Gravitational Waves

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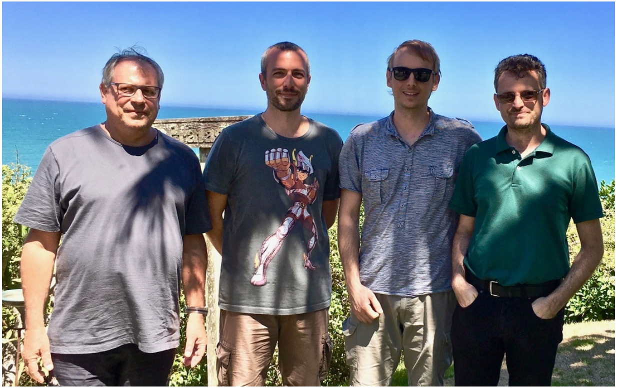

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Outline

Introduction

What are Gravitational Waves?

Gravitational Waves (hereafter GW) are ripples in the spacetime fabric propagating with the speed of light (within the framework of General Relativity). They are generated by accelerated masses.

The parallel with Electrodynamics and Electromagnetic Waves (hereafter ED and EW) is particularly profound and useful. We shall employ it extensively and because of this an overview of ED is offered.

Think about the Coulomb potential among charges and EW. In gravity we have, similarly, Newton potential and GW. Though, mind that Newton's theory is unable to predict GW. They are a prediction of General Relativity (herafter GR) and other gravity theories (which observation of GW helps us to discriminate).

Why are GW difficult to detect?

Gravity is a very weak interaction. As we shall see, the GW amplitude is proportional to:

$$h_{ij} \propto \frac{G}{c^4 r} \frac{\partial^2}{\partial t^2} D_{ij} \ , \tag{1}$$

where D_{ij} is the quadrupole mass tensor. We shall encounter D_{ij} again later. The important point now is that it depends on the mass distribution of a system of orbiting bodies and G/c^4 is a very small quantity:

$$\frac{G}{c^4} \approx 10^{-44} \frac{\mathrm{s}^2}{\mathrm{kg} \cdot \mathrm{m}} \ . \tag{2}$$

Hence we need large and rapidly moving masses in order to have a detectable effect.

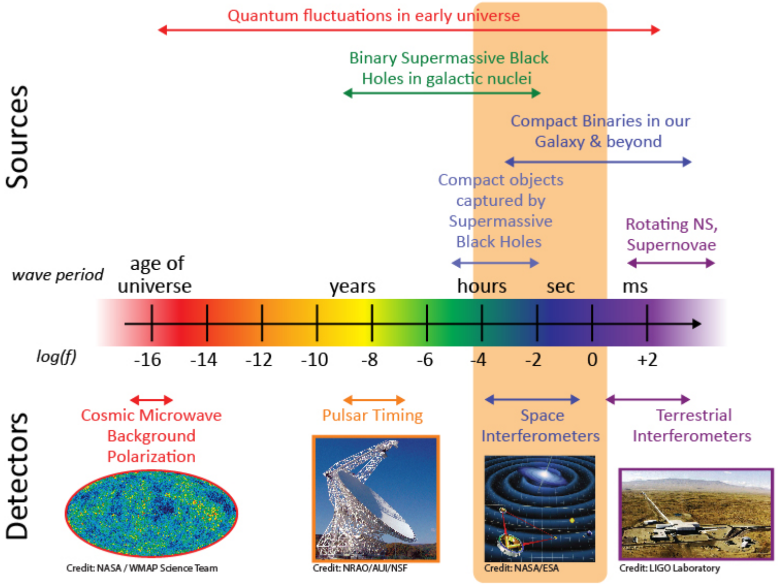
Sources of GW

Large and rapidly moving masses are found in binary systems of **Black Holes** (hereafter BH) and **Neutrons Stars** (hereafter NS).

Unfortunately (or fortunately, depending on the point of view), they are also very far away and the GW signal decays as $1/r$:

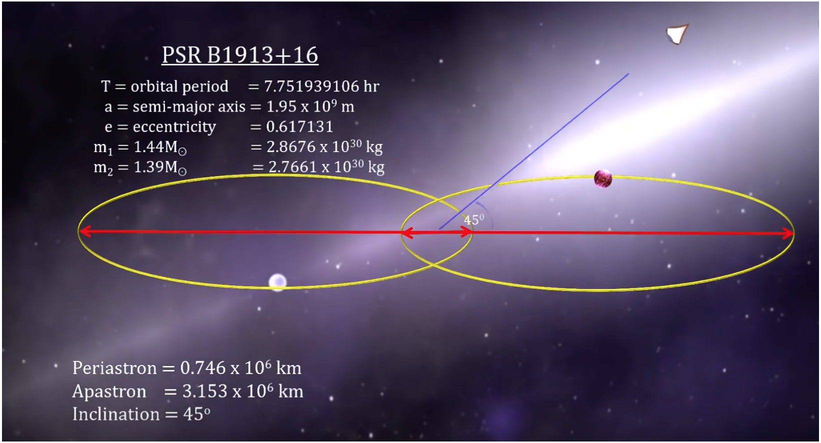
$$h_{ij} \propto \frac{G}{c^4 r} \frac{\partial^2}{\partial t^2} D_{ij} \ , \tag{3}$$

The Gravitational Wave Spectrum



Indirect detection

Hulse-Taylor pulsar (discovered in 1974)



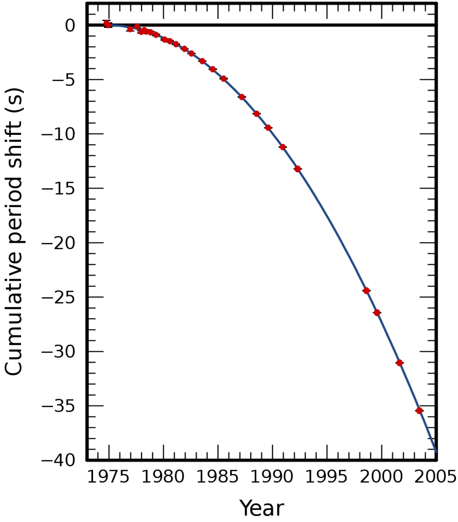
Reduction of the orbital period due to GW emission

This is the first NS-NS binary system discovered in which one of the two NS is an observable pulsar.

