Introduction to cosmology - Part 1

Oliver F. Piattella

DiSAT - Università degli Studi dell'Insubria (Como, Italy)
Núcleo Cosmo-ufes & PPGCosmo - Universidade Federal do Espírito Santo (Vitória, Brazil)
of.piattella@uninsubria.it ofp.cosmo-ufes.org

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Outline

Preliminaries

Measuring distances
Apparent and absolute magnitudes

Given a certain flux F_X in a passband X, the **apparent magnitude** is defined as:

$$m_X = -2.5 \log_{10} (F_X / \mathcal{F}_X) \tag{1}$$

where $\mathcal{F}_X$ is a reference flux (typically, that of Vega).

Absolute magnitude M: intrinsic luminosity of a source, defined as the apparent magnitude of the source if this were placed at 10 pc of distance and without extinction (loss of light).

Inverse-square law:

$$F = \frac{L}{4\pi d^2} , \tag{2}$$

with $d =$ distance to the source.

Distance modulus

Reformulating the inverse-square law with magnitudes:

$$M = -2.5 \log_{10} \left[\frac{F}{\mathcal{F}} \left(\frac{d}{10 \text{ pc}} \right)^2 \right] = m - 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right) . \quad (3)$$

Distance modulus:

$$\mu \equiv m - M = 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right) . \quad (4)$$

If a relative motion is present between source and observer, the Doppler effect would invalidate the comparison of the magnitudes of different sources in a given passband $\Rightarrow$ K correction:

$$m = M + \mu + K . \quad (5)$$

The cosmic distance ladder

In cosmology measuring distances is essential (length $\Leftrightarrow$ geometry $\Leftrightarrow$ matter).

The cosmic distance ladder is a sequence of measuring techniques allowing to determine distances up to the cosmological scales (hundreds of millions light-years).

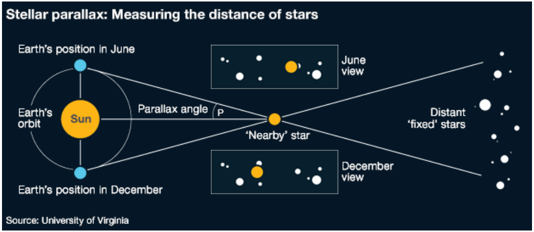
It is called "ladder" because different techniques work only within some ranges of distances which overlap, allowing thus for calibration.

So, step by step, we can reach cosmological scales.

The cosmic distance ladder
The parallax

The first step is the parallax, which is a direct method of determining distances, by trigonometry (it is the same as *triangulation* used on Earth).

Stars which are not too far from us will be seen to move with respect to the background of "fixed" stars.



The cosmic distance ladder
Parallax and the parsec

Let θ be the parallax angle. If $\theta \ll 1$, the distance to the star is:

$$d_{\rm parallax} = \frac{1\ \text{AU}}{\theta}\,, \tag{6}$$

where $1\ \text{AU} = 1.5 \times 10^{11}$ m is the astronomical unit.

If $\theta = 1$ seconds of arc, we define the **parsec**:

$$d_{\rm parallax} = \frac{6.48 \times 10^{5}\ \text{AU}}{\pi} \approx 3.09 \times 10^{16}\ \text{m} \equiv 1\ \text{pc}\,, \tag{7}$$

The ESA *Gaia* spacecraft has been able to measure distances of stars up to 100 pc.

The Milky way is about 30 kpc of diameter, though.

The cosmic distance ladder
Standard Candles

In order to measure larger distances (particularly, of objects outside the Milky way) we need to rely on indirect methods.

Standard candles are sources whose luminosity is known and whose distance can therefore be obtained, via a measurement of their flux:

$$d \propto \sqrt{L/(4\pi F)}\,. \tag{8}$$

An extra proportionality factor has to be taken into account for the largest distances, when the expansion of the universe (the so-called Hubble flow) dominates the peculiar motions.

The cosmic distance ladder

Cepheids

Cepheids variables are pulsating stars, whose period of pulsation is related to their luminosity:

$$M_X = a + b\left(\log_{10} P - 1\right). \tag{9}$$

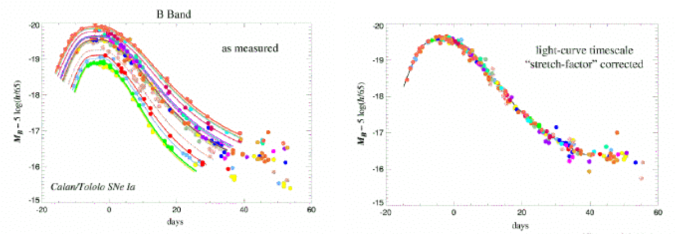
This period-luminosity relationship can be calibrated if we can determine the distance of a Cepheid variable through parallax.

Then, we can use this relationship to determine the luminosity (and so the distance) of Cepheids which are too far for using parallax.

The cosmic distance ladder
Type Ia supernovae

A type Ia supernova explosion occurs when a white dwarf accretes material from a companion until exceeding the Chandrasekhar limit.

Key feature: The faster the light curve decays from its peak, the fainter is the absolute magnitude at the peak.



The cosmic distance ladder
Type Ia supernovae as standard candles

Type Ia supernovae are standardizable candles thanks to **Phillips's relation**:

$$M_{\rm max} = a + b\,\Delta m_{15}(B)\,, \tag{10}$$

where a and b are parameters to be fitted (same for all supernovae!) and $\Delta m_{15}(B)$ is the variation of the apparent magnitude in the B band 15 days after the peak.

Calibration: know beforehand values of $M_{\rm max}$ and $\Delta m_{15}(B)$ for many type Ia supernovae in order to find a and b.

So, we do another step on the cosmic distance ladder and, since type Ia supernovae are very bright, we are able to go very far (hundreds of Mpc, in fact) in the realm of cosmology.

