_ν are bounded by one, we can rewrite the remainder term which becomes, as in the proof of the previous Theorem $\mathcal{O}\Big(\sum_{m < yz^2} d^3(m) |\mathcal{R}_m| \Big)$. But $|\mathcal{R}_m| = \mathcal{O}(2^{\omega_2(m)})$

and $2^{w_2(m)} \leq d(m)$. We know that $d(m) \ll_\epsilon m^\epsilon$ for all $\epsilon > 0$, according to Lemma 14. So, we can rewrite the remainder term as $\mathcal{O}_\epsilon(\sum_{m<yz^2} m^{4\epsilon})$, that is to say $\mathcal{O}_\epsilon((yz^2)^{1+\epsilon})$, for every $\epsilon > 0$. Finally, we prove that:

$$\forall \epsilon > 0,\ S = x\mathfrak{S} + \mathcal{O}_\epsilon((yz^2)^{1+\epsilon}).$$

4.2 Main term

We follow the proof of Theorem 2 (with our sequence $\{\zeta_r\}_{r\geq 1}$: $\zeta_r = \zeta_1$ if $r < z$ is squarefree, and $\zeta_r = 0$ either). First, let $a_1 = 1 > 0$ (in fact, we will take $a_1 = b$ for some positive real number later) and $a_d = 0$ when $d > 1$. In this case,

$$\mathfrak{S} = \Big[\sum_{m<z} \frac{\mu^2(m)}{f_1(m)}\Big]\zeta_1^2.$$

Then,

$$\sum_{n\leq x}\Big(\sum_{\substack{\nu|n(n+2)\\ \nu<z}} \lambda_\nu\Big)^2 \sim G'(z)\zeta_1^2 x \tag{4.1}$$

when x goes to infinity (for any choice of y and z which make the remainder term smaller than the main term). In a second time, we consider the sequence $\{a_d\}_{d\geq 1}$ such that $a_p = -1$ when $p < y$ is prime and $a_d = 0$ otherwise. This time, we get:

$$-\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2$$

where

$$\mathfrak{S}_1 = \sum_{\substack{m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{f_1(m)} \frac{1}{2}\Big(\sum_{\substack{r|2\\ r<z/m}} \mu(r)\Big)^2 \zeta_1^2,$$

and

$$\mathfrak{S}_2 = \sum_{m<z}\sum_{\substack{2<p<y\\ (m,p)=1}} \frac{\mu^2(m)}{f_1(m)} \frac{2}{p}\Big(\sum_{\substack{r|p\\ r<z/m}} \mu(r)\zeta_{rm}\Big).$$

Firstly, we have for $\mathfrak{S}_1$:

$$\mathfrak{S}_1 = \frac{1}{2}\sum_{\substack{z/2\leq m<z\\ m\equiv 1[2]}} \frac{\mu^2(m)}{f_1(m)}\zeta_1^2$$

which is negligible in front of $G'(z)\zeta_1^2$ because the size of $\sum_{\substack{z/2\le m<z\\ m\equiv 1 \mod 2}} \mu^2(m)/f_1(m)$ is $\log z$. Then, we study the second term $\mathfrak{S}_2$:

$$\begin{aligned}\mathfrak{S}_2 &= \sum_{m<z}\sum_{\substack{2<p<y\\(m,p)=1}} \frac{\mu^2(m)}{f_1(m)}\frac{2}{p}\Big(\sum_{\substack{r|p\\ r<z/m}}\mu(r)\zeta_{rm}\Big)\\ &= 2\sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\Big[\sum_{\substack{z/m\le p<y\\ p\nmid 2m}}\frac{1}{p}\Big]\zeta_1^2.\end{aligned}$$

We want to give an estimate of $\mathfrak{S}_2$. For this, we use the approximation (3.4) and our lemmas in the chapter 2. Before applying them, we write:

$$\frac{\mathfrak{S}_2\zeta_1^{-2}}{2} = G'(z)\sum_{p<y}\frac{1}{p} - \sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\sum_{p<z/m}\frac{1}{p} - \sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\sum_{\substack{z/m\le p<y\\ p|2m}}\frac{1}{p}$$

