Introduction to cosmology - Part 2

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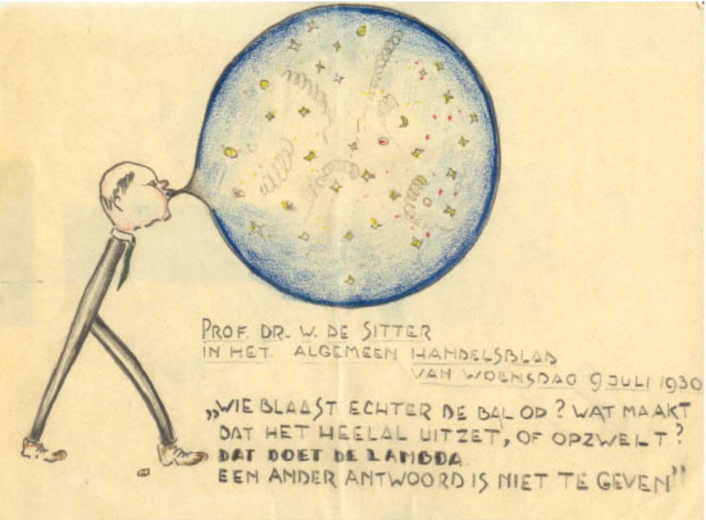
Estate Quantistica 2024

Outline

The cosmological constant



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Problems with Λ
Cosmic coincidence

Consider:

$$\frac{\rho_{\rm m}}{\rho_\Lambda} = \frac{\Omega_{\rm m0}}{\Omega_\Lambda}(1+z)^3 \,. \qquad (1)$$

At $z = 10^{32}$ if the above ratio was 10^{97} or 10^{95}, we would not observe the present acceleration of the expansion or structures would not have formed. The ratio $\rho_{\rm m}/\rho_\Lambda$ should be set at the Planck scale with a precision of 96 decimal places.[1]

Seen the other way around, as a **coincidence problem**, one might ask why the densities of matter and of the cosmological constant are of the same order at present time (Zlatev, 1998).

Then one "tries" and abandons Λ for a dynamical DE.

[1] A similar reasoning, however, applies to any ratio of different densities.

Example of dynamical DE
Quintessence

The simplest and most common way for a dynamical DE is a canonical scalar field φ:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \,, \tag{2}$$

from which one can define density and pressure on a FLRW background as:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \,, \quad P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \,. \tag{3}$$

Such simple model is also the basis for inflationary scenarios.

However, the coincidence is still there somehow “under the carpet” (hidden in the potential).

Origins of Λ

Introduced by Einstein (1917) in order to find a solution with vanishing inertia at infinity (Einstein Static Universe).

The introduction of the cosmological constant by Einstein is frequently reported to be judged by himself as "the biggest blunder of my life". This seems to be a personal comment made by Einstein to George Gamow and reported by the latter in a 1956 article on the *Scientific American*.

The dissatisfaction of Einstein about Λ can be read in a 1931 paper by Einstein himself, when he writes:
"Under these circumstances one should ask whether the observational facts can be accounted for without the inclusion of the theoretically, in all respect unsatisfactory λ-term."

ORIGINS OF Λ

Λ "revives" today as the most successful model for the accelerated expansion of the universe.

It is somehow special. There is a sort of "inevitability" of Λ:
[*S. Weinberg, Photons and gravitons in perturbation theory: Derivation of Maxwell's and Einstein's equations, Phys. Rev.* **138** *(1965), B988-B1002*], [*D. Lovelock, The Einstein Tensor and Its Generalizations, Journal of Mathematical Physics (1971) 12 (3) 498*]

$$S = \frac{c^4}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{-g} \left(R - 2\Lambda\right) + S_{\mathrm{matter}}\left[g_{\mu\nu}, \Psi\right] \ . \tag{4}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} \ . \tag{5}$$

Λ, from a mathematical viewpoint

Lovelock's theorems:

- *D. Lovelock, The Einstein Tensor and Its Generalizations, Journal of Mathematical Physics (1971) 12 (3) 498501*;
- *D. Lovelock, The Four-Dimensionality of Space and the Einstein Tensor, Journal of Mathematical Physics (1972) 13 (6) 874876*.

Given field equations in vacuum:

$$A^{\mu\nu} = 0 \,, \tag{6}$$

and the following hypothesis:

Lovelock's theorem

Hypothesis and thesis

1. $A^{\mu\nu} = A^{\nu\mu}$ (symmetry)
2. $A^{\mu\nu} = A^{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma})$
3. $\nabla_\mu A^{\mu\nu} = 0$ (divergencelessness, ∇_μ is the covariant derivative)
4. $A^{\mu\nu}$ is linear in the second derivative of the metric.

Then:

$$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \,, \tag{7}$$

where a and b are arbitrary constants and:

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \,, \tag{8}$$

is the Einstein tensor.