The last step of the ladder
The Hubble-Lemaître law

In 1929 Hubble re-examines the problem of the drift motion of the Solar System with respect to distant nebulae:

$$Kr + X\cos\alpha\cos\delta + Y\sin\alpha\cos\delta + Z\sin\delta = v\ . \qquad (11)$$

K is now known as Hubble's constant H_0;

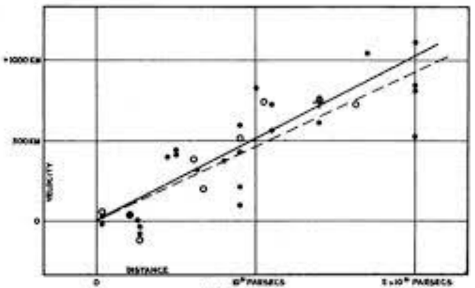


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

The Hubble-Lemaître law

For sufficiently large distances, we can neglect the peculiar relative motion:

$$v = H_0 r \; . \qquad (12)$$

This relation had already been found, on theoretical grounds, by Lemaître in 1927.[1]
Hubble's original result:

$$H_0 = (465 \pm 50)\ \mathrm{km\ s^{-1}\ Mpc^{-1}} \; . \qquad (13)$$

A previous investigation (alas, forgotten) by Lundmark in 1925:

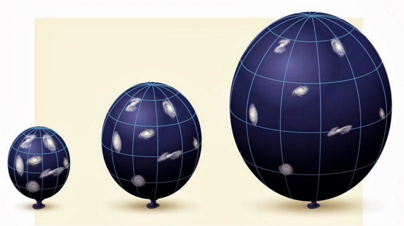
$$X \cos\alpha \cos\delta + Y \sin\alpha \cos\delta + Z \sin\delta + k + lr + mr^2 - v = 0 \; . \qquad (14)$$

yielded $l \approx 10000\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$.

[1] Implicitly, also by Friedmann in 1922.

The universe in expansion

Hubble's result shows that the universe is in expansion (more prosaically: all galaxies sufficiently far from us are receding from us).



We do not occupy the centre of the universe, but space expands $\Rightarrow$ geometric description of the cosmos $\Leftrightarrow$ General Relativity.

The universe in expansion
The Hubble radius and the redshift

At the **Hubble radius** $c/H_0 \equiv R_H$ the recessional velocity equals c. Size of the visible universe (about 4 Gpc). Solution to Olbers's paradox.

The recessional velocity of a galaxy is observed via a Doppler shift in its emission spectrum. For sufficiently large distances, the shift is always a redshift:

$$z = \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} - 1\,. \qquad (15)$$

Thanks to Hubble's law, it is customary to use z as a measure of distance (conversion $z \leftrightarrow$ Mpc requires a cosmological model, though).

Determination of Hubble's constant
Hubble tension

Sandage (1958):

$$H_0 = 50 - 100 \text{ km s}^{-1} \text{ Mpc}^{-1} . \tag{16}$$

Today:

1. Final 2018 result of the Planck mission:

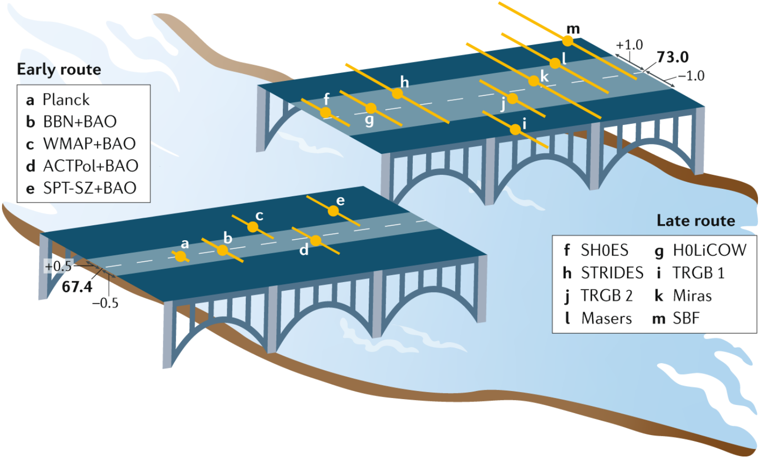
$$H_0 = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1} . \tag{17}$$

2. Hubble Space Telescope observations of 70 long-period Cepheids in the Large Magellanic Cloud (Riess, 2019):

$$H_0 = (74.22 \pm 1.82) \text{ km s}^{-1} \text{ Mpc}^{-1} . \tag{18}$$

The two determinations above are in tension. Most debated open problem in cosmology today.

The tension on H_0



Credits: Riess, Nat. Rev. Phys. 2 (2020) 10

Proposed explanations of the tension

A list of over 100 proposals: (Di Valentino et al., 2021). Among them:

- Phantom dark energy (Di Valentino et al., 2020);
- Interacting dark energy (Di Valentino et al., 2020);
- Decaying dark matter (Pandey et al., 2020);
- Running vacuum (Solá et al., 2017);
- Bulk viscosity (Yang et al., 2019)
- ...

Many proposals introduce new physics at early-times or at late-times.

Up to now, no convincing solution.

Observation

The Cosmic Microwave Background Radiation
CMB(R)

It is a relic, thermal radiation from a hot dense phase in the early evolution of our universe which has now been cooled by the cosmic expansion to 2.75 K.

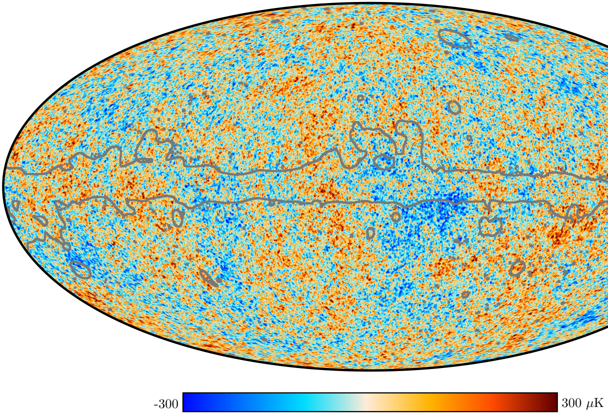
Predicted in the 1940s by Alpher and Gamow, discovered by Penzias and Wilson at Bell Labs in New Jersey in 1965.