The estimate (3.4) gives us:

$$\begin{aligned}\frac{\mathfrak{S}_2\zeta_1^{-2}}{2} &= \sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\Big(\log(1+\log y) + B_0 + \mathcal{O}\big(\frac{1}{1+\log y}\big)\Big)\\ &\quad - \sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\Big(\log(1+\log(z/m)) + B_0 + \mathcal{O}\big(\frac{1}{1+\log(z/m)}\big)\Big)\\ &\quad - \sum_{m<z}\frac{\mu^2(m)}{f_1(m)}\sum_{\substack{z/m\le p<y\\ p|2m}}\frac{1}{p}\end{aligned}$$

where $B_0 = A_0 - \sum_{p\ge 2} 1/(p(p-1))$ is a constant. Now, we apply Lemma 6, Lemma 7 and Lemma 8:

$$\frac{\mathfrak{S}_2\zeta_1^{-2}}{2} = \Big\{\log\frac{\log y}{\log z} + \frac{3}{2} + \epsilon(z)\Big\}G'(z)$$

where $\epsilon(z)$ tends to zero when z goes to infinity. Then, we have the second estimation:

$$\sum_{n\le x}\Big(\sum_{\substack{p|n(n+2)\\ p<y}}1\Big)\Big(\sum_{\substack{\nu|n(n+2)\\ \nu<z}}\lambda_\nu\Big)^2 \sim \Big\{3 + 2\log\frac{\log y}{\log z}\Big\}\frac{x}{G'(z)}\lambda_1^2 \qquad (4.2)$$

when x goes to infinity (for any choice of y and z which make the remainder term smaller than the main term) since $\zeta_1 = \lambda_1/G'(z)$ (according to (1.2)). Using the two asymptotic results (4.1) and (4.2), we conclude that, if b is a positive real number, we have (by substituting "1" for "b" in (4.1)):

$$\sum_{n\le x}\Big\{b - \sum_{\substack{p|n(n+2)\\ p<y}}1\Big\}\Big(\sum_{\substack{\nu|n(n+2)\\ \nu<z}}\lambda_\nu\Big)^2 \sim \Big\{b - 3 - 2\log\frac{\log y}{\log z}\Big\}\frac{x}{G'(z)}\lambda_1^2$$

at infinity. We choose the parameters y and z such that: $y = x^{1/2+\epsilon_0}$ and $z = x^{1/4-\epsilon_0}$, where ϵ_0 is a small positive real number, so as to make the remainder term (recall that it is $\mathcal{O}_\epsilon((yz^2)^{1+\epsilon})$, for all $\epsilon > 0$) negligible in front of the main term. Moreover, we want to choose the positive real number b as small as we can in order to give the smaller lower upper bound for $\omega(n(n+2))$ with this method. But we see that $3 + 2\log(\log y/\log z) = 3 + 2\log\big((2+2\epsilon_0)/(1-4\epsilon_0)\big) \to 3 + 2\log 2 \approx 4.4$ when ϵ_0 goes to zero with positive values. So, we can choose the real number b such that $4.5 < b < 5$. We take $b = 4.99$ which gives us the next estimate:

$$\sum_{n \le x} \Big[4.99 - \sum_{\substack{p|n(n+2) \\ p<y}} 1\Big]\Big(\sum_{\substack{\nu|n(n+2) \\ \nu<z}} \lambda_\nu\Big)^2 \sim \Big\{1.99 - 2\log\Big(\frac{2+2\epsilon_0}{1-4\epsilon_0}\Big)\Big\}\frac{x}{G'(z)}\lambda_1^2, \quad (4.3)$$

when x goes to infinity. Since the term on the right of (4.3) tends to infinity when x goes to infinity (because of the positivity of $1.99 - 2\log((2+2\epsilon_0)/(1-4\epsilon_0))$ when ϵ_0 is small) we conclude that:

$$4.99 - \sum_{\substack{p|n(n+2) \\ p<y}} 1 > 0,$$

i.e.

$$\sum_{\substack{p|n(n+2) \\ p<y}} 1 \le 4$$

for many integers n. Then, such an integer $n(n+2)$ has at most four prime factors not more than y. Moreover, n has at most $\lfloor \log x/\log y \rfloor = \lfloor 2/(1+2\epsilon_0) \rfloor = 1$ prime factor greater than y and $n+2$ has at most $\lfloor \log(x+2)/\log y \rfloor = \lfloor 2\log(x+2)/(\log x(1+2\epsilon_0)) \rfloor = 1$ prime factor greater than y. Eventually, there are infinitely many integers n with at most six prime factors, i.e. $\omega(n(n+2)) \le 6$, which establishes our Theorem.

