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Scattering of particles near a rotating black hole

How close, how much energy and how much time?

Oliver F. Piattella

Universidade Federal do Espírito Santo

April, Fools' day (dia dos bobos)

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Outline

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Work in progress

In collaboration with

Andrey A. Grib

A. Friedmann Laboratory for Theoretical Physics, St. Petersburg, Russia
Theoretical Physics and Astronomy Department, The Herzen University, St. Petersburg, Russia

Yuri V. Pavlov

A. Friedmann Laboratory for Theoretical Physics, St. Petersburg, Russia
Institute of Mechanical Engineering, Russian Academy of Science, St. Petersburg, Russia

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http://www.auger.org/

http://www.augeraccess.net

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The Pierre Auger Observatory

Cosmic Rays

- Relativistic particles (mostly nuclei, ranging from protons to uranium) constantly bombarding Earth.
- Extensive air-showers when they hit Earth's atmosphere.
- The discovery of extensive air-showers is usually credited to Pierre Auger who, in 1938, observed an unexpectedly high rate of coincidences between counters separated by a few metres.
- The Pierre Auger Observatory has been designed to study the highest energy cosmic rays.
- The Observatory must be very large as the rate of events at the very highest energies ($\sim 10^{20}$ eV) is less than 1 per km^2 per century!
- These very rare particles have mysterious origins, so that studying them may open doors to problems in exotic physics, astrophysics and cosmology.

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http://www.physics.utah.edu/~whanlon/spectrum.html

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http://www.mpi-hd.mpg.de/hfm/CosmicRay/ShowerDetection.html

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K. Greisen, Phys. Rev. Lett., 16(17):748750, 1966
G. T. Zatsepin and V. A. Kuzmin, Soviet Journal of Experimental and Theoretical Physics Letters, 4:78, 1966

- Reaction between energetic protons and cosmic background radiation photons

$$p + \gamma_{2.7K} \rightarrow \Delta^+ \rightarrow p + \pi^0 \text{ or } n + \pi^+$$

- Limit on the distance from which such energetic particles can have come: ~ 50 Mpc at 10^{20} eV (note that the Milky way has a size of ~ 30 kpc).
- Galactic magnetic field does not deflect them $\rightarrow$ cosmic origin. Where are these powerful accelerators?

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Correlation of the highest energy cosmic rays with nearby extragalactic objects

The Pierre Auger Collaboration, Science 318, 938 (2007)

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Active Galactic Nuclei

http://www.roe.ac.uk/~pnb/research.html

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Active Galactic Nuclei

http://www.auger.org/news/PRagn/about_AGN.html

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T. Padmanabhan. Theoretical Astrophysics: Galaxies and Cosmology. Cambridge Univ Pr, 2002.

- Galaxies whose core ($\lesssim 1$ pc^3) produces more radiation than the entire rest of the galaxy! $L = 10^{40} - 10^{47}$ erg/s ($L_\odot = 3.846 \cdot 10^{33}$ erg/s)
- Central engine: a supermassive black hole fed by an accretion disk.
- $M_{BH} \gtrsim 10^8\ M_\odot$.
- Very large bulk velocities, up to 10^4 km/s.
- The supermassive black hole is very rapidly spinning.

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Active Galactic Nuclei

Some evidence for a central spinning black hole

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Active Galactic Nuclei

Some evidence for a central spinning black hole

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R. P. Kerr., Phys. Rev. Lett., 11(5), 237238, 1963
B. Carter, Physical Review, 174(5), 1559-1571, 1968
Kerr Solution in Boyer-Lindquist coordinates:

$$ds^2 = dt^2 - \frac{2Mr\left(dt - a\sin^2\theta d\phi\right)^2}{r^2 + a^2\cos^2\theta}$$

$$- \left(r^2 + a^2\cos^2\theta\right)\left(\frac{dr^2}{\Delta} + d\theta^2\right) - \left(r^2 + a^2\right)\sin^2\theta d\phi^2$$

with

$$\Delta \equiv r^2 - 2Mr + a^2\,, \qquad M > 0\,, \qquad a \in [0, M]$$

R.H. Boyer and R.W. Lindquist., J. math. phys., 8:265, 1967.