Convincing evidence that the cosmos emerged from a **Hot Big Bang** more than 10 billion years ago.

Remarkable uniformity of the CMB radiation, at a temperature of 2.7 Kelvin in all directions, with a small ± 3.3 mK dipole due to our local motion.

CMB sky

Planck 2018 temperature map



https://www.cosmos.esa.int/web/planck/picture-gallery

The Cosmic Microwave Background Radiation
Origins

According to the **Hot Big Bang** model, the early universe was in a hot, dense state called the primordial plasma.

Known (and possibly unknown) particles species interacted at a so high rate ($\Gamma \gg H$) that thermal equilibrium could be attained.

The expansion of the universe cools down the primordial plasma, causing the various species to decouple ($\Gamma \sim H$).

The last species remained (that we can observe) are photons, the CMB, left over when Thomson scattering ($e^- + \gamma \longleftrightarrow e^- + \gamma$) became inefficient against the expansion of the universe (about 400 thousands years after the Big Bang).

CMB power spectra

Temperature-temperature correlation and power spectrum

Calling $\Theta \equiv \Delta T/T$ and $\hat{n}$ the direction of the line of sight:

$$\boxed{\Theta(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{T,\ell m} Y_\ell^m(\theta,\phi)} \tag{19}$$

The $a_{T,\ell m}$'s are assumed to be Gaussian stochastic variables:

$$\boxed{\langle a_{T,\ell m} \rangle = 0} \qquad \boxed{\langle a_{T,\ell m} a^*_{T,\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_{TT,\ell}} \tag{20}$$

$C_{TT,\ell} = \langle |a_{T,\ell m}|^2 \rangle$ is the CMB TT power spectrum.

Under space inversion: $a_{T,\ell m} \to (-1)^\ell a_{T,\ell m}$.

CMB power spectra
Polarization

Thomson scattering polarizes CMB photons.

Using the Stokes's parameters Q and U:

$$(Q+iU)(\hat{n}) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} a_{P,\ell m}\, {}_2Y_\ell^m(\hat{n}) \,, \tag{21}$$

and c.c. for $(Q-iU)(\hat{n})$.

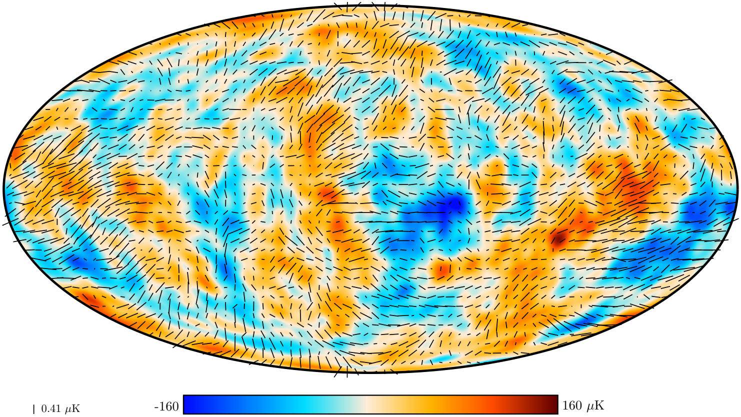
It is convenient to use the following combinations of $a_{P,\ell m}$:

$$a_{E,\ell m} = -(a_{P,\ell m} + a^*_{P,\ell -m})/2 \,, \quad a_{B,\ell m} = i(a_{P,\ell m} - a^*_{P,\ell -m})/2 \,. \tag{22}$$

Under space inversion: $a_{E,\ell m} \to (-1)^\ell a_{E,\ell m}$ and $a_{B,\ell m} \to -(-1)^\ell a_{B,\ell m}$

Planck 2018

Polarization



CMB power spectra

Polarization

The only four spectra that we can build from CMB temperature and polarization measurement are thus:

$$\boxed{\langle a_{T,\ell m} a^*_{T,\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{TT,\ell}} \tag{23}$$

$$\boxed{\langle a_{T,\ell m} a^*_{E,\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{TE,\ell}} \tag{24}$$

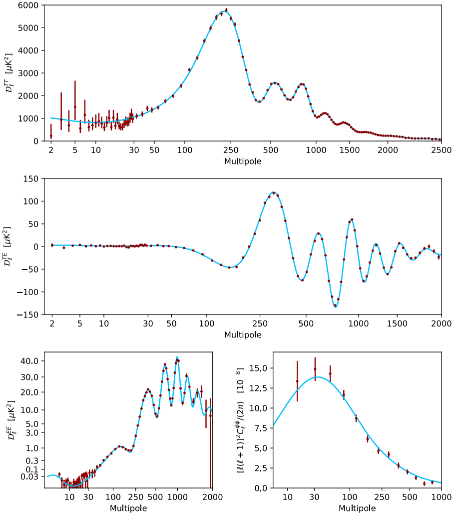
$$\boxed{\langle a_{E,\ell m} a^*_{E,\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{EE,\ell}} \tag{25}$$

$$\boxed{\langle a_{B,\ell m} a^*_{B,\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{BB,\ell}} \tag{26}$$

The BB power spectrum is sourced by tensor perturbations (primordial gravitational waves).

No TB and EB correlations if we assume a primordial distribution function invariant under space inversion.