4.3 A lower bound

We showed in the previous section that

$$|\tilde{\mathcal{N}_0}| = \#\{n \le x | \omega(n(n+2)) \le 6\} \to +\infty$$

when x goes to infinity and we want to quantify this fact. We start with (4.3). We note that the function of n: $4.99 - \sum_{\substack{p|n(n+2) \\ p<y}} 1$ is raised by five times the characteristic function of $\mathcal{N}_0$; moreover, the weights λ_ν are bounded in absolute value by one (with the choice $\lambda_\nu = 1$; see Lemma 12) and there is

at most 2^4 squarefree divisors of $n \in \tilde{\mathcal{N}_0}$ (the divisors ν of n which are not squarefree give a contribution zero in the sum: $\lambda_\nu = 0$). Then, the square of the sum $\sum_{\substack{\nu | n(n+2) \\ \nu < z}} \lambda_\nu$ is uniformly bounded by 2^8 when $n \in \tilde{\mathcal{N}_0}$. So,

$$\sum_{n \leq x} \Big[4.99 - \sum_{\substack{p | n(n+2) \\ p < y}} 1\Big] \Big(\sum_{\substack{\nu | n(n+2) \\ \nu < z}} \lambda_\nu \Big)^2 \leq 1280 \sum_{\substack{n \leq x \\ \omega(n(n+2)) \leq 6}} 1.$$

Consequently, since $G'(z) \sim \beta_1 \log^2 z$, we have:

$$\#\{n \leq x | \omega(n(n+2)) \leq 6\} \gg \frac{x}{\log^2 x}. \tag{4.4}$$

Chapter 5

Without small prime factors

5.1 On the sequence $\{p+2\}_{p\geq 2}$

In this section, we consider prime numbers p such that all prime factors p' of $p+2$ are greater than p^a, where $a>0$ is small. We want to determine such an admissible value of a: there should be infinitely many primes p with at most four prime factors of $p+2$ greater than p^a and no prime factors less than p^a (i.e. with our notations: $\omega(p+2)\leq 4$ and $p\in\mathcal{P}_a$). More precisely, we want to prove that:

Theorem 7. *For $a=1/17$, we have:*

$$\sum_{\substack{p\leq x\\ p\in\mathcal{P}_a\\ \omega(p+2)\leq 4}} 1 \gg \frac{x}{\log^2 x}.$$

Proof. We follow the proof of Theorem 2. Firstly, we keep the same values of a_d: $a_1=2$ and $a_d=0$ when $d>1$ which gives the estimation (3.3). Lastly, we choose the sequence $\{a_d\}_{d\geq 1}$ such that $a_{p'}=-2$ when $2<p'<x^a$ and $a_{p'}=-1$ when $x^a\leq p'<y$. Then,

$$\begin{aligned}-\mathfrak{S} &= \sum_{\substack{m<z\\ m\equiv 1[2]}}\sum_{\substack{2<p<y\\ (m,p)=1}}\frac{\mu^2(m)}{\phi_1(m)}\frac{a_p}{p-1}\Big[\sum_{\substack{r|d\\ r<z/m}}\mu(r)\zeta_{rm}\Big]^2\\ &= \mathfrak{S}_1+\mathfrak{S}_2,\end{aligned}$$

with

$$\mathfrak{S}_1=\sum_{\substack{m<z\\ m\equiv 1[2]}}\frac{\mu^2(m)}{\phi_1(m)}\Big(\sum_{\substack{z/m\leq p<y\\ p\nmid 2m}}\frac{1}{p-1}\Big)\zeta_1^2$$

and

$$\mathfrak{S}_2=\sum_{\substack{m<z\\ m\equiv 1[2]}}\frac{\mu^2(m)}{\phi_1(m)}\Big(\sum_{\substack{z/m\leq p<x^a\\ p\nmid 2m}}\frac{1}{p-1}\Big)\zeta_1^2,$$

where we apply again Lemma 2, Lemma 3 and Lemma 4 which give us (with this second choice of $\{a_d\}_{d\geq 1}$):