Resurrection of Λ

Cosmology

Friedmann-Lemaître-Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \ , \tag{9}$$

Friedmann equations with Λ:

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3}\rho_{\rm tot} + \frac{\Lambda}{3} - \frac{K}{a^2} \ , \tag{10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho_{\rm tot} + 3p_{\rm tot}) + \frac{\Lambda}{3} \ . \tag{11}$$

A positive Λ works as antigravity. It also can be seen as a perfect fluid with equation of state:

$$w_\Lambda \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1 \ . \tag{12}$$

Accelerated expansion of the universe

Type Ia supernovae are standard candles which allowed to extend the cosmic distance ladder to large redshifts ($z \sim 1$) and from which the accelerated expansion of the universe was discovered.

- *A. G. Riess et al. [Supernova Search Team], Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant , Astron. J.* **116** *(1998) 1009 [astro-ph/9805201]*
- *S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Measurements of Omega and Lambda from 42 High-Redshift Supernovae , Astrophys. J.* **517** *(1999) 565 [astro-ph/9812133]*

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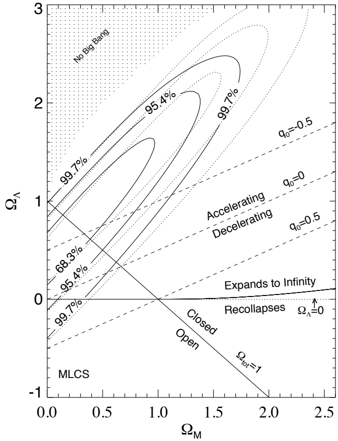
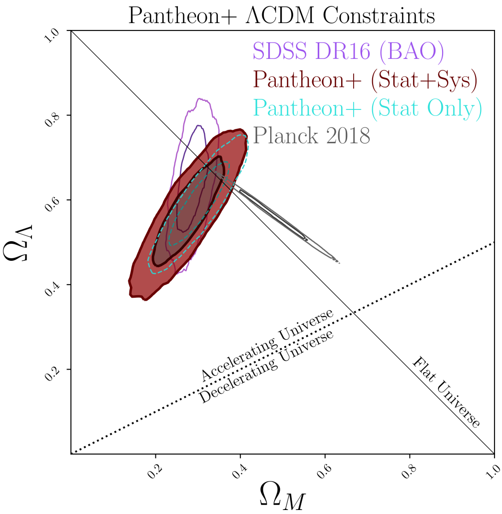


Recent Supernovae type Ia data

D. Brout, D. Scolnic, B. Popovic, A. G. Riess, J. Zuntz, R. Kessler, A. Carr, T. M. Davis, S. Hinton and D. Jones, et al., The Pantheon+ Analysis: Cosmological Constraints,[arXiv:2202.04077 [astro-ph.CO]].



Λ from the large-scale structure

It was already clear before type Ia SN that a pure CDM model was incomplete and that (perhaps) Λ was necessary.

From Efstathiou, Sutherland and Maddox (1990):

The cold dark matter (CDM) model for the formation and distribution of galaxies in a universe with exactly the critical density is theoretically appealing and has proved to be durable, but recent work suggests that there is more cosmological structure on very large scales... We argue here that the successes of the CDM theory can be retained and the new observations accommodated in a spatially flat cosmology in which as much as 80% of the critical density is provided by a positive cosmological constant, which is dynamically equivalent to endowing the vacuum with a non-zero energy density...

Λ and cosmology

If Λ is the cause of the accelerated expansion, data require:

$$\rho_\Lambda \sim \Omega_\Lambda \rho_{\rm cr} \sim 10^{-47}\ \mathrm{GeV}^4 \sim 10^{-52}\ \mathrm{m}^{-2}\ . \tag{13}$$

What is the problem with Λ? None, if you avoid to frame it within particle physics.

If you do, some questions arise:

- Huge discrepancy with the predictions coming from quantum field theory (old cosmological constant problem);
- Why ρ_Λ has the above tiny value? (new cosmological constant problem).

The first question was raised by Zel'dovich in the framework of Sakharov's induced gravity.[2]

[2] Y. B. Zel'dovich, JETP letters 6 (1967), 316-317; A. D. Sakharov, Dokl. Akad. Nauk SSSR (1967) 177, 70-71

Λ AS THE WEIGHT OF VACUUM

Quantum vacuum:

- The Lamb shift (*W. E. Lamb, R. C. Retherford, Physical Review. 72 (1947) (3): 241243*);
- The Casimir effect (*H. B. G. Casimir, D. Polder, Physical Review. 73 (1948) (4): 360372*),

(see also *R. L. Jaffe, Casimir effect and the quantum vacuum, PRD 72, 021301(R) (2005)*)

Our concern is however to understand whether and how vacuum energy gravitates.