The observability of one of the 2 NS (as a pulsar) allows to determine the Keplerian and Post-Keplerian parameter, leading to the discovery of the reduction of the orbital period.

GR calculations match perfectly this reduction, suggesting that the binary system is losing energy via GW emission.

Reduction of the orbital period due to GW emission



Merger

If the orbital period is diminishing, this means that the binary system is shrinking. Eventually, the two NS will merge one into another. Close to merger, we expect the GW signal to be the largest possible.

Direct detections

LIGO-Virgo detections

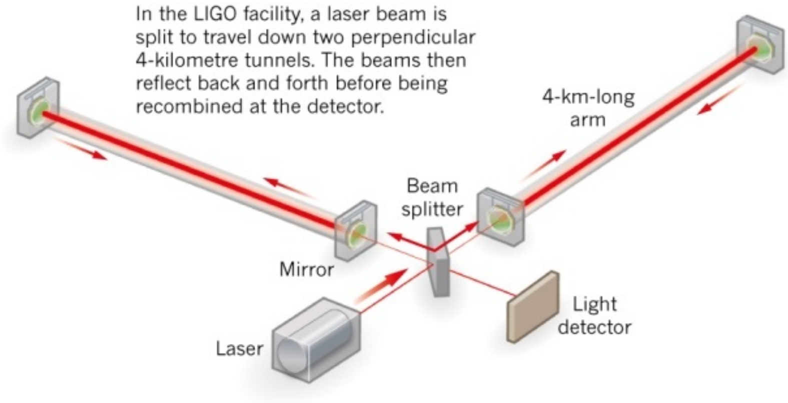
All the following are direct detections of GW resulting from BH-BH mergers:

- GW150914: First GW direct detection
- GW151226
- GW170104
- GW170608
- GW170814

The event GW170817 is the first GW signal coming from a NS-NS merger. Its electromagnetic counterpart was also observed as a Gamma Ray Burst (GRB 170817A), 1.7 seconds after the GW signal.

Michelson Interferometer

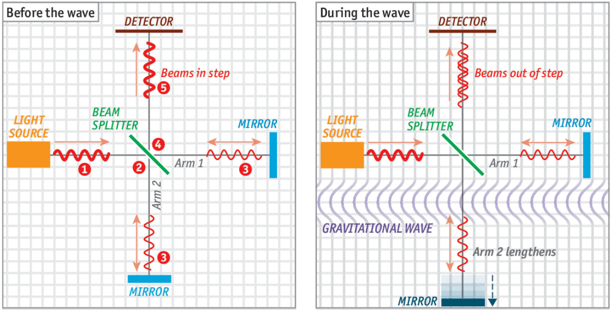
LIGO = Laser Interferometer Gravitational-Wave Observatory



How a GW changes the length of the arms

Catching a wave

How a laser-interferometer observatory works



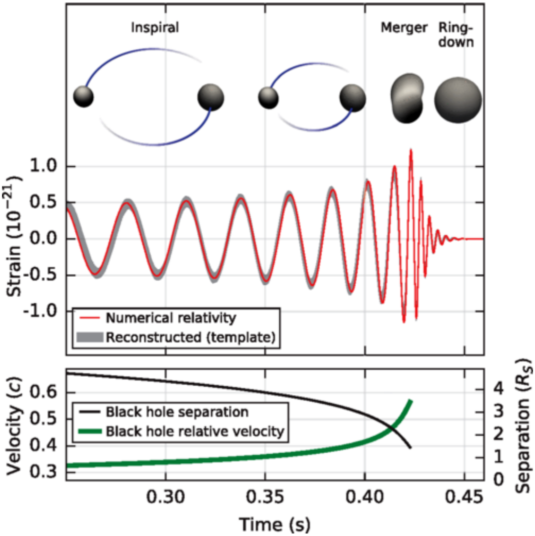
The light source sends out a beam (1) that is divided by a beam splitter (2). The half-beams produced follow paths of identical length (3), reflecting off mirrors to recombine (4), then travel in step to the detector (5).

When a *gravitational wave* arrives, it disturbs space-time, lengthening (in this example) the light's path along **arm 2**; when the beams recombine and arrive at the detector, they are no longer in step.

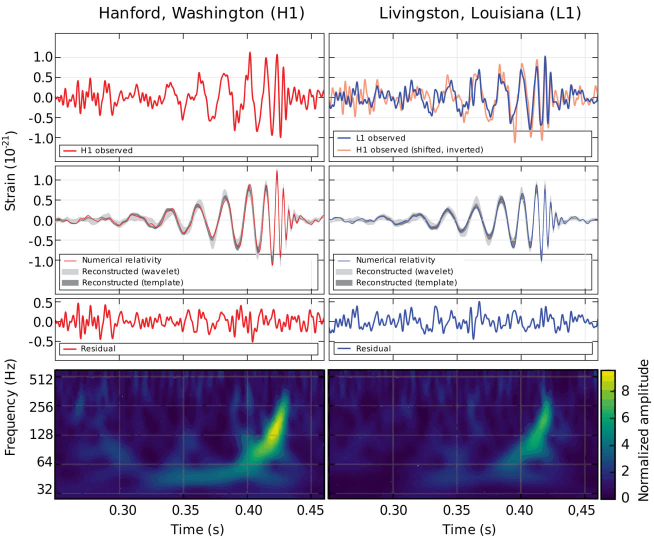
Source: *The Economist*

Economist.com

Typical GW signal



GW150914



Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
512
256
128
64
32
Normalized amplitude
8
6
4
2
0
0.30
0.35
0.40
0.45
0,45
Time (s)

Importance of GW for Astronomy

Multi-messenger astronomy

GW provide a new window through which observe the Universe.