CMB spectra



$\mathcal{D}_\ell^{TT}$ [μK^2]
6000
5000
4000
3000
2000
1000
0
2 5 10 30 50 100 250 500 1000 1500 2000 2500
Multipole
$\mathcal{D}_\ell^{TE}$ [μK^2]
150
100
50
0
−50
−100
−150
2 5 10 30 50 100 250 500 1000 1500 2000
Multipole
$\mathcal{D}_\ell^{EE}$ [μK^2]
40.0
30.0
20.0
10.0
5.0
3.0
1.0
0.3
0.1
0.03
10 30 100 250 500 1000 2000
Multipole
$[\ell(\ell+1)]^2 C_\ell^{\phi\phi}/(2\pi)$ [10^{-8}]
15.0
12.5
10.0
7.5
5.0
2.5
0.0
10 30 100 250 500 1000
Multipole

Cosmic variance

Observationally, the only average that we can do is the angular one:

$$C_\ell^{\rm obs} = \frac{1}{4\pi}\int d^2\hat{n}\; d^2\hat{n}'\; \mathcal{P}_\ell(\hat{n}\cdot\hat{n}')\Theta(\hat{n})\Theta(\hat{n}') = \frac{1}{2\ell+1}\sum_m a_{\ell m}a_{\ell,-m}\,. \tag{27}$$

The cosmic variance is defined then as:

$$\sigma^2_{C_\ell} = \left\langle\left(\frac{C_\ell - C_\ell^{\rm obs}}{C_\ell}\right)^2\right\rangle = 1 - 2\frac{\langle C_\ell^{\rm obs}\rangle}{C_\ell} + \frac{1}{C_\ell^2}\langle C_\ell^{{\rm obs}\,2}\rangle\,. \tag{28}$$

For Gaussian perturbations:

$$\boxed{\sigma^2_{C_\ell} = \frac{2}{2\ell+1}} \tag{29}$$

Redshift surveys

Redshift surveys determine angular positions (declination and right ascension), redshifts and spectra of galaxies.

The purpose is the study of the correlations among the positions (angular or spatial):

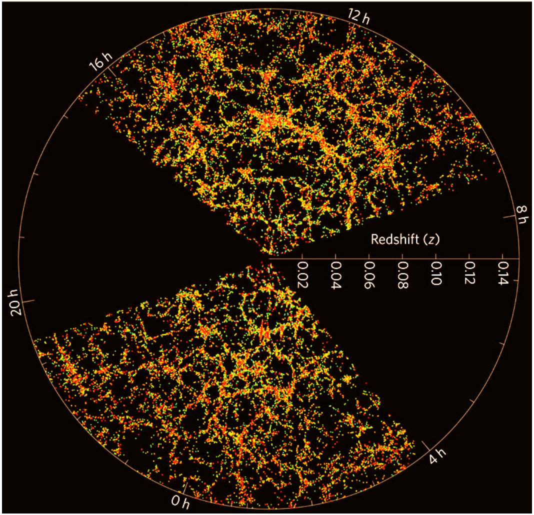
$$\xi(|\mathbf{x}_1 - \mathbf{x}_2|) = \langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2) \rangle \,, \tag{30}$$

and of the power spectrum:

$$\xi(r) = \frac{1}{2\pi^2} \int dk k^2 P(k) \frac{\sin(kr)}{kr} \,. \tag{31}$$

These quantities allow to infer information about gravity and the physics of structure formation.

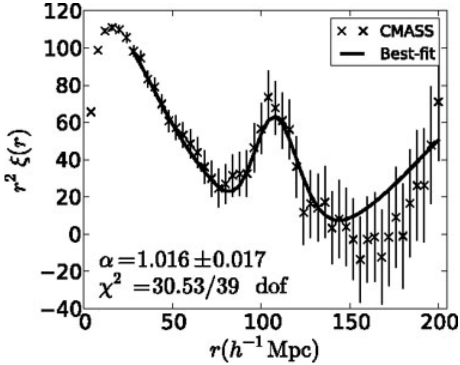
Galaxy distribution



https://www.sdss3.org/science/gallery_sdss_pie2.php

Large-scale correlation
Baryon Acoustic Oscillations (BAO)

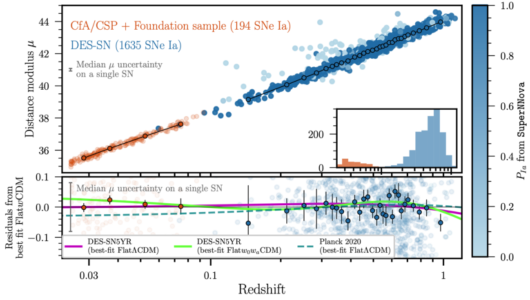
The BAO are a feature of the correlation function containing precious informations on the early-universe (before recombination).



https://ned.ipac.caltech.edu/level5/March14/Percival/Percival4.html

Type Ia supernovae

Extend the Hubble diagram to the farthest distances: the expansion of our universe is **accelerated**.



https://arxiv.org/pdf/2401.02929

But gravity "as we know it" is an attractive force, therefore a universe filled with matter "as we know it" should cause the expansion to decelerate.

Dark Energy

So, what does cause the acceleration of the expansion?

Maybe gravity is a repulsive force at very large scales, those on which we do cosmology. If this is the case, GR must be modified or extended. The minimum modification is to recover Einstein's cosmological constant Λ which, if of the correct sign, provides the sought for effect of anti-gravity.

Or maybe there exists a new form of matter, or rather energy, which acts as anti-gravity. This is known as **Dark Energy** (DE).

The most simple and successful candidate[2] for DE is precisely the cosmological constant Λ, which can be interpreted as both.

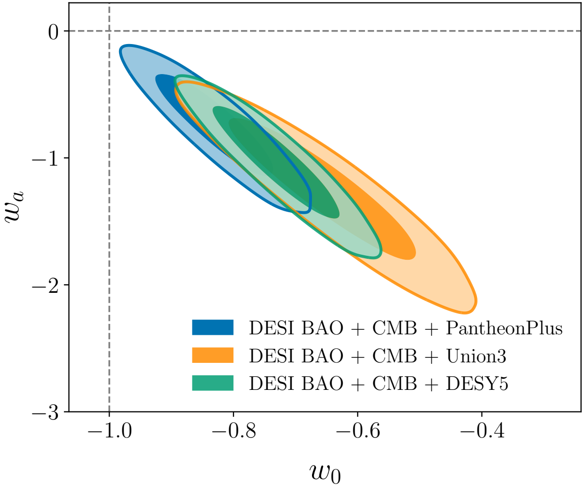
[2] This primacy is being challenged by recent data of the Dark Energy Spectroscopic Instrument (2024).