$$-\sum_{p\leq x}\Big(\sum_{\substack{d|p+2\\ d<y}} a_d\Big)\Big(\sum_{\substack{\nu|p+2\\ \nu<z}}\lambda_\nu\Big)^2 \sim \Big\{2+\log\frac{\log y}{\log z}+\log(a\frac{\log x}{\log z})\Big\}\frac{x}{G(z)\log x}\lambda_1^2 \tag{5.1}$$

when x goes to infinity (morally, we have added (3.5) to (3.5) with $y = x^a$). Thus, the estimates (3.3) and (5.1) give us:

$$\sum_{p\leq x}\Big[2-\sum_{\substack{p'|p+2\\ 2<p'<x^a}} 2-\sum_{\substack{p'|p+2\\ x^a\leq p'<y}} 1\Big]\Big(\sum_{\substack{\nu|p+2\\ \nu<z}}\lambda_\nu\Big)^2 \sim \Big\{-\log\frac{\log y}{\log z}-\log(a\frac{\log x}{\log z})\Big\}\frac{x}{G(z)\log x}\lambda_1^2,$$

at infinity. We take back the parameters y and z of the proof of Theorem 2, i.e. $y = x^{1/4+\epsilon_0}$ and $z = x^{1/8-\epsilon_0}$ ($\epsilon_0 > 0$ small), which gives us:

$$\sum_{p\leq x}\Big[2-\sum_{\substack{p'|p+2\\ 2<p'<x^a}} 2-\sum_{\substack{p'|p+2\\ x^a\leq p'<y}} 1\Big]\Big(\sum_{\substack{\nu|p+2\\ \nu<z}}\lambda_\nu\Big)^2 \sim \log\Big(\frac{(1-8\epsilon_0)^2}{8a(2+8\epsilon_0)}\Big)\frac{x}{G(z)\log x}\lambda_1^2. \tag{5.2}$$

Here, we take the constant a in order to make the logarithm in the right hand side of (5.2) positive, which is equivalent to make the function of a in the logarithm greater than 1. In short, we want to determine a such that $a < (1-8\epsilon_0)^2/(16(1+4\epsilon_0))$ where this function (of the variable ϵ_0) decrease on $]0, 1/8[$ with a limit in zero equals to $1/16 = 0.0625$. So, we can choose for a every value in $]0, 1/16[$; for instance, $a = 1/17$ is admissible. Now, the right hand side of (5.2) tends to infinity when x goes to infinity for this value of a. Then, we conclude that

$$2-\sum_{\substack{p'|p+2\\ 2<p'<x^a}} 2-\sum_{\substack{p'|p+2\\ x^a\leq p'<y}} 1 > 0$$

for a lot of primes p. This inequality implies that the sum over prime factors of $p+2$ not more than x^a is zero. Finally, a prime number p which satisfies the previous inequality is such that $p+2$ has no prime factor less than x^a, one prime factor p' in $[x^a, y[$ (or no prime factor in this interval) and at most three prime factors bigger than y. By the same arguments as in the proof of (3.7), we conclude that:

$$\sum_{\substack{p\leq x\\ p\in\mathcal{P}_{1/17}\\ \omega(p+2)\leq 4}} 1 \gg \frac{x}{\log^2 x}.$$

In particular, there is infinitely many prime numbers p such that $p+2$ has at most four prime factors not more than $p^{1/17}$ and no prime factors lower than $p^{1/17}$. □

5.2 On the sequence $\{n(n+2)\}_{n\geq 1}$

Here, we consider integers n such that all prime factors of $n(n+2)$ are greater than $n^{a'}$, where $a' > 0$ is small. We want to determine such an admissible value of a': there should be infinitely many integers n such that $n(n+2)$ has at most six prime factors greater than $n^{a'}$ and no prime factors less than $n^{a'}$(i.e. with our notations: $\omega(n(n+2)) \leq 6$ and $n \in \mathcal{Q}_{a'}$). More precisely, we have:

Theorem 8. *For $a' = 1/17$, we have:*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{Q}_{a'}\\ \omega(n(n+2))\leq 6}} 1 \gg \frac{x}{\log^2 x}.$$