We have always gathered informations mostly by collecting photons (i.e. via EW), and still do.

Some regions of the sky are however obscure, i.e. we do not receive any EW signal from them but this does not mean that nothing is there.

GW allow us to see e.g. BH-BH binaries, in their final moments of existence. There is no EW counterpart to these events. Other potential messengers are neutrinos and cosmic rays.

Electrodynamics

Electrodynamics

Our starting point are Maxwell's equations (ME):

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \text{(Gauss's law)}\ , \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday-Neumman-Lenz law)}\ , \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's law for magnetism)}\ , \tag{6}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \qquad \text{(Ampère's law)}\ . \tag{7}$$

These are usually complemented by the Lorentz Force equation:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\ . \tag{8}$$

Scalar and Vector potentials

Since $\mathbf{B}$ is divergenceless, we can write it in terms of a **Vector Potential**:

$$\mathbf{B} = \nabla \times \mathbf{A} \,. \tag{9}$$

Hence, Faraday's law becomes:

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \,, \tag{10}$$

and being the field between parenthesis irrotational, it can be written as a gradient of a **Scalar Potential**:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \,. \tag{11}$$

Potential formulation of ME

The definitions $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ automatically account for Gauss's law for magnetism and for Faraday's law. One can show that Gauss's law can be written as:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \qquad \text{(Poisson's equation)}\ , \tag{12}$$

and Ampère's law can be written as:

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \mathbf{J}\ . \tag{13}$$

We shall come back later to these equations.

Gauge Freedom

From the definitions $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ it is easy to see that the new potentials:

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda \,, \qquad V' = V - \frac{\partial \lambda}{\partial t} \,, \tag{14}$$

where λ is a generic function, leave unchanged the electric and magnetic fields.

These are **Gauge Transformations** and the invariance of the electric and magnetic fields under them is called **Gauge Invariance**. The freedom that allows us to choose any λ that satisfy the above relations is called **Gauge Freedom**.

ME in vacuum

Consider the source-free ME and apply the curl to the $\nabla \times \mathbf{E}$ equation:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \ . \tag{15}$$

Using vector calculus and the other ME it is straightforward to show that:

$$-\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla^2 \mathbf{E} = 0 \ . \tag{16}$$

Similarly, for **B**:

$$-\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} + \nabla^2 \mathbf{B} = 0 \ . \tag{17}$$

These are **Wave Equations**.

Electromagnetic waves

Electric and magnetic fields oscillate sustaining each other and forming an **Electromagnetic Wave**. They are e.g. light, radiowaves, microwaves, γ-rays, X-rays,...

The speed of propagation of an EW is:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c \, , \tag{18}$$

the speed of light.

From now on we employ:

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \, , \tag{19}$$

which is called **D'Alembert Operator**.

Plane-wave solutions:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \,, \qquad \mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \,, \tag{20}$$

where $\mathbf{k}$ is the **wavenumber** and ω is the **angular frequency of the wave** and they are related by:

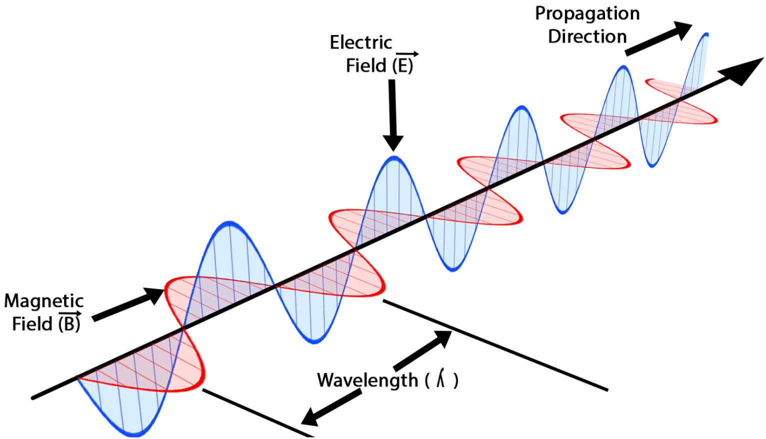
$$|\mathbf{k}| = \omega/c \,. \tag{21}$$

In order to have the real electric and magnetic fields recall to take the real part of the above expressions. Alternatively, you can add the complex conjugate to the above expressions.

It is not difficult to show, using ME in vacuum, that $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ are mutually orthogonal and that $|\mathbf{B}_0| = |\mathbf{E}_0|/c$.

So, EW are **transversal** and do not need a medium in order to propagate.