Dark Energy

DESI 2024 VI: Cosmological Constraints from the Measurements of Baryon Acoustic Oscillations

https://arxiv.org/abs/2404.03002

Cosmological constant: $w_0 = -1$, $w_a = 0$.



Dark Matter

Observations of different nature and from different sources at different distance scales point out the existence of another dark component, called Dark Matter (DM).

This is also not matter "as we know it", in the sense that, *if it is made up of fundamental particles*, we cannot identify these within the Standard Model of particles. Certainly, such DM particles do not interact electromagnetically. Perhaps, they interact weakly.

One could also try and explain observations with a different theory of gravity (a dichotomy similar to the one concerning DE), but it is more difficult due to the various evidences at different scales of distance.

Dark Matter
Observational evidences

The dynamics of galaxies in clusters. The pioneering applications of the virial theorem to the Coma cluster by Fritz Zwicky (1933) resulted in a virial mass 400 times the observed one (which can be estimated by the light emission).

Page 125 of (Zwicky, 1933):
"In order to obtain, as observed, a mean Doppler effect of 1000 km/s or more, the average density in the Coma system should be at least 400 times larger of that derived on the basis of observation of luminous matter. If this were the case, then we would be presented with the surprising result that dark matter is present with a density much larger than that of luminous matter."

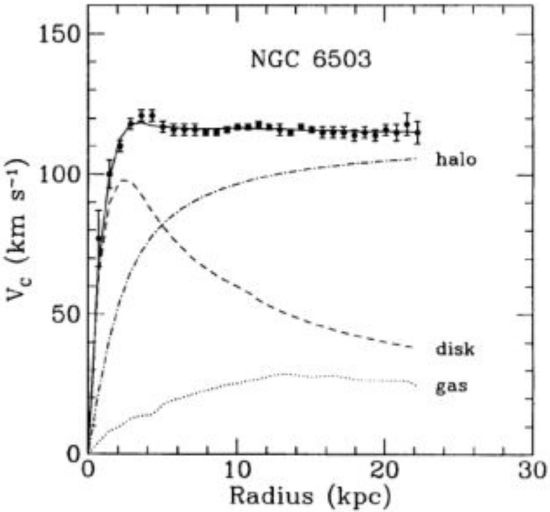
To my knowledge, this is the first instance in which the terminology "dark matter" ("dunkle Materie", in German) is used.

Dark Matter
Observational evidences

Rotation curves of disc galaxies. By measuring the spectral displacement in the emission of a galaxy (from stars and interstellar gas) it is possible to build a velocity curve, i.e. a plot of velocity *vs* distance from the centre of the galaxy. One expects, far from the galactic bulge, $V \propto 1/\sqrt{R}$. What has been found instead, is a flattish velocity profile.

It must be stressed that the inner velocity profile of disc galaxies can be perfectly well explained within Newtonian gravity, assuming an appropriate model for the mass distribution (a bulge is necessary) and *without* DM. The necessity for the latter comes if one is able to extend observation very far from the centre of the galaxy. Here, one cannot see anymore stars, but one rather observes the intergalactic medium HI 21 cm radio emission.

Dark Matter
Observational evidences



https://ned.ipac.caltech.edu/level5/Sept17/Freese/Freese2.html

Dark Matter
Observational evidences

The formation of structures in the universe. Within galaxies and clusters of galaxies the density of matter is far larger than the cosmological average: $\delta_{\rm b} \gg 1$.

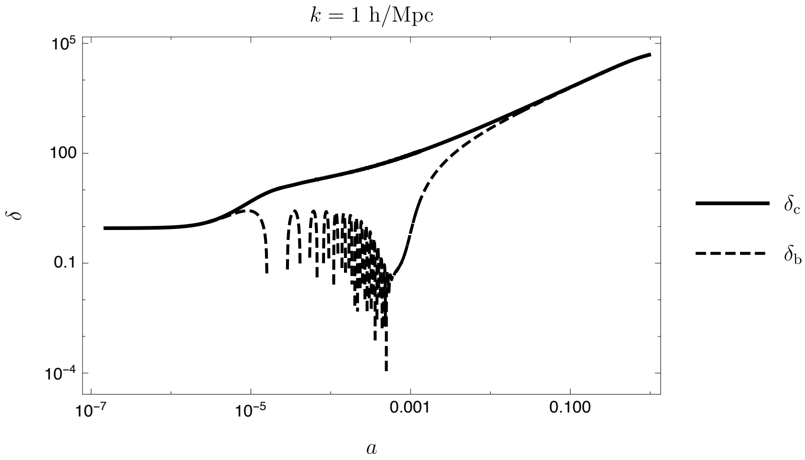
Relativistic cosmology predicts that $\delta_{\rm b}$ *alone* would grow by a factor 10^3 between recombination and the present time.

From the analysis of the CMB spectrum we infer that the value of $\delta_{\rm b}$ at recombination is $\delta_{\rm b} \approx 10^{-5}$.

Combining these two informations we infer that today $\delta_{\rm b} \approx 10^{-2}$, which means that no structure could have formed. In this instance, DM is required in order to enhance the growth of $\delta_{\rm b}$ after recombination.