Proof. We follow the proof of Theorem 3. We keep the coefficients $a_1 = b$ and $a_d = 0$ when $d > 1$ at the begining and we get the estimate (4.1). Next, let $a_p = -b$ when $p < x^a$ and $a_p = -1$ when $x^a \leq p < y$ (for some positive real number b). Then, we have in this second case:

$$\begin{aligned}
-\mathfrak{S} &= \sum_{m<z} \sum_{\substack{p<y\\ (m,p)=1}} \frac{\mu^2(m)}{f_1(m)} \frac{a_p}{p} 2^{\omega_2(p)} \Big(\sum_{\substack{r|p\\ r<z/m}} \mu(r)\zeta_{rm} \Big)\\
&= \mathfrak{S}_1 + \mathfrak{S}_2 + \epsilon(z)G'(z)\zeta_1^2
\end{aligned}$$

where

$$\frac{\mathfrak{S}_1\zeta_1^{-2}}{2} = (b-1)\sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \sum_{\substack{z/m\leq p<x^a\\ p\nmid 2m}} \frac{1}{p},$$

$$\frac{\mathfrak{S}_2\zeta_1^{-2}}{2} = \sum_{m<z} \frac{\mu^2(m)}{f_1(m)} \sum_{\substack{z/m\leq p<y\\ p\nmid 2m}} \frac{1}{p}$$

and $\epsilon(z)$ tends to zero when z goes to infinity. We apply Lemma 6, Lemma 7 and Lemma 8 (morally, we add $(b-1)$ times the estimate (4.2) for to (4.2) with $y = x^a$) which give us (for this second choice of $\{a_d\}_{d\geq 1}$):

$$\begin{aligned}
&-\sum_{n\leq x} \Big(\sum_{\substack{d|n(n+2)\\ d<y}} a_d \Big) \Big(\sum_{\substack{\nu|n(n+2)\\ \nu<z}} \lambda_\nu \Big)^2\\
&\qquad \sim \Big\{ 3(b-1) + 2(b-1)\log\big(a\frac{\log x}{\log z}\big) + 3 + 2\log\frac{\log y}{\log z} \Big\} \frac{x}{G'(z)}\lambda_1^2 \qquad (5.3)
\end{aligned}$$

at infinity. Thus, with the two choices of the sequence $\{a_d\}_{d\geq 1}$, we have (accorrding to (4.1) and (5.3)):

$$\sum_{n\leq x}\Big[b-\sum_{\substack{p|n(n+2)\\ p<x^a}} b-\sum_{\substack{p|n(n+2)\\ x^a\leq p<y}} 1\Big]\Big(\sum_{\substack{\nu|n(n+2)\\ \nu<z}}\lambda_\nu\Big)^2$$
$$\sim\Big\{-2b+2(b-1)\log\big(\frac{1-4\epsilon_0}{4a}\big)\big)+2\log\big(\frac{1-4\epsilon_0}{2+4\epsilon_0}\big)\Big\}\frac{x}{G'(z)}\lambda_1^2 \quad (5.4)$$

where we have kept the same expressions of our parameters $y=x^{1/2+\epsilon_0}$ and $z=x^{1/4-\epsilon_0}$. We want to make the term into embraces positive (we take b in $]4,5[$ as in the proof of Theorem 3). So, we want to have:

$$\log(4a)<-\frac{b}{b-1}+\log(1-4\epsilon_0)+\frac{1}{b-1}\log\big(\frac{1-4\epsilon_0}{2+4\epsilon_0}\big),$$

i.e.

$$a<\frac{1}{4e}(1-4\epsilon_0)\big(\frac{1-4\epsilon_0}{2+4\epsilon_0}\big)^{1/(b-1)}e^{-1/(b-1)}$$

where the term on the right hand side is a decreasing function of ϵ_0 (for a given $b\in]4,5[$) which tends to $e^{-1-1/(b-1)}2^{-2-1/(b-1)}$, say $f(b)$. We see that f is an increasing function of b and we find that $a'=1/17$ is admissible (we look at b near to 5). Now, the right hand side of (5.4) tends to infinity when x goes to infinity. Then, we conclude that

$$b-\sum_{\substack{p|n(n+2)\\ p<x^a}} b-\sum_{\substack{p|n(n+2)\\ x^a\leq p<y}} 1>0$$

for many integers n. This inequality implies that the sum over prime factors of $n(n+2)$ not more than x^a is zero. Finally, an integer n which satisfies the previous inequality is such that $n(n+2)$ has no prime factor less than x^a, at most four prime factors in $[x^a,y[$ and at most two prime factors bigger than y. By the same argument as in the proof of (4.4), we conclude that:

$$\sum_{\substack{n\leq x\\ n\in\mathcal{Q}_{1/17}\\ \omega(n(n+2))\leq 6}} 1\gg\frac{x}{\log^2 x}.$$