Electromagnetic Wave



Energy of an EW

The energy flux carried by an EW is given by the modulus of the **Poynting Vector**:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \parallel \mathbf{k} \,. \tag{22}$$

For example, if $\mathbf{E}_0 \parallel \hat{\mathbf{x}}$ and $\mathbf{B}_0 \parallel \hat{\mathbf{y}}$ then we have:

$$S_x = 0 \,, \quad S_y = 0 \,, \quad S_z = \epsilon_0 c E_{0x}^2 \cos^2(\omega t - kz + \phi) \,. \tag{23}$$

Averaged over many periods (π/ω) one gets:

$$\langle S_z \rangle = \epsilon_0 c E_{0x}^2 / 2 \,. \tag{24}$$

Spherical EW

Far from a system of charge we expect an EW to propagate **spherically**. Hence:

$$\mathbf{S} \propto \hat{\mathbf{r}} \, . \tag{25}$$

The EW power through a closed surface $\mathcal{S}$ is:

$$P = \oint_{\mathcal{S}} \mathbf{S} \cdot \mathbf{dA} \, . \tag{26}$$

Since the wave is spherical, choose $\mathcal{S}$ to be a sphere and thus $dA = r^2 d\Omega \hat{\mathbf{r}}$. Then, we must have:

$$|\mathbf{E}| \propto 1/r \, , \quad |\mathbf{B}| \propto 1/r \quad \Rightarrow \quad \langle|\mathbf{S}|\rangle \propto 1/r^2 \, , \tag{27}$$

so that P does not depend on r, which means that energy can be carried to $r \to \infty$ and thus we have an EW.

EW in the potential formulation

Exploiting gauge freedom and choosing λ such that:

$$\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \qquad \text{(Lorentz gauge)}\ , \tag{28}$$

we obtain the following equations for the potentials:

$$\Box \mathbf{A} = -\mu_0 \mathbf{J}\ , \qquad \Box V = -\rho/\epsilon_0\ . \tag{29}$$

In absence of sources, these are again wave equations. Within sources, they are **Inhomogeneous Wave Equations**.

Retarded potentials

The inhomogeneous wave equations can be formally solved in terms of the **Retarded Potentials**:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}', t_R)}{|\mathbf{r} - \mathbf{r}'|} dV' \,, \tag{30}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}', t_R)}{|\mathbf{r} - \mathbf{r}'|} dV' \,, \tag{31}$$

where the integration is performed over the volume V containing charges and currents and:

$$t_R \equiv t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \,, \tag{32}$$

is the **Retarded Time**. This takes into account that the EW signal travels with finite speed c.

Sources of EW

A static charge cannot radiate EW because its electric field lines are radial and $\mathbf{B} = 0$.

A charge moving with constant velocity also cannot radiate because its electric field lines are again radial. Moreover, the electric and magnetic field strengths fall as $1/r^2$. Hence $\langle|\mathbf{S}|\rangle \propto 1/r^4$ and $E \propto 1/r^2$.
The retarded potentials computed in this case are called **Liénard-Wiechert Potentials**.

In order to produce EW we need an accelerating charge.

Dipole Approximation

Assuming $r' \ll r$ and $d \ll c/\omega$, i.e. the size of the distribution of charges is much smaller than the EW wavelength, the potentials can be approximated as:

$$V(\mathbf{r},t) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{r}\cdot(d\mathbf{p}/dt)(t_0)}{rc}\right] , \qquad (33)$$

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi}\frac{(d\mathbf{p}/dt)(t_0)}{r} , \qquad (34)$$

where

$$Q \equiv \int_V \rho(\mathbf{r}', t_0) dV' , \qquad \mathbf{p}(t_0) = \int_V \mathbf{r}'\rho(\mathbf{r}', t_0) dV' , \qquad (35)$$

are the **Monopole Moment** (i.e. the total charge) and **Dipole Moment** of the charge distribution and:

$$t_0 \equiv t - \frac{r}{c} . \qquad (36)$$

Fields in the Dipole Approximation

From the previous solutions we can compute the electric and magnetic fields. Keeping only the $\propto 1/r$ terms:

$$\mathbf{E}(\mathbf{r},t) = \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})\right] = \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}} \,, \tag{37}$$

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi r c}\left[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}\right] = \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\frac{\sin\theta}{r}\hat{\boldsymbol{\phi}} \,. \tag{38}$$

They are orthogonal, as expected and the Poynting vector is:

$$\mathbf{S} = \frac{\mu_0 \ddot{p}(t_0)^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{r}} \,. \tag{39}$$

The irradiated power is then:

$$P = \frac{\mu_0 \ddot{p}(t_0)^2}{6\pi c} \,. \tag{40}$$

Covariant Formulation of Electrodynamics

Recap of Special Relativity

Relativity Principle: Physics is the same in reference frames in relative uniform motion (inertial frames).

The speed of light c is an invariant. Galileo transformations are substituted by **Lorentz Transformations**.

Lorentz transformations contain the usual 3-dimensional rotations and moreover also space-time rotations, better known as **Boosts**.

For example, a boost of velocity V in the $\hat{\mathbf{x}}$ direction is written as:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} , \tag{41}$$

where

$$\beta \equiv \frac{V}{c} , \tag{42}$$

and

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}} , \tag{43}$$

is the **Lorentz Factor**.

Extension of the definition of vector: from 3-vectors, defined under rotations:

$$x'^i = R^i{}_j x^j \ , \qquad (i, j = 1, 2, 3) \tag{44}$$

to 4-vectors:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \ , \qquad (\mu, \nu = 0, 1, 2, 3) \tag{45}$$

defined under Lorentz transformations $\Lambda^\mu{}_\nu$. The Einstein summation convention is employed here: repeated high and low indices are summed.

Position 4-vector in Cartesian coordinates:

$$x^\mu = (ct, x, y, z) \ . \tag{46}$$

Tensors are defined as:

$$T^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma} \ . \tag{47}$$

Minkowski metric

Minkowski metric $\eta_{\mu\nu}$:

$$\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma} \ . \tag{48}$$

In matrix form:

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) \ . \tag{49}$$

Line element:

$$ds^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu} = -c^2dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{50}$$

The line element is invariant under Lorentz transformations.
Raising and lowering indices:

$$x_{\mu} = \eta_{\mu\nu}x^{\nu} \ . \tag{51}$$

ME in $3+1$ spacetime

Defining the **Electromagnetic Field Tensor**:

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E^x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} , \tag{52}$$

and the current 4-vector:

$$J_\nu \equiv (c\rho, J^x, J^y, J^z) , \tag{53}$$

one can show that the ME can be written as:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu , \tag{54}$$

$$\partial_\alpha F^{\mu\nu} + \partial_\mu F^{\nu\alpha} + \partial_\nu F^{\alpha\mu} = 0 , \tag{55}$$

$(\partial_\mu \equiv \partial/\partial x^\mu)$.