Dark Matter

Baryons falling in the CDM potential wells

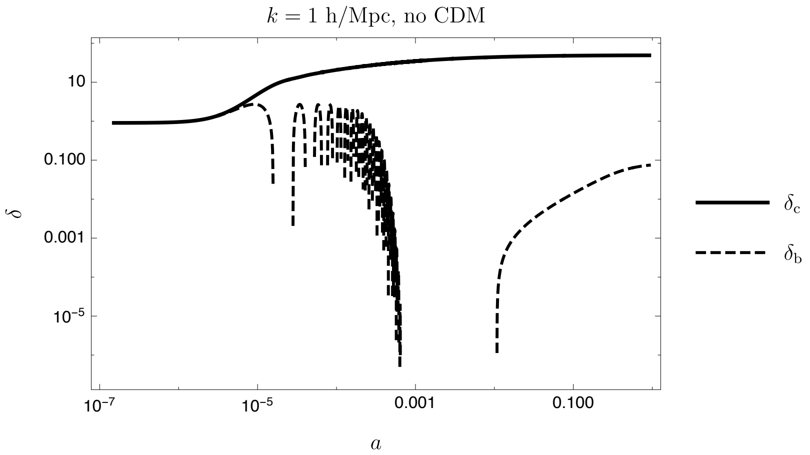


done with CLASS (the Cosmic Linear Anisotropy Solving System):

`https://lesgourg.github.io/class_public/class.html`

Dark Matter

Baryons NOT falling in the CDM potential wells

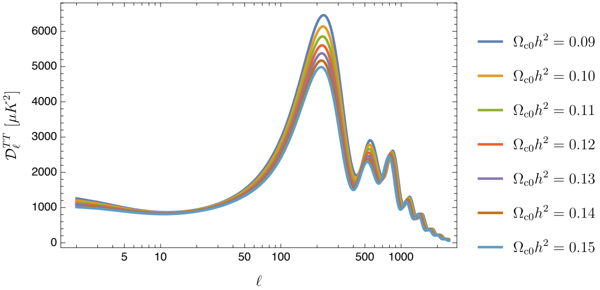


done with CLASS (the Cosmic Linear Anisotropy Solving System):

`https://lesgourg.github.io/class_public/class.html`

Dark Matter
Observational evidences

The peaks of the CMB spectrum. The height of the first and the third peak of the CMB spectrum requires the presence of DM.



done with CLASS (the Cosmic Linear Anisotropy Solving System):

`https://lesgourg.github.io/class_public/class.html`

Dark Matter
Observational evidences

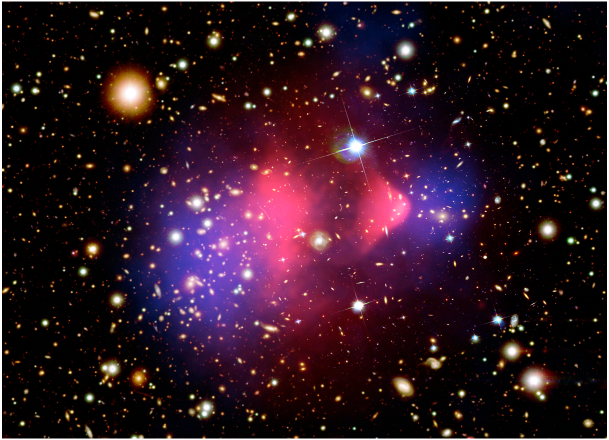
Gravitational Lensing and the Bullet Cluster. (Clowe, 2006).

X-ray maps: collision between the hot gases of two galaxy clusters.

Gravitational lensing maps: the gases lag behind the centres of mass of the respective clusters.

Explanation: the DM halos went through one another unscathed, leaving behind the hot gas. This is considered a direct empirical proof of the existence of DM forming a massive halo and a DM gravitational potential well in which gas and galaxies lie.

Dark Matter
The Bullet Cluster



https://www.esa.int/ESA_Multimedia/Images/2007/07/The_Bullet_Cluster2

Dark Matter
Candidates

Some candidates:

- Weakly Interacting Massive Particles (WIMPs);
- Axions;
- Sterile Neutrinos;
- Primordial Black Holes;
- ...

Dark Matter
Small-scale anomalies

On sub-galactic scales, of about 1 kpc, the CDM paradigm displays some difficulties. They stem from the results of numerical simulations of the formation of structures:

1. The *core-cusp problem* or *cuspy halo problem*. The CDM distribution in the centre of the halo has a cusp profile, whereas observation suggests a core one;
2. The *Missing satellites problem* or *dwarf galaxy problem*. Numerical simulations predict a large number of satellite structures, which are not observed;
3. The *Too big to fail problem*. The sub-structures predicted by the simulations are too big not to be seen.

Relativistic Cosmology

Gravity

On the largest scales the dominant fundamental interaction is gravity.

Hence, in order to do cosmology we need a theory of gravity. The General Theory of Relativity (GR) is our framework.[3]

It turns out that Newtonian physics works surprisingly well. It is also surprising that attempts of doing cosmology with Newtonian gravity are well posterior to relativistic cosmology itself.

[3] Although extensions of GR are investigated today in relation with Dark Energy.

Field equations in GR
In a nutshell

Einstein equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \; ; \qquad (32)$$

Ricci tensor:

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\mu_{\rho\nu} + \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} - \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\mu\sigma} \; ; \qquad (33)$$

Christoffel symbols of the Levi-Civita connection:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\rho g_{\mu\nu} \right) \; , \qquad (34)$$

and $g_{\mu\nu}$ is the metric. $T_{\mu\nu}$ is the energy-momentum or stress-energy tensor, and describes the matter content.
We have seen indications about the matter content, but which is the metric which describes the universe on very large scales?

The cosmological principle

The **cosmological principle**[4] states that the universe is *spatially* isotropic and homogeneous, i.e., there is no way of identifying a special direction or a special position.

A more formal definition can be found in (Weinberg, 1972) and is based on the following two requirements:

1. The hypersurfaces with constant cosmic standard time are **maximally symmetric subspaces** of the whole of the spacetime;
2. The global metric and all the cosmic tensors such as the stress-energy one $T_{\mu\nu}$ are form-invariant with respect to the isometries of those subspaces.

[4] It seems that the terminology "cosmological principle" was used for the first time by E. A. Milne in 1935.