And if Λ can indeed be interpreted as vacuum energy.

Standard argument leading to the problem

In Minkowski space we have that

$$\langle T_{\mu\nu} \rangle \propto \eta_{\mu\nu} \ . \tag{14}$$

Hence by the equivalence principle, in curved space one has:

$$\langle T_{\mu\nu} \rangle = -\rho_{\rm vac} g_{\mu\nu} \ . \tag{15}$$

The vacuum energy density becomes the cosmological constant.

Calculation of $\rho_{\rm vac}$
Scalar field on flat space

For a massive non-interacting scalar field on flat space:

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left(a_k e^{ik^\mu x_\mu} + a_k^\dagger e^{-ik^\mu x_\mu}\right) , \qquad (16)$$

with

$$\omega_k \equiv k^0 = \sqrt{k^2 + m^2} . \qquad (17)$$

The vacuum expectation value of the energy-momentum tensor is:

$$\langle T_{\mu\nu} \rangle = \int \frac{d^3k}{(2\pi)^3 2k^0} k_\mu k_\nu . \qquad (18)$$

CALCULATION OF $\rho_{\rm vac}$

PUTTING A CUTOFF

Considering a UV cutoff M:

$$\langle\rho\rangle = \frac{1}{4\pi^2}\int_0^M dk k^2\sqrt{k^2+m^2} =$$

$$\frac{M^4}{16\pi^2}\left[\sqrt{1+\frac{m^2}{M^2}}\left(1+\frac{m^2}{2M^2}\right) - \frac{m^4}{2M^4}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)\right]$$

$$= \frac{M^4}{16\pi^2}\left(1+\frac{m^2}{M^2}+\ldots\right). \quad (19)$$

The standard argument goes as: M = Planck mass, so $\rho_{\rm vac} \sim 10^{76}$ GeV^4. On the other hand, $\rho_\Lambda \sim 10^{-47}$ GeV^4. So we have a discrepancy of 123 orders of magnitude!

Note that this problem would remain in dynamical DE models.

Induced Gravity

The previous conclusion is based on an incomplete discussion of the problem: the renormalization procedure is missing.

On the other hand, it can be taken as good in the framework of *induced gravity*:

$$e^{iS_{\rm ind}[g]} = \int \mathcal{D}\Phi e^{iS_{\rm m}[\Phi,g]} . \tag{20}$$

To lowest order:

$$S_{\rm ind}[g] = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa_{\rm ind}}R - \rho_{\Lambda,{\rm ind}} + \dots\right) . \tag{21}$$

From here one can show:

$$\frac{1}{2\kappa_{\rm ind}} \sim M^2 , \quad \rho_{\Lambda,{\rm ind}} \sim M^4 , \tag{22}$$

implying $\rho_{\Lambda,{\rm ind}} \sim M_{\rm Pl}^4$.

The problem when using a UV cutoff

For the pressure we can compute:

$$\langle p \rangle = \frac{1}{3}\frac{1}{4\pi^2}\int_0^M dk \frac{k^4}{\sqrt{k^2+m^2}} =$$
$$\frac{1}{3}\frac{M^4}{16\pi^2}\left[\sqrt{1+\frac{m^2}{M^2}}\left(1-\frac{3m^2}{2M^2}\right)+\frac{3m^4}{2M^4}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)\right]$$
$$= \frac{1}{3}\frac{M^4}{16\pi^2}\left(1-\frac{m^2}{M^2}+\dots\right) . \quad (23)$$

So, at the leading order $\langle p \rangle = \langle \rho \rangle/3$, as radiation does. Indeed, putting a cutoff spoils Lorentz invariance.

The logarithmic terms instead give the expected behaviour for vacuum: $\langle p \rangle = -\langle \rho \rangle$.

Recovering the vacuum equation of state

Dimensional regularisation

Using dimensional regularisation one gets:

$$\langle\rho\rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}}\frac{1}{2}\int_0^\infty dk k^{d-2} d^{d-2}\Omega\omega_k$$
$$= \frac{\mu^4}{2(4\pi)^{d-1}}\frac{\Gamma(-d/2)}{\Gamma(-1/2)}\left(\frac{m}{\mu}\right)^d , \qquad (24)$$

with μ an arbitrary scale. Similarly

$$\langle p\rangle = \frac{\mu^4}{4(4\pi)^{d-1}}\frac{\Gamma(-d/2)}{\Gamma(1/2)}\left(\frac{m}{\mu}\right)^d , \qquad (25)$$

Now $\langle p\rangle = -\langle\rho\rangle$ as expected.