The 4-vector potential

Defining the **Potential Four-vector**:

$$A^\mu = (V/c, A_x, A_y, A_z) \ , \tag{56}$$

the electromagnetic field tensor can be written as:

$$F^{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu \ . \tag{57}$$

Gauge-invariance is now expressed as:

$$A^\mu \rightarrow A^\mu + \partial_\mu \lambda \ . \tag{58}$$

Lorentz gauge:

$$\partial_\mu A^\mu = 0 \ . \tag{59}$$

And ME can be written as:

$$\Box A^\mu = -\mu_0 J^\mu \ . \tag{60}$$

Plane-wave solution for A_μ in vacuum

Plane-wave solution:

$$A_\mu = e_\mu e^{ik_\nu x^\nu} + e^*_\mu e^{-ik_\nu x^\nu} \ . \tag{61}$$

The complex conjugate is introduced in order to make A_μ real.
e_μ is the **Polarization Vector**.
Using $\Box A_\mu = 0$ we get:

$$k_\mu k^\mu = 0 \ , \tag{62}$$

i.e. the EW propagates with the speed of light. Using the Lorentz gauge condition $\partial^\mu A_\mu = 0$, we get:

$$k_\mu e^\mu = 0 \ , \tag{63}$$

which gives us transversality. This latter condition tells us that only 3 out or 4 components of e_μ are actually independent.

Exploiting gauge freedom

Now, thanks to gauge freedom we can change A_μ to:

$$A'_\mu = e_\mu e^{ik_\nu x^\nu} + e^*_\mu e^{-ik_\nu x^\nu} + \partial_\mu \lambda \ . \tag{64}$$

But we still want to stay in the Lorentz gauge $\partial^\mu A'_\mu = 0$. This demands that:

$$\Box \lambda = 0 \ , \tag{65}$$

and we choose as solution:

$$\lambda = i\epsilon e^{ik_\nu x^\nu} - i\epsilon^* e^{-ik_\nu x^\nu} \ , \tag{66}$$

so that the new 4-potential reads:

$$A'_\mu = e'_\mu e^{ik_\nu x^\nu} + e'^*_\mu e^{-ik_\nu x^\nu} \ , \tag{67}$$

with

$$e'_\mu = e_\mu - \epsilon k_\mu \ . \tag{68}$$

Two polarizations

Since ϵ is arbitrary, we can choose it in order to fix a component of e'_μ as we please. Hence, only $3 - 1 = 2$ components of the polarization vector e_μ are physically independent. Take, for example:

$$k^\mu = (k, 0, 0, k) \ , \tag{69}$$

i.e. a wave propagating in the z direction. The condition $k^\mu e_\mu = 0$ gives us:

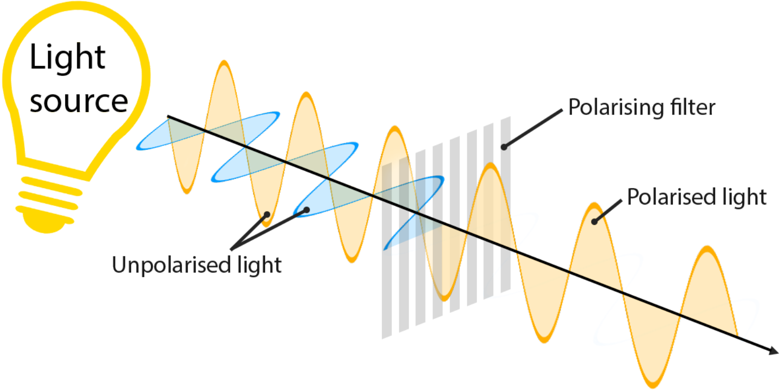
$$e_0 + e_3 = 0 \ , \tag{70}$$

and gauge freedom allows us to establish:

$$e'_3 = e_3 - \epsilon k \ . \tag{71}$$

Choosing ϵ suitably we can make e_0 and e_3 to vanish. Hence, only e_1 and e_2 carry physical significance.

The two polarizations represents physically the two degrees of freedom with which the electric field can oscillate in the plane perpendicular to the direction of propagation.



Helicity

Any plane wave ψ which is transformed into:

$$\psi' = e^{ih\theta}\psi \ , \tag{72}$$

by a rotation of θ about the propagation axis is said to have **helicity** h. For our EW with $k^\mu = (k, 0, 0, k)$ then:

$$e'_\mu = R^\nu{}_\mu e_\nu \ , \tag{73}$$

with:

$$R^\nu{}_\mu \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{74}$$

a rotation about the z axis.

Hence, applying the rotation we have:

$$e'_0 = e_0 \ , \qquad e'_1 = \cos\theta e_1 + \sin\theta e_2 \ , \tag{75}$$

$$e'_2 = -\sin\theta e_1 + \cos\theta e_2 \ , \qquad e'_3 = e_3 \ . \tag{76}$$

Defining:

$$e_\pm \equiv e_1 \mp i e_2 \ , \tag{77}$$

we have:

$$e'_\pm = e^{\pm i\theta} e_\pm \ . \tag{78}$$

Hence, an EW can be decomposed into two parts with helicity ± 1 and one with helicity zero. The latter however is not physical since it can be made vanishing thanks to gauge freedom.

References

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Gravitational Waves

General Relativity

Tensor are now defined with respect to any coordinate transformation:

$$T'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} T^{\rho\sigma} \ . \tag{79}$$

Einstein equations (EE):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \ . \tag{80}$$

Highly non-linear and very difficult to solve for strong fields. In this case we need **Numerical General Relativity**.