In particular, there are infinitely many integers n such that $n(n+2)$ has at most six prime factors bigger than $n^{1/17}$ and no prime factors lower than $n^{1/17}$. □

Chapter 6

Appendix

6.1 A combinatorial result

In the proof of Theorem 2, we have admitted the following fact.

Lemma 9. *There is an integer k (equal to 2^6 for instance) such that $d^6(m) \leq d_k(m)$, for every integer $m \geq 1$.*

Proof. It suffices to show the result on the power of prime numbers because the function d_k is multiplicative (for this, we note that when $(m,n) = 1$ then all divisor c of mn take the form bc, where a is a divisor of m, b is a divisor of n and this decomposition is unique). Then, we want to find an integer k such that $d^6(p^r) \leq d_k(p^r)$, that is to say $(r+1)^6 \leq d_k(p^r)$, for every prime number p and every integer $r \geq 1$. But $d_k(p^r)$ is the coefficient of z^r in the series' expansion of $1/(1-z)^k$. The derivation rule gives us $d_k(p^r) = \binom{r+k-1}{k-1}$. The case $r = 1$ gives a good idea of the candidate for k. Indeed, $d^6(p) = 2^6$ and $d_k(p) = k$ so $k \geq 2^6$ necessarily and we prove that $k = 2^6$ is admitted. For this, we can say that $(r+1)^6 \leq (r+1)\dots(r+6)$ and we see that $(2^6-1)! \leq (r+2^6-1)\dots(r+7)$ when $r \geq 3$. Finally, we check that $(r+1)^6 \leq \binom{r+2^6-1}{2^6-1}$ for $r = 2$ and the lemma follows readily. □

6.2 Some useful estimates

We give below two estimates that we used in each of the proofs of the Theorems and that can be found in [3].

Lemma 10. *For every $t > 0$,*

$$\sum_{n<t} \frac{1}{n} = \log t + \gamma + \mathcal{O}(t^{-1/3}).$$

Proof. Let t be a real number greater than 1. We study the sums $\sum_{n\le t} 1/n$ since $0 \le \sum_{n\le t} 1/n - \sum_{n<t} 1/n \le 1/n$. Then,

$$\begin{aligned}\sum_{n\le t}\frac{1}{n} = \sum_{n\le t}\Big(\int_n^t \frac{du}{u^2} + \frac{1}{t}\Big) &= \int_1^t \Big[\sum_{n\le u} 1\Big]\frac{du}{u^2} + 1 - \frac{\{t\}}{t}\\ &= \int_1^t (u - \{u\})\frac{du}{u^2} + 1 - \frac{\{t\}}{t}\\ &= \log t + \gamma + \mathcal{O}(1/t) = \log t + \gamma + \mathcal{O}(t^{-1/3})\end{aligned}$$

since $\gamma = 1 - \int_1^\infty \{u\}du/u^2$ and $\int_t^\infty \{u\}du/u^2 \ll 1/t$. When $t \le 1$, the result follows to the fact that $t^{1/3}\log t$ goes to 0 when t goes to 0. □

Lemma 11. *For every $t > 0$,*

$$\sum_{p\le t}\frac{1}{p-1} = \log(1+\log t) + A_0 + \mathcal{O}(\frac{1}{1+\log t})$$

where A_0 is a numerical constant.