Extract a finite result

Considering $d = 4 - \epsilon$ one can easily investigate the pole structure of the Gamma function and see that:

$$\langle \rho \rangle = -\frac{m^4}{64\pi^2}\left[\frac{2}{\epsilon} + \frac{3}{2} - \gamma - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] + \dots \,. \qquad (26)$$

By eliminating the divergent term one has:

$$\langle \rho \rangle = \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) \,. \qquad (27)$$

In general one can show the same result for any free field, provided a minus sign for the fermionic ones. Hence:

$$\langle \rho_{\rm tot} \rangle = \frac{1}{64\pi^2}\sum_i (-1)^{2S_i} g_i m_i^4 \ln\left(\frac{m_i^2}{\mu^2}\right) \,. \qquad (28)$$

Pauli Sum Rules

Pauli already observed in 1951 (ETH lectures) that even using a UV cutoff, no weight for vacuum is obtained if the following conditions are met:

$$\sum_n (-1)^{2S_n} g_n = 0 \, , \sum_n (-1)^{2S_n} g_n m_n^2 = 0 \, , \sum_n (-1)^{2S_n} g_n m_n^4 = 0 \, . \tag{29}$$

Visser shows how these conditions provide a bridge between the finiteness of the zero-point energy and Lorentz invariance. He also speculates on the consequences of taking these relations to be valid non-perturbatively, leading to the necessity of physics beyond the standard model (*M. Visser, Phys. Lett. B* **791** *(2019) 43 [arXiv:1808.04583 [hep-th]]*).

A more complete argument
Semiclassical gravity

In semiclassical gravity, quantum fields are considered on a dynamical, but classical, geometry. Quantum effects have a backreaction on the latter.

For the generating functional of the Green functions:

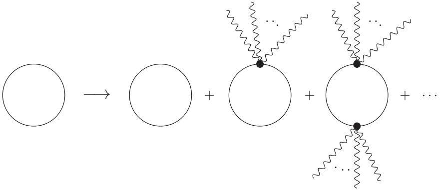
$$Z[J,g] = \mathcal{N} e^{iS_{\rm vac}[g]} \int \mathcal{D}\Phi e^{iS_{\rm m}[\Phi,g]+i\Phi J} \,. \tag{30}$$

The metric is a classical external field. For $S_{\rm vac}[g] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R$:

$$G_{\mu\nu} = \kappa\langle T_{\mu\nu}\rangle \,. \tag{31}$$

Semiclassical gravity
Divergences and renormalization

On flat space bubble diagrams (vacuum-vacuum amplitudes) are irrelevant. In the curved case, however, they couple to the external graviton ($g = \eta + h$).



Adding vertices with external gravitons reduces the superficial degree of divergence (quartic, quadratic, logarithmic). We have then qualitatively new divergences which must be compensated by *geometric* counterterms in the vacuum action.

Semiclassical gravity
Geometric counterterms

Since h is dimensionsless, in order to preserve the dimension of the diagram we must use derivatives of h and the mass of the field in the bubble.

$$S_{\mathrm{vac}}[g] \leftarrow 2\Lambda + \alpha_1 Rmn^2 + \alpha_2 Rc^2 + \alpha_3 R^2 + \alpha_4 \Box R \,. \qquad (32)$$

Radiative corrections are then provided for the cosmological constant (and also Newton's constant). These come only from massive fields and are $\propto m^2$ or m^4.

The latter can be made compatible with observation by using a suitable renormalization condition. So, no actual problem from this side.

Cosmological constant induced by phase transitions

Consider a simple example:

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi + V(\Phi) \right] . \tag{33}$$

If the field rolls down to a minimum of its potential:

$$\langle T_{\mu\nu} \rangle = -V(\Phi_{\min}) g_{\mu\nu} . \tag{34}$$

Then, we have a cosmological constant behaviour.

Electroweak phase transition

After the electroweak phase transition we have ($\lambda \simeq 0.1$):

$$V(H) = -\frac{\lambda v^4}{4} + \frac{1}{2}\lambda v^2 H^2 + \frac{\lambda}{2}\frac{v}{\sqrt{2}}H^3 + \frac{\lambda}{16}H^4 \,, \qquad (35)$$

with $m_H^2 = \lambda v^2$ being the Higgs mass and $v = \langle H \rangle$. Then:

$$\rho_{\rm ind} = -\frac{1}{4}m_H^2 v^2 \,, \qquad v^2 = \frac{\sqrt{2}}{4G_F^2} \,, \qquad (36)$$

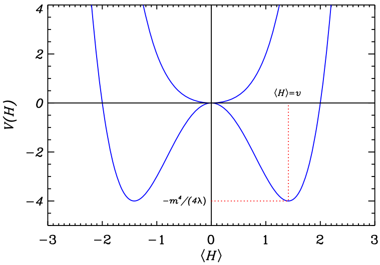
lead to:

$$\rho_{\rm ind} = -\frac{\sqrt{2}}{16}\frac{m_H^2}{G_F^2} \approx -1.2 \times 10^8 \ {\rm GeV}^4 \,. \qquad (37)$$

Here $G_F \simeq 1.16 \times 10^{-5}$ GeV^{-2} is Fermi's constant and $m_H \approx 125$ GeV.