We shall treat them in the **Weak-Field Limit** only, linearizing them. So, mind that the GW we are going to describe are also subject to this regime.

Linearized EE

Weak field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| \ll 1 \ . \tag{81}$$

Keeping only the terms linear in $h_{\mu\nu}$ one can show that:

$$R_{\mu\nu} = -\frac{1}{2}\left(\partial_\gamma\partial_\nu h^\gamma{}_\mu + \partial_\gamma\partial_\mu h^\gamma{}_\nu - \partial_\mu\partial_\nu h^\gamma{}_\gamma - \Box h_{\mu\nu}\right) \ , \tag{82}$$

and the EE can be thus written:

$$\Box h_{\mu\nu} - \partial_\gamma\partial_\nu h^\gamma{}_\mu - \partial_\gamma\partial_\mu h^\gamma{}_\nu + \partial_\mu\partial_\nu h^\gamma{}_\gamma = -\frac{16\pi G}{c^4} S_{\mu\nu} \ , \tag{83}$$

with $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda{}_\lambda$. Note that $h^\mu{}_\nu = \eta^{\mu\rho}h_{\rho\nu}$, since we are in the weak-field limit.

General covariance

GR is characterized by **General Covariance**, i.e. EE are valid in any reference frame. Thus, consider the coordinate transformation:

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x) \ , \tag{84}$$

with small $\varepsilon^\mu(x)$, so that the field remains weak. Since the metric is a tensor, it transforms as:

$$g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} g^{\rho\sigma} \ . \tag{85}$$

And thus we have:

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \varepsilon_\mu - \partial_\mu \varepsilon_\nu \ , \tag{86}$$

with $\varepsilon_\mu \equiv \eta_{\mu\nu}\varepsilon^\nu$. So, if $h_{\mu\nu}$ is solution of the EE, then $h'_{\mu\nu}$ also is. This is reminiscent of the gauge invariance of EM (put A_μ instead of $h_{\mu\nu}$).

Harmonic coordinate system

In EM we exploit gauge invariance in order to obtain a wave equation for A_μ. The Lorentz gauge is particularly appropriated for this.
In GR we do something similar, by choosing the **Harmonic Coordinate System**, for which:

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \,, \tag{87}$$

and which in the weak-field approximation gives us:

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\lambda{}_\lambda \,. \tag{88}$$

This choice is equivalent to make a coordinate transformation with:

$$\Box\varepsilon_\nu \equiv \partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\lambda{}_\lambda \,. \tag{89}$$

Linearized Gravity Solution

The harmonic condition allows us to put the EE in the following form:

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} S_{\mu\nu} \; , \tag{90}$$

i.e. as an inhomogeneous wave-equation, similar to the one we found for A_μ. We already know how to solve this:

$$\boxed{h_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{c^4} \int \frac{S_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|/c)}{|\mathbf{x}' - \mathbf{x}|} dV'} \tag{91}$$

being this the retarded potential and $t - |\mathbf{x}' - \mathbf{x}|/c$ the retarded time. This shows that gravity propagates with the speed of light, differently from Newton's theory where the action at a distance is instantaneous.

The above solution for $h_{\mu\nu}$ contains, for example, Newton's theory. Suppose a static distribution of masses, with mass density $\mu(\mathbf{x}')$. Then:

$$S_{00} = \frac{\mu c^2}{2} \ , \tag{92}$$

is the only nonvanishing component of $S_{\mu\nu}$, and

$$h_{00}(\mathbf{x}) = \frac{2G}{c^2} \int \frac{\mu(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} dV' \ . \tag{93}$$

This is the expression for the Newtonian potential generated by a distribution of masses (times 2 and divided by c^2). For a single point mass M, $\mu(\mathbf{x}') = \delta^{(3)}(\mathbf{x}')$ and so:

$$h_{00}(\mathbf{x}) = \frac{2GM}{c^2 r} \ . \tag{94}$$

This is not a GW, of course.

Linearized GW

We interpret as a GW solutions of the homogeneous wave equation, in the harmonic gauge:

$$\Box h_{\mu\nu} = 0 \,, \qquad \partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\lambda{}_\lambda \,, \tag{95}$$

because this has a general solution as a superposition of plane waves which can propagate to infinity:

$$h_{\mu\nu}(x) = e_{\mu\nu} e^{ik_\lambda x^\lambda} + e^*_{\mu\nu} e^{-ik_\lambda x^\lambda} \,, \tag{96}$$

with:

$$k_\mu k^\mu = 0 \,, \qquad k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\lambda{}_\lambda \,, \tag{97}$$

and $e_{\mu\nu} = e_{\nu\mu}$ is the **Polarization Tensor**. In general $h_{\mu\nu}$ (and consequently $e_{\mu\nu}$) has 10 independent components. However, this number is lowered to 6 due to the relation $k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\lambda{}_\lambda$, coming from the harmonic gauge.