The cosmological principle
Geometric interpretation

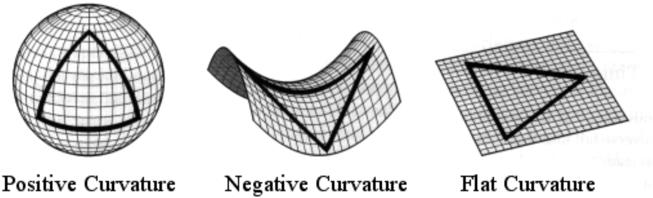
There exist "privileged" coordinate systems in which the spatial sections (constant time) are maximally symmetric (therefore, isotropic and homogeneous).

In this foliation the time is called **cosmic time**. Physical quantities depend only on the cosmic time.

The cosmological principle can be formulated as the requirement of isotropy plus the Copernican principle, since a space that is isotropic at any point is also homogeneous.[5]

[5] The Copernican principle states that we are no privileged observer in the universe and that any other observer would see, on average, the same universe from their vantage point as we do from ours.

Geometries of the spatial sections



https://pages.uoregon.edu/jschombe/cosmo/lectures/lec15.html

The cosmological principle
Compatibility with observation

The cosmological principle seems to be compatible with observations at very large scales.

According to (Wu, 1998): *on a scale of about 100 h^{-1} Mpc the rms density fluctuations are at the level of ∼10% and on scales larger than 300 h^{-1} Mpc the distribution of both mass and luminous sources safely satisfies the cosmological principle of isotropy and homogeneity.*[6]

In a recent work (Sarkar, 2016) it is found that the quasar distribution is homogeneous on scales larger than 250 h^{-1} Mpc.

[6] Here h is the Hubble constant in units of 100 km s^{-1} Mpc^{-1}.

The FLRW metric

Friedmann-Lemaître-Robertson-Walker metric:

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right) . \tag{35}$$

The time coordinate used here is called **cosmic time**, whereas the spatial coordinates are called **comoving coordinates**;

$a(t)$ is called **scale factor**, since it tells us how the distance between two points scales with time:

$$dl^2 = a^2 \gamma_{ij} dx^i dx^j = a^2 dx^2 . \tag{36}$$

The Friedmann equations

From the Einstein equations:

$$\boxed{H^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3c^2} T_{00} + \frac{\Lambda c^2}{3}} \tag{37}$$

$$\boxed{g_{ij}\left(H^2 + 2\frac{\ddot{a}}{a} + \frac{Kc^2}{a^2} - \Lambda c^2\right) = -\frac{8\pi G}{c^2} T_{ij}} \tag{38}$$

About $T_{\mu\nu}$:

- $G_{0i} = 0$ implies that $T_{0i} = 0$, i.e., there cannot be a flux of energy in any direction (it would violate isotropy);
- since $G_{ij} \propto g_{ij}$, then $T_{ij} \propto g_{ij}$.
- since $G_{\mu\nu}$ depends only on t, so does $T_{\mu\nu}$.

The Friedmann equations
The energy-momentum tensor

Let us stipulate that:

$$T_{00} = \rho(t)c^2 = \varepsilon(t) \ , \qquad T_{0i} = 0 \ , \qquad T_{ij} = g_{ij}P(t) \ , \qquad (39)$$

where $\rho(t)$ is the rest mass density, $\varepsilon(t)$ is the energy density and $P(t)$ is the pressure.

Matter described by an stress-energy tensor for which there exists a reference frame in which $T^{\mu}{}_{\nu}$ is diagonal and with all equal spatial entries is called a **perfect fluid**.

Introducing the 4-velocity of the fluid element, which has to have coordinates $u_\mu = (-c, 0, 0, 0)$ in our cosmic time-comoving coordinates:

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}} \qquad (40)$$

The Friedmann equations

The Friedmann equation becomes:

$$\boxed{H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{Kc^2}{a^2}} \quad (41)$$

The acceleration equation is the following:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}} \quad (42)$$

Note that $a > 0$ by definition. However, $\dot{a}, \ddot{a} \gtreqless 0$.

From the Friedmann equation: $\dot{a}$ can vanish only if $K > 0$, i.e., for a spatially closed universe. If $K \leqslant 0$ and if there exists an instant for which $\dot{a} > 0$, then the universe will expand forever.

The continuity equation

The conservation equation

$$\boxed{\nabla_\nu T^{\mu\nu} = 0} \tag{43}$$

is encapsulated in GR through invariance under diffeomorphisms and it is compatible with the **Bianchi identities** $\nabla_\nu G^{\mu\nu} = 0$.

It is not independent from the Friedmann equations and it has a particularly simple form:

$$\boxed{\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0} \tag{44}$$

This is the $\mu = 0$ component of $\nabla_\nu T^{\mu\nu} = 0$ and it is also known from fluid dynamics as **continuity equation**.

Equations of state

Assume an **equation of state** $P = w\rho c^2$, with w constant.:

$$\boxed{\rho = \rho_0 a^{-3(1+w)}} \qquad (w = \text{ constant}) \ , \tag{45}$$

where $\rho_0 \equiv \rho(a_0 = 1)$.

- Cold matter (DM and known matter): $w = 0$, for which $\rho = \rho_0 a^{-3}$;
- Hot matter (photons and early neutrinos): $w = 1/3$, for which $\rho = \rho_0 a^{-4}$. The adjective hot refers to the fact that particles making up this kind of matter are relativistic;
- Vacuum energy (Λ): $w = -1$, i.e., $P = -\rho c^2$ and ρ is a constant.

Critical density and density parameters

The value of the total ρ such that $K = 0$ is called **critical energy density**:

$$\boxed{\rho_{\rm cr} \equiv \frac{3H^2}{8\pi G}} \qquad (46)$$

Its present value:

$$\boxed{\rho_{\rm cr,0} = 1.878\ h^2 \times 10^{-29}\ {\rm g\ cm}^{-3}} \qquad (47)$$

depends only on the value of H_0, which is determined observationally.

Instead of densities, it is useful to employ the density parameter Ω:

$$\boxed{\Omega \equiv \frac{\rho}{\rho_{\rm cr}} = \frac{8\pi G\rho}{3H^2}} \qquad (48)$$

i.e., the energy density normalised to the critical one.