Electroweak phase transition

Higgs Potential (plots taken from Martin's review)



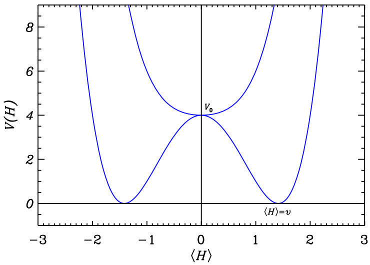


Fig. 2. Effective potential of the Higgs boson before and after the electroweak phase transition. The left panel corresponds to a situation where the vacuum energy vanishes at high temperature. As a consequence ρ_{vac} is negative at temperature smaller than the critical temperature. This is the situation treated in the text where the quantity $-m^4/(4\lambda)$ is explicitly calculated. On the right panel, the off-set parameter V_0 is chosen such that the vacuum energy is zero after the transition. As a consequence, it does not vanish at high temperatures.

The cosmological constant problem

Then:

$$\rho_{\Lambda,\mathrm{obs}} = \rho_{\Lambda,\mathrm{vac}} + \rho_{\Lambda,\mathrm{ind}} \;, \tag{38}$$

with $\rho_{\Lambda,\mathrm{ind}} \sim 10^8\ \mathrm{GeV}^4 \gg \rho_{\Lambda,\mathrm{obs}} \sim 10^{-47}\ \mathrm{GeV}^4$.

The problem is then: the renormalization condition on $\rho_{\Lambda,\mathrm{vac}}$ has to be chosen with a precision of 56 significant digits. This suggests a sort of hierarchy problem between the electroweak scale and the Hubble scale H_0.

Note that the same problem does not happen for G, because $G^{-1}_{\mathrm{obs}} \gg G^{-1}_{\mathrm{ind}}$, so $G^{-1}_{\mathrm{obs}} \sim G^{-1}_{\mathrm{vac}} \sim M^2_{\mathrm{Pl}}$.

Cosmological perturbations

Small fluctuations

Observation of CMB reveals that the early universe (z larger than 1100) was close to homogeneity and isotropy, with relative deviations of order 10^{-5}.

Since these are so small, we can treat them as **small** perturbations around a perfectly homogeneous and isotropic FLRW background.

As for the late universe, a perturbative approach would allow us to understand the evolution of the universe on very large scales, but not to fully describe how structures form. This ultimately needs powerful machines and numerical simulations.

Perturbed metric

Let:

$$\bar{ds}^2 = \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = a^2(\eta)(-d\eta^2 + \delta_{ij} dx^i dx^j) \ . \qquad (39)$$

Define:

$$\delta g_{\mu\nu}(x(\bar{x})) = g_{\mu\nu}(x(\bar{x})) - \bar{g}_{\mu\nu}(\bar{x}) \ . \qquad (40)$$

The components of the full metric $g_{\mu\nu}$ still are functions of the background coordinates $\bar{x}$. The choice of $x(\bar{x})$ is arbitrary and establishes a **gauge**:

$$g_{\mu\nu} = a^2(\eta) \left\{ \begin{array}{cc} -[1 + 2\psi(\eta, \mathbf{x})] & w_i(\eta, \mathbf{x}) \\ w_i(\eta, \mathbf{x}) & \delta_{ij}[1 + 2\phi(\eta, \mathbf{x})] + \chi_{ij}(\eta, \mathbf{x}) \end{array} \right\} . \qquad (41)$$

Gauge transformation of perturbations

In a system of coordinate x we define a *perturbation* of Q as:

$$\delta Q(x) = Q(x) - \bar{Q}(x)\,, \tag{42}$$

where $\bar{Q}(x)$ is the *background counterpart* of Q. The crucial point is that $\bar{Q}$ *is not a geometric quantity, but a fixed function of the coordinates*. This makes the above splitting *not covariant*, and so also δQ *is not a geometric quantity*.
Upon a change of coordinates $x \to \hat{x}$, $Q(x)$ changes to $\hat{Q}(\hat{x})$ according to its tensorial properties, but $\bar{Q}(x)$ simply turns into $\bar{Q}(\hat{x})$. So, the perturbation changes as:

$$\hat{\delta} Q(\hat{x}) = \hat{Q}(\hat{x}) - \bar{Q}(\hat{x})\,. \tag{43}$$

The **gauge transformation** is the change in the functional form of δQ.