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The condition $\Delta = 0$ ($g_{rr} \to \infty$) gives two special distances: the event horizon

$$r_H = M + \sqrt{M^2 - a^2}$$

and the Cauchy horizon

$$r_C = M - \sqrt{M^2 - a^2}$$

Asking $g_{tt} = 0$ you will find instead

$$r_0 = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$

the so-called *static limit*

The spacetime region $r_H < r < r_0$ is called *ergosphere*. Here the Penrose process may take place

R. Penrose, Riv. Nuovo Cim., 1:252276, 1969

Kerr solution

http://www.oulu.fi/astronomy/astrophysics/pr/black_holes.html

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Kerr solution
Penrose Diagram

http://www.engr.mun.ca/~ggeorge/astron/blackholes.html

For $\theta = \pi/2$

$$\frac{dt}{d\tau} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\varepsilon - \frac{2Ma}{r}L\right]$$

$$\frac{d\phi}{d\tau} = \frac{1}{\Delta}\left[\frac{2Ma}{r}\varepsilon + \left(1 - \frac{2M}{r}\right)L\right]$$

$$\left(\frac{dr}{d\tau}\right)^2 = \varepsilon^2 + \frac{2M}{r^3}(a\varepsilon - L)^2 + \frac{a^2\varepsilon^2 - L^2}{r^2} - \frac{\Delta}{r^2}\delta_1$$

where τ is the proper time, $\delta_1 = 0$ for photons whereas $\delta_1 = 1$ for massive particles.

We are here interested in $\delta_1 = 1$. In this case, ε and L are the particle specific energy and specific angular momentum (*specific* means normalised to the mass $\rightarrow$ equivalence principle).

Demanding $dt/d\tau \geq 0$, so that the particle does not go "backward in time"

$$l \leq \left(\frac{x^3 + A^2 x + 2A^2}{2A} \right) \varepsilon$$

We use hereafter the "normalised" variables $l \equiv L/M$, $x \equiv r/M$ and $A \equiv a/M$.
We also assume $\varepsilon = 1$ for a particle coming from infinity.
In this case, there are limiting specific angular momenta for which the particles can achieve the horizon

$$-2\left(1+\sqrt{1+A}\right) = l_L \leq l \leq l_R = 2\left(1+\sqrt{1-A}\right)$$

Note the asymmetry.

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Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy

Is it possible to extract ultra-high energies from the ergosphere of a rapidly rotating BH? Maybe yes, but with two conditions: first, the BH must be extremal and, second, the scattering must take place exactly on the horizon.

The Energy in the centre of mass

For two particles of mass m in the centre of mass reference frame in Minkowski space

$$p^\mu_{tot} = mu^\mu_1 + mu^\mu_2 = (E_{cm}, 0, 0, 0)$$

with u^μ_1, u^ν_2 four-velocities.
Using $\eta_{\mu\nu}u^\mu u^\nu = -1$ we get

$$E_{cm} = m\sqrt{2}\sqrt{1 - \eta_{\mu\nu}u^\mu_1 u^\nu_2}$$

The equivalence principle should allow $\eta_{\mu\nu} \to g_{\mu\nu}$, therefore

$$E_{cm} = m\sqrt{2}\sqrt{1 - g_{\mu\nu}u^\mu_1 u^\nu_2}$$

A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, Problem Book in Relativity and Gravitation (Princeton Univ. Press, New Jersey, 1975)

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The Energy in the centre of mass
Schwarzschild case

With normalised units and $\varepsilon_1 = \varepsilon_2 = 1$:

$$\frac{E_{cm}^2}{2m^2} = \frac{2x^2(x-1) - l_1 l_2 (x-2) - \sqrt{2x^2 - l_1^2(x-2)}\sqrt{2x^2 - l_2^2(x-2)}}{x^2(x-2)}$$

For $x \to x_H = 2$ we get

$$E_{cm}(x \to 2) = \frac{m}{2}\sqrt{(l_2 - l_1)^2 + 16}$$

When $l_1 = l_2$ we have $E_{cm} = 2m$.
When $l_1 = 4$ and $l_2 = -4$ (the maximum angular momenta allowed) we have $E_{cm} = 2\sqrt{5}m$.
With a rotating BH could we have a more efficient process?