Proof. We use Mertens Theorem which says that, for every $u \ge 2$,

$$\sum_{p\le u}\frac{\log p}{p} = \log u + \mathcal{O}(1).$$

Let $t > 0$. If we denote by R the bounded function defined by:
$R(u) = \sum_{p\le u} \log p/p - \log u$, we have:

$$\begin{aligned}\sum_{p\le t}\frac{1}{p} = \sum_{p\le t}\frac{\log p}{p}\frac{1}{\log p} &= \sum_{p\le t}\frac{\log p}{p}\Big(\int_p^t \frac{du}{u(\log u)^2} + \frac{1}{\log t}\Big)\\ &= \sum_{p\le t}\frac{\log p}{p}\int_p^t \frac{du}{u(\log u)^2} + \frac{1}{\log t}\sum_{p\le t}\frac{\log p}{p}\\ &= \int_2^t \Big[\sum_{p\le u}\frac{\log p}{p}\Big]\frac{du}{u(\log u)^2} + \frac{1}{\log t}\sum_{p\le t}\frac{\log p}{p}\\ &= \int_2^t \frac{du}{u\log u} + \int_2^t \frac{R(u)}{u(\log u)^2}du + 1 + \frac{R(t)}{\log t}\\ &= \log\log t + c + \mathcal{O}(\frac{1}{1+\log t})\end{aligned}$$

because $\int_t^\infty R(u)/(u\log^2 u)du = \mathcal{O}(1/\log t)$ and where

$$c = \int_2^{+\infty} R(u)\frac{du}{u(\log u)^2} + 1 - \log\log 2 \approx 0,26.$$

We end the proof by writing:

$$\sum_{p\le t}\frac{1}{p-1}=\sum_{p\le t}\frac{1}{p}+\sum_{p\ge 2}\frac{1}{p(p-1)}+\mathcal{O}(\frac{1}{1+\log t})$$

which gives us the lemma with the constant $A_0 = c + \sum_{p\ge 2}\frac{1}{p(p-1)}$. □

6.3 On the weights λ_ν

In the proofs of our Theorems, we used the following result:

Lemma 12. *In the case where $\zeta_r = \zeta_1$ when $r < z$ is squarefree and $\zeta_r = 0$ otherwise, the weights λ_ν are bounded in absolute value by $|\lambda_1|$.*

Proof. We recall that

$$\lambda_\nu = \mu(\nu) f(\nu) \sum_{r<z/\nu}\frac{\mu^2(r\nu)}{f_1(r\nu)}\zeta_{r\nu}.$$

Let ν be a squarefree integer. With the hypothesis, we get:

$$\begin{aligned}\lambda_\nu &= \mu(\nu) f(\nu)\Big[\sum_{r<z/\nu}\frac{\mu^2(r\nu)}{f_1(r\nu)}\Big]\zeta_1\\ &= \frac{\mu(\nu)f(\nu)}{f_1(\nu)}\frac{G_\nu(z)}{G_1(z)}\lambda_1\end{aligned}$$

since $\zeta_1 = \lambda_1/G_1(z)$ and where

$$G_t(z) = \sum_{\substack{r<z/t\\(r,t)=1}}\frac{\mu^2(r)}{f_1(r)}$$

Now,

$$\begin{aligned}G_1(z) &= \sum_{\delta|\nu}\sum_{\substack{s<z\\(s,\nu)=\delta}}\frac{\mu^2(s)}{f_1(s)} = \sum_{\delta|\nu}\frac{\mu^2(\delta)}{f_1(\delta)}\sum_{\substack{r<z/\nu\\(r,\nu)=1}}\frac{\mu^2(r)}{f_1(r)}\\ &\ge \Big[\sum_{\delta|\nu}\frac{\mu^2(\delta)}{f_1(\delta)}\Big]G_\nu(z)\end{aligned}$$

since $\delta \le \nu$. Since the function μ^2/f_1 is multiplicative, we have:

$$\sum_{\delta|\nu}\frac{\mu^2(\delta)}{f_1(\delta)} = \prod_{p|\nu}(1+\frac{1}{f_1(p)}) = \frac{f(\nu)}{f_1(\nu)}.$$

Eventually

$$|\lambda_\nu| \le |\lambda_1|$$

and the lemma follows readily. □

6.4 On the function ϕ

In order to control the remainder term in the proof of Theorem 2, we claim that $m/\phi(m) \ll \log m$. In fact,