Gauge transformation induced by coordinates transformation

The change in the functional form of δQ can be made explicit if we consider a coordinate transformation:

$$x \to \hat{x} = x + \xi(x)\,, \tag{44}$$

where ξ is considered as small as δQ, in order to preserve the linear order of the perturbations. Then:

$$\hat{\delta} Q(x+\xi) - \delta Q(x) = \hat{Q}(x+\xi) - Q(x)\,. \tag{45}$$

Since δQ and ξ are small, $\hat{\delta} Q(x+\xi) = \hat{\delta} Q(x)$. Therefore:

$$\hat{\delta} Q(x) - \delta Q(x) = \mathcal{L}_\xi Q(x)\,, \tag{46}$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ.

The problem of the gauge

Since the choice of the gauge is arbitrary, we might find one for which:

$$g_{\mu\nu}(x(\bar{x})) = \bar{g}_{\mu\nu}(\bar{x})\,, \tag{47}$$

and then conclude that there are no perturbations, even if g is a different metric. Conversely, we might have $g = \bar{g}$ and choosing a gauge such that:

$$\bar{g}_{\mu\nu}(x(\bar{x})) \neq \bar{g}_{\mu\nu}(\bar{x})\,, \tag{48}$$

concluding that there are perturbations, even if there are none. The problem of the gauge is the very dependence of perturbations on the gauge, which does not allow to define them unambiguously. This issue is overcome by using **gauge-invariant variables** (Bardeen, 1980).

The Scalar-Vector-Tensor decomposition

We can write w_i as follows:

$$\boxed{w_i = \partial_i w + S_i} \tag{49}$$

Here, w is the **scalar** part of w_i and S_i (which is divergenceless) is the **vector** part of w_i.
We can write χ_{ij} in the following form:

$$\boxed{\chi_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) 2\mu + \partial_j A_i + \partial_i A_j + \chi_{ij}^T} \tag{50}$$

with A_i divergenceless. The transverse part χ_{ij}^T cannot be decomposed in any scalar or divergenceless vector. It constitutes a **tensor perturbation**.

Scalar perturbations and their gauge-invariant combinations

The **Bardeen's potentials**:

$$\boxed{\Psi = \psi + \frac{1}{a}\left[\left(w - \mu'\right) a\right]'} \quad \boxed{\Phi = \phi + \mathcal{H}\left(w - \mu'\right) - \frac{1}{3}\nabla^2 \mu} \quad (51)$$

The **comoving curvature perturbation**:

$$\boxed{\mathcal{R} \equiv \phi + \mathcal{H} v - \frac{1}{3}\nabla^2 \mu} \quad \boxed{\zeta \equiv \phi + \frac{\delta\rho}{3(\rho + P)} - \frac{1}{3}\nabla^2 \mu} \quad (52)$$

Here $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta}$ (the conformal Hubble factor); v is the (total) velocity perturbation and $\delta\rho$ the (total) density perturbation.

Linearised Einstein equations for scalar perturbations in the Newtonian gauge

Relativistic Poisson equation

Newtonian gauge: choose $w = \mu = 0$. The perturbed metric is written as:

$$g_{00} = -a^2(1 + 2\Psi) \,, \quad g_{0i} = 0 \,, \quad g_{ij} = a^2\delta_{ij}(1 + 2\Phi) \,, \qquad (53)$$

Relativistic Poisson equation (00 linearised Einstein equation):

$$\boxed{3\mathcal{H}\Phi' - 3\mathcal{H}^2\Psi + k^2\Phi = 4\pi G a^2 \left(\rho_{\rm c}\delta_{\rm c} + \rho_{\rm b}\delta_{\rm b} + \rho_\gamma\delta_\gamma + \rho_\nu\delta_\nu\right)} \qquad (54)$$

The model considered here is Λ + CDM (c) + baryons (b) + photons (ν) + neutrino (ν). $\delta_x = \delta\rho_x/\rho_x$ is the **density contrast**.

Equation for the anisotropic stress

Spatial traceless part of the field equations:

$$\boxed{k^2(\Phi + \Psi) = 12\pi G a^2 \hat{k}_i \hat{k}^j \pi^i{}_j} \tag{55}$$

This equation tells us that $\Phi = -\Psi$, unless a quadrupole moment of the energy content distribution is present.

For example, when CDM dominated the universe then $\Phi = -\Psi$ but this is not the case in the early universe, because of neutrinos.

Even when CDM or DE dominates but the underlying theory of gravity is not GR one might have $\Phi \neq -\Psi$. One can probe the value of $\Phi + \Psi$ via weak lensing.

Equation for tensor perturbations

For:

$$g_{00} = -a^2 \,, \qquad g_{0i} = 0 \,, \qquad g_{ij} = a^2(\delta_{ij} + h_{ij}^T) \,, \tag{56}$$

one obtains:

$$\boxed{h_{ij}^{T''} + 2\mathcal{H} h_{ij}^{T'} + k^2 h_{ij}^T = 16\pi G a^2 \pi_{ij}^T} \tag{57}$$

where π_{ij}^T is the tensorial part of the anisotropic stress.

These are gravitational waves in the expanding universe.