The Energy in the centre of mass
Kerr case

With normalised units and $\varepsilon_1 = \varepsilon_2 = 1$:

$$\frac{E_{cm}^2}{2m^2} = \frac{1}{x(x^2 - 2x + A^2)} \left[2A^2(1+x) - 2A(l_1 + l_2) - l_1 l_2 (x-2) \right.$$

$$\left. + 2x^2(x-2) - \sqrt{2(A - l_1)^2 - l_1^2 x + 2x^2}\sqrt{2(A - l_2)^2 - l_2^2 x + 2x^2} \right]$$

This time, the horizon dwells at $x_H = 1 + \sqrt{1 - A^2}$.
For $A = 1$ and $x \to x_H = 1$

$$\frac{E_{cm}}{\sqrt{2}m_0}(A = 1, x \to 1) = \sqrt{\frac{l_2 - 2}{l_1 - 2} + \frac{l_1 - 2}{l_2 - 2}}$$

When l_1 or l_2 tend to 2, the centre of mass energy diverges.

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The allowed specific angular momenta

For $A = 1$, $x_H = 1 + \sqrt{1 - A^2} = 1$.
The condition for achieving the horizon [recall $(dr/d\tau)^2 > 0$]

$$-2\left(1 + \sqrt{1 + A}\right) = l_L \leq l \leq l_R = 2\left(1 + \sqrt{1 - A}\right)$$

becomes

$$-2\left(1 + \sqrt{2}\right) = l_L \leq l \leq l_R = 2$$

So $l = 2$ is a limiting case, but still allowed.
From the "forward in time" condition $dt/d\tau > 0$ we have also

$$l \leq \frac{x^3 + A^2 x + 2A^2}{2A} \qquad \Rightarrow \qquad l \leq \frac{x^3 + x + 2}{2}$$

So that for $x = 1$ we have $l \leq 2$.

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Why extremality?

For a generic A, on the horizon we have

$$l \leq \frac{x_H^3 + A^2 x_H + 2A^2}{2A} = \frac{2x_H}{A} \equiv l_H$$

On the other hand, in order to achieve the horizon we know

$$-2\left(1+\sqrt{1+A}\right) = l_L \leq l \leq l_R = 2\left(1+\sqrt{1-A}\right)$$

The formula for the centre of mass energy on x_H for generic A is

$$\frac{E_{cm}}{2m_0}(x \to x_H) = \sqrt{1 + \frac{(l_1 - l_2)^2(4 + l_H^2)}{16(l_1 - l_H)(l_2 - l_H)}}$$

It is not difficult to prove that $l_R \leq l_H$. Therefore, only the case $A = 1$ could provide a diverging E_{cm}.

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Thorne's limit $A \lesssim 0.9980 \pm 0.0002$

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$$\Delta\tau = M \int_{x_f}^{x_i} dx \frac{x^{3/2}}{\sqrt{2x^2 - l^2 x + 2(A-l)^2}}$$

For $A = 1$ and $l = 2$ we get

$$\Delta\tau = \frac{M}{3\sqrt{2}} \left[2\sqrt{x}(3+x) + 3\ln\frac{\sqrt{x}-1}{\sqrt{x}+1} \right]_{x_f}^{x_i}$$

diverging when $x_f \to x_H = 1$.

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Type of orbit

For $A = 1$ and $l = 2$ the particles have to wait an infinite amount of proper time.
What do they do in the meantime?
Consider

$$\Delta\phi = \int_{x_f}^{x_i} dx \frac{\sqrt{x}[lx + 2(A-l)]}{(x - x_H)(x - x_C)\sqrt{2x^2 - l^2 x + 2(A-l)^2}}$$

You can see that $\Delta\phi \to \infty$ for $x \to x_H = 1$ when $A = 1$ and $l = 2$
The particle therefore commit infinite revolutions around the black hole before scattering

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Given

$$\frac{E_{cm}}{2m}(x \to x_H) = \sqrt{1 + \frac{(l_1 - l_2)^2(4 + l_H^2)}{16(l_1 - l_H)(l_2 - l_H)}}$$

we would like l_1 or l_2 to achieve l_H. Unless for $A = 1$, that is not possible because the horizon is out of reach. For $A = 1$ we have seen that the process is indeed possible, but it requires infinite proper time as well.