Lemma 13. *We have:*

$$m/\phi(m) \ll \log\log m.$$

Proof. We expand the multiplicative function Id/ϕ as a product:

$$\frac{m}{\phi(m)} = \prod_{p|m} \frac{1}{1-1/p}.$$

Let P be a parameter that we will choose later. We get:

$$\begin{aligned}
\log(\frac{m}{\phi(m)}) &= -\sum_{p|m} \log(1-\frac{1}{p}) = \sum_{p|m} \frac{1}{p} + \mathcal{O}(\sum_{p|m} \frac{1}{p}) \\
&= \sum_{\substack{p|m \\ p\le P}} \frac{1}{p} + \sum_{\substack{p|m \\ p>P}} \frac{1}{p} + \mathcal{O}(1) \\
&\le \log\log P + \frac{\log m}{P\log P} + \mathcal{O}(1)
\end{aligned}$$

according to Lemma 11. The choice $P = \log m$ gives us:

$$\log(\frac{m}{\phi(m)}) \le \log\log\log m + \mathcal{O}(1)$$

and we easily conclude. $\square$

6.5 On the function d

In the proof of Theorem 3 we use the following estimation of the function d.

Lemma 14. *For every $\epsilon > 0$, $d(n) \ll_\epsilon n^\epsilon$.*

Proof. Let n be an integer. It is well-known that $d(n) = \prod_{p^\alpha \| n}(\alpha+1)$. Thus, for a parameter P that we will choose later, we have:

$$\log d(n) = \sum_{\substack{p^\alpha \| n \\ p\le P}} \log(\alpha+1) + \sum_{\substack{p^\alpha \| n \\ p>P}} \log(\alpha+1).$$

Now,

$$\sum_{\substack{p^\alpha \| n \\ p>P}} \log(\alpha+1) \le \sum_{\substack{p^\alpha \| n \\ p>P}} \alpha \le \frac{\log n}{\log P}$$

and

$$\sum_{\substack{p^\alpha \| n \\ p \le P}} \log(\alpha+1) \le \log\big(1+\frac{\log n}{\log 2}\big)\sum_{p\le P} 1 \ll \log\log n \frac{P}{\log P}.$$

We take $P = \log n/\log\log n$ and we get:

$$\log d(n) \le C\frac{\log n}{\log\log n}$$

for some constant $C > 0$. Finally,

$$d(n) \le n^{C/\log\log n}$$

and the lemma follows readily. □

Notations

Notations used troughout these notes are the following:

$\longrightarrow$ The use of the letters p and p' always refer to prime numbers.

$\longrightarrow$ $[m, n]$ stands for the lcm and (m, n) for the gcd of n and m.

$\longrightarrow$ $|\mathcal{A}|$ or $\#\mathcal{A}$ stands for the cardinality of the set $\mathcal{A}$.

$\longrightarrow$ $\omega(d)$ is the number of prime factors of d, counted without multiplicity.

$\longrightarrow$ $\omega_2(d)$ is the number of odd prime factors of d, counted without multiplicity.

$\longrightarrow$ $\phi(d)$ is the Euler totient function.

$\longrightarrow$ $\mu(d)$ is the Möbius function, that is 0 when d is divisible by a square > 1 and otherwise $(-1)^r$, where r is the number of prime factors of d.

$\longrightarrow$ $d(n)$ is the number of divisors of n and $d_k(n)$ is the number of representations of n as $a_1 \ldots a_k = n$ where $a_1, \ldots, a_k$ are positive integers.

$\longrightarrow$ The notation of Vinogradov $f \ll g$ means that $|f(t)| \leq cg(t)$ for some constant c independant of the variable t.

$\longrightarrow$ The notation $f = \mathcal{O}_A(g)$ means that there exists a constant B such that $|f| \leq Bg$ but that this constant may depend on A.

$\longrightarrow$ The notation $f \ll_A g$ means that $f \ll g$ with a constant c which may depend on A.

$\longrightarrow$ The notation $f \star g$ denotes the arithmetical convolution of f and g, that is to say the function h on positive integers such that $h(d) = \sum_{q|d} f(q)g(d/q)$.

$\longrightarrow$ The integer part of the real number x is defined by $\lfloor x \rfloor$.

$\longrightarrow$ The notation $a \equiv b[q]$ means that q divides $a - b$, for any three integers a, b and q.

$\longrightarrow$ The logarithmic integral function Li is defined by $\mathrm{Li}(x) = \int_0^x dt/\log t$ and satisfies $\mathrm{Li}(x) \ll x/\log x$.

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