Equation for vector perturbations

For:

$$g_{00} = -a^2 \,, \qquad g_{0i} = 0 \,, \qquad g_{ij} = a^2(\delta_{ij} + h^V_{ij}) \,, \tag{58}$$

with:

$$h^V_{ij} = \partial_i A_j + \partial_j A_i \,, \qquad \partial_i A^i = 0 \,. \tag{59}$$

One has:

$$h^{V''}_{ij} + 2\mathcal{H} h^{V'}_{ij} = 0 \,. \tag{60}$$

With the Laplacian missing, the last equation is no more a wave equation. With no vector sources, in the early, radiation-dominated universe, for which $\mathcal{H} = 1/\eta$, one has:

$$h^V_{ij} \propto 1/\eta^2 \,, \tag{61}$$

and hence vector perturbations vanish.

Equations for the various matter components

The linearized Einstein's equations are alone not enough for completely describing the evolution of the perturbative quantities.

To those one adds the Boltzmann equations for the various species.

Inflation

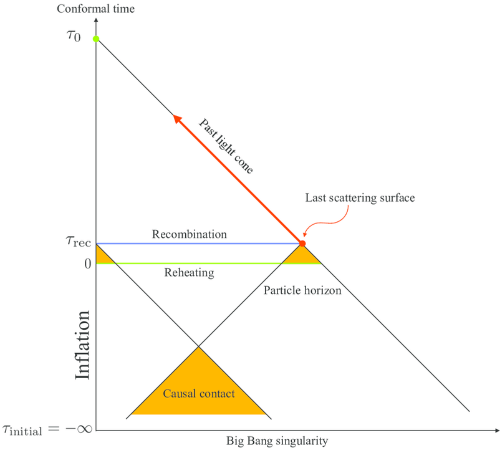
Problems in the standard model of cosmology

We have already encountered the flatness problem.

The **horizon problem** is an issue that appears when we calculate the angular size of the particle horizon at recombination and notice that it is only a small portion of the CMB sky.

How is it possible that the latter is so isotropic if no causal process could have provided the conditions to be so?

The horizon problem



Debono, Ivan and Smoot, George. (2016). General Relativity and Cosmology: Unsolved Questions and Future Directions. Universe. 2.

The horizon problem

The proper particle-horizon distance is the following:

$$d_{\rm H} = a(t) \int_0^t \frac{dt'}{a(t')} = a \int_0^a \frac{da'}{H(a')a'^2} \,, \tag{62}$$

whereas the angular diameter distance has the following form:

$$d_{\rm A} = a(t) \int_t^{t_0} \frac{dt'}{a(t')} = a \int_a^1 \frac{da'}{H(a')a'^2} \,. \tag{63}$$

The horizon problem

In an universe dominated by matter and radiation:

$$\frac{d_{\rm H}}{d_{\rm A}} = \frac{\sqrt{\Omega_{\rm m0} a + \Omega_{\rm r0}} - \sqrt{\Omega_{\rm r0}}}{\sqrt{\Omega_{\rm m0} + \Omega_{\rm r0}} - \sqrt{\Omega_{\rm m0} a + \Omega_{\rm r0}}} \, . \tag{64}$$

This ratio tends to zero for $a \to 0$ and at recombination it is equal to:

$$\frac{d_{\rm H}}{d_{\rm A}}(a_{\rm rec} = 10^{-3}) = 0.018 \, , \tag{65}$$

which corresponds to about 1° in the CMB sky.

Therefore, we have roughly $4\pi/(0.018)^2 \approx 10^4$ causally disconnected regions in the sky.

Solution
Inflation

Assume $H = H_I$ constant before the radiation-dominated epoch:

$$a(t) = a_i e^{H_I(t-t_i)} \ . \tag{66}$$

Now:

$$d_{\mathrm{H}} \sim \frac{a}{a_I H_I}(e^N - 1) \ . \tag{67}$$

Since $d_{\mathrm{A}} \approx a/H_0$ for small scale factors, we can conclude that:

$$\frac{d_{\mathrm{H}}}{d_{\mathrm{A}}} \approx \frac{H_0}{a_I H_I} e^N \ , \tag{68}$$

and so, in order to have $d_{\mathrm{H}} > d_{\mathrm{A}}$, we obtain the condition:

$$\boxed{\frac{a_I H_I}{H_0} < e^N} \tag{69}$$

N is called the number of *e-folds*.