We have however a residual gauge freedom, because we can choose another $h'_{\mu\nu}$, related to the former by

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \varepsilon_\mu - \partial_\mu \varepsilon_\nu \ , \tag{98}$$

with

$$\Box \varepsilon_\nu \equiv \partial_\mu h^\mu{}_\nu - \frac{1}{2} \partial_\nu h^\lambda{}_\lambda \ . \tag{99}$$

This new $h'_{\mu\nu}$ still solves EE and the harmonic gauge condition, so it describes the same GW. Let's choose:

$$\varepsilon^\mu(x) = i\varepsilon^\mu e^{ik_\lambda x^\lambda} - i\varepsilon^{*\mu} e^{-ik_\lambda x^\lambda} \ . \tag{100}$$

Then:

$$h'_{\mu\nu}(x) = e'_{\mu\nu} e^{ik_\lambda x^\lambda} + e'^{*}_{\mu\nu} e^{-ik_\lambda x^\lambda} \ , \tag{101}$$

with

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu \ . \tag{102}$$

We have freedom of suitably choosing the four ε_μ, making thus 4 out of the 6 components of $e_{\mu\nu}$ to vanish. Hence, only 2 out of 10 components are physically significant. Consider:

$$k^\mu = (k, 0, 0, k) \ . \tag{103}$$

Then $k^\mu e_{\mu\nu} = \frac{1}{2} k_\nu e^\lambda{}_\lambda$ tells us that:

$$e_{00} + e_{30} = -(-e_{00} + e_{11} + e_{22} + e_{33})/2 \ , \tag{104}$$

$$e_{01} + e_{31} = 0 \ , \qquad e_{02} + e_{32} = 0 \ , \tag{105}$$

$$e_{03} + e_{33} = (-e_{00} + e_{11} + e_{22} + e_{33})/2 \ . \tag{106}$$

Therefore:

$$e_{01} = -e_{31} \ , \quad e_{02} = -e_{32} \ , \quad e_{03} = -(e_{00} + e_{33})/2 \ , \quad e_{22} = -e_{11} \ . \tag{107}$$

Moreover, the residual gauge freedom allows us to modify the 6 independent components as follows:

$$e'_{00} = e_{00} - 2k\varepsilon_0 \ , \qquad e'_{11} = e_{11} \ , \tag{108}$$

$$e'_{12} = e_{12} \ , \qquad e'_{13} = e_{13} + k\varepsilon_1 \ , \tag{109}$$

$$e'_{23} = e_{23} + k\varepsilon_2 \ , \qquad e'_{33} = e_{33} + 2k\varepsilon_3 \ . \tag{110}$$

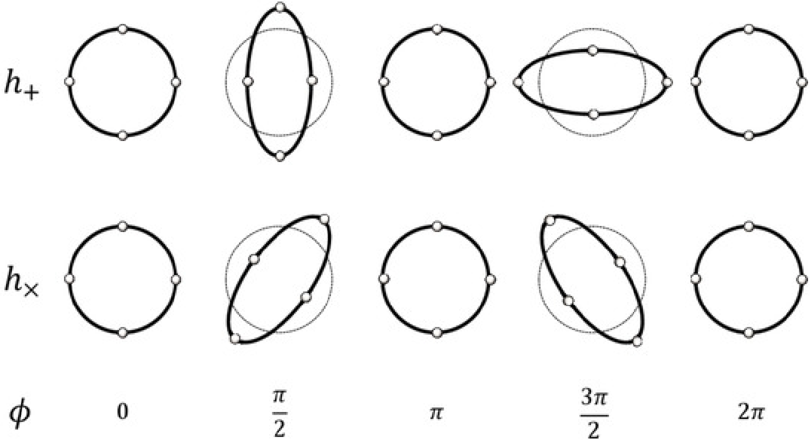
We can choose ε_μ such that all the components above vanish, except for e_{11} and e_{12}, which are the physical degrees of freedom of a GW representing its polarizations.

Typically the following notation is used in the literature:

$$e_{11} \equiv h_+ \ , \qquad e_{12} \equiv h_\times \ . \tag{111}$$

GW polarization

GW polarization is more complicated than EW one because $e_{\mu\nu}$ is a tensor.



Helicity of a GW

Performing a rotation about the direction of propagation, as we did for the EW case:

$$e'_{\mu\nu} = R_\mu{}^\rho R_\nu{}^\sigma e_{\rho\sigma} \ , \tag{112}$$

we get that:

$$e'_{00} = e_{00} \ , \qquad e'_{33} = e_{33} \ , \tag{113}$$

$$f'_\pm = e^{\pm i\theta} f_\pm \ , \qquad e'_\pm = e^{\pm 2i\theta} e_\pm \ , \tag{114}$$

with

$$f_\pm \equiv e_{31} \mp i e_{32} = -e_{01} \pm i e_{02} \ , \tag{115}$$

$$e_\pm \equiv e_{11} \mp i e_{12} = -e_{22} \pm i e_{12} \ . \tag{116}$$

Hence, the physically significant components are those with helicity ± 2.

Energy and momentum of GW

For EW we defined energy and momentum thought the Poynting vector. What is the correspondent quantity in GR?

The energy-momentum of a GW can be computed by making us of the **Pseudo-Tensor of the Gravitational Field**. We do not enter in detailed calculations here (see e.g. Weinberg, section 10.3 or Chapter 13 of Landau-Lifshitz), but simply state the result:

$$t_{\mu\nu} = \frac{c^4}{32\pi G} h^{\lambda}{}_{\rho,\mu} h^{\rho}{}_{\lambda,\nu} \ . \tag{117}$$

For a plane-wave GW propagating in the $\hat{\mathbf{z}}$ direction:

$$\langle t_{\mu\nu} \rangle = \frac{c^4 k_\mu k_\nu}{8\pi G} \left(|e_{11}|^2 + |e_{12}|^2 \right) = \frac{c^4 k_\mu k_\nu}{16\pi G} \left(|e_+|^2 + |e_-|^2 \right) \ . \tag{118}$$

The average is made over spacetime regions much larger than $1/k$ (this allows to kill all the imaginary exponentials, since are oscillating functions).