A possible way out is the multiple scattering. The particles have initial momenta

$$-2\left(1 + \sqrt{1 + A}\right) = l_L \leq l \leq l_R = 2\left(1 + \sqrt{1 - A}\right)$$

getting close to the horizon. There they scatter once, achieving say $l_H - \delta$ and scatter twice, getting very high energies.

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Disconnected motions

Start from the equation for $(dr/d\tau)^2$ written as:

$$x^3 \left(\frac{dx}{d\tau}\right)^2 = 2\,(A - l)^2 - l^2 x + 2x^2$$

For sufficiently large angular momenta, i.e. $l < l_L$ or $l_R < l < l_H$, the geodesics motion is characterised by two disconnected motions:

- an internal one, with an aphelion say x_A
- an external one with perihelion say x_P

These motions are disconnected in the sense that the region $x_A < x < x_P$ is forbidden since $(dx/d\tau)^2 < 0$
The first scattering must take place in the internal region.

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Example of disconnected motion for the Schwarzschild case

Aphelion and Perihelion

▸ Curiosity

From

$$x^3 \left(\frac{dx}{d\tau}\right)^2 = 2\,(A-l)^2 - l^2 x + 2x^2 = 0$$

you can calculate that

$$x_{A,P} = \frac{l^2 \pm \sqrt{l^4 - 16(A-l)^2}}{4}$$

For $l = l_{\mathrm{H}}$ you get

$$x_A = x_{\mathrm{H}} \qquad x_P = \frac{x_{\mathrm{H}}^2}{2 - x_{\mathrm{H}}} = \frac{l_H^2}{4} x_H$$

The internal orbit does not exist.
The trick is not to employ $l = l_H$, but $l_H - \delta$. This would leave some room outside the horizon.

Aphelion and Perihelion for $l_H - \delta$

For $l = l_{\mathrm{H}} - \delta$, the aphelion x_A grows a bit whereas the perihelion x_P decreases a bit

$$x_A = x_{\mathrm{H}} + \frac{(2 - x_{\mathrm{H}})^2}{4x_{\mathrm{H}}(x_{\mathrm{H}} - 1)}\delta^2 + O(\delta^3)$$

$$x_P = \frac{x_{\mathrm{H}}^2}{2 - x_{\mathrm{H}}} - \frac{2\sqrt{x_{\mathrm{H}}}}{\sqrt{2 - x_{\mathrm{H}}}}\delta + O(\delta^2)$$

The particle has now a small room to live between the horizon and x_A.

Let

$$\eta \equiv \frac{(2 - x_{\mathrm{H}})^2}{4x_{\mathrm{H}}(x_{\mathrm{H}} - 1)}\delta^2 \;, \qquad \eta' \equiv \frac{2\sqrt{x_{\mathrm{H}}}}{\sqrt{2 - x_{\mathrm{H}}}}\delta$$

for simplicity.

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Centre of mass energy upon the second scattering

From $l = l_{\mathrm{H}} - \delta$ Grib and Pavlov find

$$E_{cm} \approx \frac{m}{\sqrt{\delta}} \sqrt{\frac{2(l_H - l_2)}{1 - \sqrt{1 - A^2}}}$$

One realizes that if $\delta \to 0$, the energy in the centre of mass system diverges.
So, if infinite energies are not the case in the single collision event, but are they in the multiple ones?
No, achieving the horizon always requires infinite coordinate time.
Nevertheless, the multiple scattering could provide much larger energies.

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How much time from $x_H + \eta$ to x_H?