Single scalar field slow-roll inflation

Consider a canonical scalar field:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \ . \tag{70}$$

In the background FLRW metric the energy density and pressure are:

$$\rho_\varphi = -T^0{}_0 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \ , \quad P_\varphi = \frac{1}{3}\delta^i{}_j T^j{}_i = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \ . \tag{71}$$

Moreover:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0 \ , \tag{72}$$

and:

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] \ , \qquad \frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\dot{\varphi}^2 - V(\varphi)\right] \ . \tag{73}$$

Slow-roll condition

In order for H to vary slowly:

$$\frac{|\dot{H}|}{H^2} \ll 1 \,, \tag{74}$$

Using Friedmann equation and the expression for $\dot{H}$ we can write the above condition as:

$$\boxed{\dot{\varphi}^2 \ll V(\varphi)} \tag{75}$$

which is the first **slow-roll condition**. When the kinetic term of the scalar field is negligible with respect to the potential one, one has:

$$P_\varphi \approx -\rho_\varphi \approx -V(\varphi) \approx \text{constant} \,. \tag{76}$$

That is, the scalar field potential, when it dominates over the kinetic term, behaves as a cosmological constant.

Slow-roll parameters

The condition of slow-roll is parametrized as:

$$\boxed{\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right)} \tag{77}$$

The derivative can be written as:

$$\dot{\epsilon} = 2H\epsilon(\epsilon - \eta) \ , \tag{78}$$

where

$$\boxed{\eta \equiv -\frac{1}{H}\frac{\ddot{\varphi}}{\dot{\varphi}}} \tag{79}$$

is a second slow-roll parameter. The smallness of η gives us:

$$\boxed{3H\dot{\varphi} \approx -V_{,\varphi}} \tag{80}$$

Production of primordial modes

Spectral indices

Quantum fluctuations in the inflaton field are amplified and turn classical, providing the seeds for scalar and tensor perturbations:

$$\Delta_S^2 \equiv \Delta_{\mathcal{R}}^2 \equiv \frac{k^3 P_{\mathcal{R}}(k)}{2\pi^2} = \left.\frac{H^2}{8\pi^2 M_{\rm Pl}^2 \epsilon}\right|_{k=aH} \equiv A_S \left(\frac{k}{k_*}\right)^{n_S(k)-1} , \quad (81)$$

$$\Delta_T^2 \equiv 2\Delta_h^2 \equiv \frac{k^3 P_h(k)}{\pi^2} = \left.\frac{2H^2}{\pi^2 M_{\rm Pl}^2}\right|_{k=aH} \equiv A_T \left(\frac{k}{k_*}\right)^{n_T(k)} , \quad (82)$$

where the general k-dependence (given by the specific model of inflation) is embedded in $n_S(k)$ and $n_T(k)$, which are known as **scalar spectral index** and **tensor spectral index**.

These can be measured in the CMB.

Relation to the inflationary model

One can determine:

$$\boxed{n_T = -2\epsilon} \tag{83}$$

and:

$$\boxed{n_S - 1 = -4\epsilon + 2\eta = -6\epsilon_V + 2\eta_V} \tag{84}$$

And the **tensor-to-scalar ratio**:

$$\boxed{r_* \equiv \frac{\Delta_T^2(k_*)}{\Delta_S^2(k_*)} = \frac{A_T}{A_S} = 16\epsilon = -8n_T} \tag{85}$$

The energy scale of inflation:

$$\boxed{V_* = \frac{3\pi^2 M_{\mathrm{Pl}}^4}{2} r_* A_S} \tag{86}$$

Observational results

For the scalar spectral index at 68% CL:

$$n_S = 0.9586 \pm 0.0056 \,, \quad \frac{dn_S}{d\ln k} = 0.009 \pm 0.010 \,, \tag{87}$$

$$\frac{d^2 n_S}{d(\ln k)^2} = 0.025 \pm 0.013 \,, \tag{88}$$

using the pivot scale $k_* = 0.05\ \mathrm{Mpc}^{-1}$. For the scalar amplitude at 68% CL:

$$\ln(10^{10} A_S) = 3.094 \pm 0.034 \,. \tag{89}$$

For the tensor-to-scalar ratio:

$$r_{0.002} < 0.10 \,, \tag{90}$$

at 95% confidence level. The energy scale of inflation:

$$V_* = \left(1.88 \times 10^{16}\ \mathrm{GeV}\right)^4 \frac{r}{0.10} \,. \tag{91}$$

The Starobinsky model
(1979, 1980, 1983)

The Starobinsky model is:

$$f(R) = R + \frac{R^2}{6M^2} \, . \tag{92}$$

As any $f(R)$ theory, it can be framed into GR plus a canonical scalar field. The potential is:

$$\boxed{U(\chi) = \frac{3}{4} M_{\rm Pl}^2 M^2 \left(1 - e^{-\sqrt{2/3}\chi/M_{\rm Pl}}\right)^2} \tag{93}$$

The scalar spectral index and the tensor-to-scalar ratio are:

$$n_S = 1 - \frac{2}{N} \, , \qquad r = \frac{12}{N^2} \, . \tag{94}$$

Substituting $N = 50$ and 60, the predictions obtained are in excellent agreement with the Planck constraints.

Planck constraints