Critical density and density parameters

Friedmann equation

We can then rewrite Friedmann equation:

$$1 = \Omega - \frac{Kc^2}{H^2 a^2} \; . \tag{49}$$

Defining:

$$\Omega_K \equiv -\frac{Kc^2}{H^2 a^2} \; , \qquad \rho_K \equiv -\frac{3Kc^2}{8\pi G a^2} \; , \tag{50}$$

one has:

$$1 = \Omega + \Omega_K \; . \tag{51}$$

Therefore, the sum of all the density parameters, *the curvature one included*, is always equal to unity.[7]

[7] This is not so surprising, since the 00 Einstein equation is a *constraint*.

Critical density and density parameters

Friedmann equation

It is also common normalisation to the *present-time* critical density:

$$\boxed{\Omega \equiv \frac{\rho}{\rho_{\mathrm{cr},0}} = \frac{8\pi G\rho}{3H_0^2}} \tag{52}$$

This leaves manifest the dependence on a of each material component:

$$\frac{H^2}{H_0^2} = \sum_x \Omega_{x0} f_x(a) + \frac{\Omega_{K0}}{a^2}\,, \tag{53}$$

where $f_x(a_0 = 1) = 1$. Consistently:

$$\boxed{\Omega_0 + \Omega_{K0} = 1} \qquad \Omega_0 \equiv \sum_x \Omega_{x0}\,, \tag{54}$$

also known as **closure relation**.

Spatial curvature
The flatness problem

From the latest Planck data (2018):

$$\boxed{\Omega_{K0} = 0.0007 \pm 0.0019} \tag{55}$$

at 68% of confidence level for the joint analysis of lensed CMB temperature and polarisation spectra and BAO.

Such very small spatial curvature implies a fine-tuning problem in K known as the **flatness problem**.

The flatness problem

Since $|\Omega_{K0}| < 1$, then:

$$|\Omega_K| = \left| -\frac{K}{H^2 a^2} \right| = \left| -\frac{K}{H^2 a^2} \frac{H_0^2}{H_0^2} \right| = |\Omega_{K0}| \frac{H_0^2}{H^2 a^2} < \frac{H_0^2}{H^2 a^2} \, . \quad (56)$$

For sufficiently small a the radiation component dominates and it makes H to scale as $H \propto a^{-2}$. Therefore:

$$\frac{H_0^2}{H^2 a^2} \propto a^2 \, , \qquad \text{radiation domination.} \quad (57)$$

That is, the more in the past we go the closer to zero Ω_K gets.

The flatness problem

During matter domination we have $H \propto a^{-3/2}$, so:

$$\frac{H_0^2}{H^2 a^2} \propto a \,, \qquad \text{matter domination.} \tag{58}$$

Instead, during Λ domination H is a constant. So:

$$\frac{H_0^2}{H^2 a^2} \propto \frac{1}{a^2} \,, \qquad \Lambda \text{ domination.} \tag{59}$$

Looked the other way around, $|\Omega_K|$ grows for almost the entire history of the universe, *proportionally to a power of the scale factor*. Yet, today its value is smaller than 1 instead of being huge.

The flatness problem

Let us make another calculation, the other way around. Suppose that the earliest time at which our theory is reliable is the Planck scale, where $a_P = 10^{-32}$.

Suppose that $\Omega_{K,P}$ has some unknown value, which we consider as the *initial value* for the curvature density parameter. Then, $\Omega_{K0} = 10^{60}\Omega_{K,P}$

If for some reason $\Omega_{K,P} \simeq 10^{-59}$, then $\Omega_{K0} \simeq 10$, in complete disagreement with observation.

We conclude that, in order to match observation, $\Omega_{K,P}$ has to be determined by some physical mechanism to be zero with a precision of at least 60 significative digits! This is an example of **fine-tuning**.

The flatness problem
Inflation

Fine tunings might not be severe problems. After all, a fine-tuned theory is not *wrong*, in the sense that it is not falsified by observation. On the other hand, fine-tuning conveys a sense of *ad hoc*-ness to the theory, something unnatural, or not fully understood, that we would like to explain more convincingly.

For the flatness problem, such an explanation is possibly provided by the inflationary theory.

How it works for the flatness problem can be seen from the above equation $H_0^2/(H^2a^2) \sim 1/a^2$: if H is almost constant, the curvature density parameter *decreases*. So, if before radiation-domination an evolutionary phase exists in which H is almost constant for sufficiently long time, then we might be able to explain why the curvature density parameter was so small to begin with.

The ΛCDM model

The most successful cosmological model:

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \frac{\Omega_{\rm c0}}{a^3} + \frac{\Omega_{\rm b0}}{a^3} + \frac{\Omega_{\rm r0}}{a^4} + \frac{\Omega_{K0}}{a^2} \; . \tag{60}$$

Using $H_0 = 67.66 \pm 0.42$ km s^{-1} Mpc^{-1}:

$$\boxed{\Omega_\Lambda = 0.6889 \pm 0.0056 \,, \qquad \Omega_{\rm m0} = 0.3111 \pm 0.0056} \tag{61}$$

where $\Omega_{\rm m0} = \Omega_{\rm c0} + \Omega_{\rm b0}$. CMB tells us:

$$\boxed{\Omega_{\rm b0} h^2 = 0.02242 \pm 0.00014 \,, \qquad \Omega_{\rm c0} h^2 = 0.11933 \pm 0.00091} \tag{62}$$

The radiation content:

$$\boxed{\Omega_{\gamma 0} h^2 \approx 2.47 \times 10^{-5} \,, \qquad \Omega_{\nu 0} h^2 \approx 1.68 \times 10^{-5}} \tag{63}$$

Recalling the closure relation, we can conclude that today 69% of our universe is made of cosmological constant, 26% of CDM and 5% of baryons. Radiation and spatial curvature are negligible.