Some calculations lead to

$$\Delta\tau = M \frac{x_H^{3/2}\sqrt{\eta}}{\sqrt{2}\left(\frac{x_{\rm H}^2}{2-x_{\rm H}} - \eta' - x_H\right)^{1/2}} + O(\eta^2)$$

and

$$\Delta t = \frac{M x_H \sqrt{2 x_H}\ln\left(\frac{\eta}{x_f - x_H}\right)}{\sqrt{(x_H - 1)(2 - x_H)}\left(\frac{x_{\rm H}^2}{2-x_{\rm H}} - \eta' - x_H\right)^{1/2}} + O(\eta^2)$$

$\Delta\tau$ is finite and vanishes for $\eta \to 0$ (the particle is already on the horizon!)
The coordinate time presents a logarithmic divergence.
Extracting infinite energy from a rotating BH requires infinite time.

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Centre of mass energy (very) close to the horizon

Scattering on the horizon requires infinite time. But we do not really need it, as we shall see. Just stay close:

$$\frac{E_{cm}^2}{4m^2} = 1 + \frac{(l_1 - l_2)^2}{2(2 - x_H)(l_H - l_1)(l_H - l_2)}$$

$$+\alpha \frac{(l_1 - l_2)^2 \left[x_H(x_H - 1)(l_1 + l_2)^2 + 2A(l_1 + l_2)(4x_H - l_1 l_2) + (2 - x_H)(l_1^2 l_2^2 - 4x_H l_H^2)\right]}{2(2 - x_H)^3 (l_H - l_1)^3 (l_H - l_2)^3}$$

$$+O(\alpha^2)$$

where

$$\eta = \frac{(2 - x_{\rm H})^2}{4x_{\rm H}(x_{\rm H} - 1)} \delta^2 = \alpha$$

When you ask $l \to l_H$, you automatically demand also $\delta \to 0$ which implies $\alpha \to 0$ and the evaluation of E_{cm} on the horizon with consequent $E_{cm}, \Delta t \to \infty$.

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How close for a given E_{cm}?

Here is a generalisation of Grib and Pavlov's formula:

$$E_{cm}^2 = \frac{m^2}{\delta}\left[\frac{2(l_H - l_2)}{2 - x_H}\right.$$

$$\left. + \frac{x_H(x_H-1)(l_H+l_2)^2 + 2A(l_H+l_2)(4x_H - l_H l_2) + (2-x_H)l_H^2(l_2^2 - 4x_H)}{2(2-x_H)x_H(x_H-1)(l_H-l_2)^3}\right]$$

Consider $A = 0.998$, which implies $x_H \approx 1.063$ and $l_H \approx 2.131$, and $l_2 = l_L = -4.83$.
The above formula becomes:

$$E_{cm}^2 = \frac{14.937 m^2}{\delta}, \qquad \Rightarrow \qquad E_{\rm cm} = \frac{3.865 m}{\sqrt{\delta}}$$

Remember that δ cannot be taken to zero.

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How close for a given E_{cm}?

Consider scattering of protons, $m \sim 1$ GeV.
Demanding an energy in the centre of mass $E = 10^{12}$ GeV we find

$$\sqrt{\delta} \approx 10^{-12}\,, \qquad \Rightarrow \qquad \eta \approx 3.45 \cdot 10^{-48}$$

which is pretty consistent with our approximation, i.e. small δ.
Indeed, the scattering has to be very close to the horizon!
But it is not necessary for it to take place exactly on the horizon. The fundamental fact here is that the divergence in Δt is logarithmic and therefore the smallness of η is unimportant.

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How much time to get to $x_H + \eta$?

A little... ish

An approximate calculation for a particle coming from $x_i = 10^2 x_H$ (approximatively where the disk is)

$$\Delta t \approx \sqrt{2}M \frac{l_H x_H}{\sqrt{x_H - 1}\sqrt{l_H^2 - 4\eta' - 4x_H}} \ln\left(\frac{10^2 x_H - x_H}{3.45 \cdot 10^{-48}}\right)$$

Given $A = 0.998$, $x_H \approx 1.06$ and $l_H \approx 2.12$, we obtain

$$\Delta t \approx 2.93 \cdot 10^3 M$$

Restoring the proper units we have, in seconds

$$\Delta t \approx 2.93 \cdot 10^3 \frac{GM}{c^3} \text{ sec}$$

For a typical BH which is believed to dwell in the centre of an AGN, $M = 10^8 \cdot M_\odot = 2 \cdot 10^{38}$ kg. So we find

$$\Delta t \approx 1.45 \cdot 10^6 \text{ sec} = 16.76 \text{ d}$$

And how much time to escape and come to us?

The particle has now to come to Earth, in order to be detected. How much time does it take its travel to Earth?
This is easy because it is ultrarelativistic.
Now, because of the GZK effect, the relevant distance to an AGN has to be $\lesssim 75$ Mpc. From this number we derive that

$$\Delta t \approx 2.25 \cdot 10^{13} M \approx 1.11 \cdot 10^{16} \text{ sec} \approx 3.53 \cdot 10^{8} \text{ y}$$

The time-scale for the scattering process is of the order of a week, whereas the particle takes a wealth of time to arrive to Earth.

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Am I correct?

Perhaps

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Conclusions

The main result we find may be summarised as

$$\frac{E_{cm}}{m} \propto \eta^{-1/4} \,, \qquad \Delta t \propto -M \ln \eta$$

which can be combined in order to obtain

$$\Delta t \propto M \ln \left(\frac{E_{cm}}{m} \right)$$

i.e. the time required for a scattering producing a centre of mass energy E_{cm}/m is proportional to the order of magnitude of the latter.
For example, if it takes one week for a proton to achieve an energy of 10^{12} GeV, it would take ten days in order to achieve an energy of 10^{16} GeV, i.e. the grand unification scale (GUT).

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Perspectives

- We have discussed the scattering as a given fact, but we need to investigate in detail the feasibility of the process. That is, investigate a cross-section (which would certainly depend on the accretion disk physics)
- The escape function, i.e. the distribution of outcoming angular momenta (from their values depend if a particle may or not escape from the singularity)
- Zasvlaskii's paper: charged black holes may possibly act as a counterpart of rotating ones for high energy scattering processes (Q and A exchange their roles)
- Another Zasvlaskii's paper: general explanation of high energy scattering processes

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Let $\mathcal{S}$ be a space-like surface on a manifold $\mathcal{M}$; $D^+(\mathcal{S})$ and $D^-(\mathcal{S})$ are the future and the past of $\mathcal{S}$, respectively.
When there are no closed time curves, then

$$D^-(\mathcal{S}) \cap D^+(\mathcal{S}) = \varnothing$$

$\mathcal{S}$ is a Cauchy surface if $D^-(\mathcal{S}) \cap D^+(\mathcal{S}) = \varnothing$ and

$$D^-(\mathcal{S}) \cup \mathcal{S} \cup D^+(\mathcal{S}) = \mathcal{M}$$

In practice, all the events on the manifold are determined by informations (initial consitions) on $\mathcal{S}$ (imagine the Cauchy problem)
If $D^-(\mathcal{S}) \cup \mathcal{S} \cup D^+(\mathcal{S}) \neq \mathcal{M}$, then there are regions not determined by $\mathcal{S}$. Their border is the Cauchy horizon.

When you consider a particle at rest, then r, θ and ϕ are constant. This implies

$$ds^2 = g_{tt} dt^2$$

for which $g_{tt} \geq 0$.
The case $g_{tt} = 0$ determines for Kerr metric

$$r_0 = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$

and is called static limit because it is the closest distance at which you may find a photon at rest.

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Write Kerr metric in the following form

$$ds^2 = \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right) dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} \left(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}} dt\right)^2$$

You can see the angular velocity

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mra}{(r^2 + a^2 \cos^2\theta)(r^2 + a^2) + 2Mra^2 \sin^2\theta}$$

measured by an observer at infinity and induced by the metric. This is the frame-dragging, or *Lense-Thirring effect.*
J. Lense, H. Thirring, Physikalische Zeitschrift 19, 156 (1918)

Strictly speaking, "helion" refers to the Sun.
For a black hole, more suitable terminology would be

- Perimelasma - Apomelasma (from G. Landis)
- Peribothra - Apobothra
- Perinigricon - Aponigricon

The latter used in
R. Schodel et al, Nature 419, 694-696 (2002)