Radiation of GW

Let us recover the solution for $h_{\mu\nu}$ in presence of matter, i.e.

$$h_{\mu\nu}(\mathbf{x},t) = \frac{4G}{c^4}\int \frac{S_{\mu\nu}(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|/c)}{|\mathbf{x}'-\mathbf{x}|}dV' \,. \tag{119}$$

Recall that this solution satisfies the harmonic gauge condition

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\lambda{}_\lambda \,. \tag{120}$$

It is convenient to work with:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\lambda{}_\lambda \,. \tag{121}$$

The solution for $\bar{h}_{\mu\nu}$ is then:

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|/c)}{|\mathbf{x}' - \mathbf{x}|} dV' \ , \tag{122}$$

i.e. the stress-energy momentum appears again!
The harmonic gauge condition for $\bar{h}_{\mu\nu}$ is now:

$$\partial_\mu \bar{h}^\mu{}_\nu = 0 \ , \tag{123}$$

which applied to the solution gives us:

$$\partial_\mu T^\mu{}_\nu = 0 \ , \tag{124}$$

replacing the usual GR relation $\nabla_\mu T^\mu{}_\nu = 0$.

Now we do basically what we did for EW. We go in the **Wave Zone**, meaning $r' \ll r$ and $\omega r' \ll c$, and so expand:

$$\frac{1}{|\mathbf{x}' - \mathbf{x}|} \approx \frac{1}{r}\left(1 + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^2}\right) , \tag{125}$$

and

$$T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|/c) \approx T_{\mu\nu}\left(\mathbf{x}', t - \frac{r}{c} + \frac{\mathbf{x}\cdot\mathbf{x}'}{cr}\right) , \tag{126}$$

and we expand again the energy-momentum tensor about $t_0 \equiv t - r/c$:

$$T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|/c) \approx T_{\mu\nu}(\mathbf{x}', t_0) + \partial_t T_{\mu\nu}(\mathbf{x}', t_0)\frac{\mathbf{x}\cdot\mathbf{x}'}{cr} , \tag{127}$$

dropping higher order contributions.

Let's consider the $T_{\mu\nu}$ of a system of bodies and keep just the dominant term in the previous expansion. This amounts to assume that the velocities of the bodies are small $v/c \ll 1$. Then we have:

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) \approx \frac{4G}{c^4 r} \int T_{\mu\nu}(\mathbf{x}', t_0) dV' \, . \tag{128}$$

We now write the right hand side in a more suggestive form. From $\partial_\mu T^\mu{}_\nu = \eta^{\mu\rho}\partial_\rho T_{\mu\nu} = 0$ we get:

$$\partial_j T_{ij} - \partial_0 T_{i0} = 0 \, , \qquad \partial_j T_{0j} - \partial_0 T_{00} = 0 \, . \tag{129}$$

Multiply the first equation by x^k and integrate over all space:

$$\partial_0 \int T_{i0} x^k dV = \int (\partial_j T_{ij}) x^k dV = \int \partial_j (T_{ij} x^k) dV - \int T_{ik} dV \, . \tag{130}$$

Hence, dropping the surface term:

$$\int T_{ik} dV = -\frac{1}{2}\partial_0 \int (T_{i0}x^k + T_{k0}x^i) dV \ . \tag{131}$$

From the other equation we have, multiplying by $x^k x^l$:

$$\partial_0 \int T_{00} x^k x^l dV = \int (\partial_j T_{0j}) x^k x^l dV \ , \tag{132}$$

and integrating by parts we have:

$$\partial_0 \int T_{00} x^k x^l dV = -\int (T_{l0}x^k + T_{k0}x^l) dV \ . \tag{133}$$

So, we can write:

$$\int T_{ik} dV = \frac{1}{2}\frac{\partial^2}{\partial x_0^2} \int T_{00} x^k x^l dV \ . \tag{134}$$

The T_{00} component is the mass energy density μc^2, then we can write:

$$\bar{h}_{ik} = \frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2}\int \mu x^k x^l dV \; . \tag{135}$$

For our plane GW, propagating in the $\hat{\mathbf{z}}$ direction, we can then write:

$$h_{ik} = \frac{2G}{3c^4 r}\frac{\partial^2}{\partial t^2}\int \mu(3x^k x^l - \delta_{ij} r^2) dV \equiv \frac{2G}{3c^4 r}\ddot{D}_{ij} \; , \tag{136}$$

where we have introduced the **Mass Quadrupole Tensor** D_{ij}. Note that we have extracted from the full solution for $h_{\mu\nu}$ only the part corresponding to the GW.

So we need at least a quadrupole moment of the mass distribution and we need it to be time-dependent in order to produce GW.

Energy loss via emission of GW

The energy flux in the direction $\hat{\mathbf{z}}$ can be computed as:

$$ct_{0z} = \frac{G}{36\pi c^5 r^2}\left(\dddot{D}_{11}^2 + \dddot{D}_{12}^2\right) . \tag{137}$$

This can be generalized to any direction, giving the infinitesimal intensity of a GW into a solid angle $d\Omega$:

$$dI = \frac{G}{72\pi c^5}\left(\dddot{D}_{\mu\nu}e_{\mu\nu}\right)^2 d\Omega . \tag{138}$$

Averaging over all polarizations and integrating over the solid angle we get:

$$-\frac{d\mathcal{E}}{dt} = \frac{G}{45c^5}\dddot{D}_{ij}^2 . \tag{139}$$

It is a very small energy loss!

Energy loss in GW in a binary system

Problems at the end of Chapter 13 of Landau-Lifshitz

Assuming circular trajectories:

$$-\frac{d\mathcal{E}}{dt} = \frac{32G^4 m_1^2 m_2^2 (m_1 + m_2)}{45c^5 r^2} . \tag{140}$$

From the virial theorem we know that:

$$\mathcal{E} = -\frac{G m_1 m_2}{2r} , \tag{141}$$

and hence the orbital radius shrinks with time at the rate:

$$\dot{r} = -\frac{64G^3 m_1 m_2 (m_1 + m_2)}{5c^5 r^3} , \tag{142}$$

leading eventually to the merger of the two bodies.
Again, beware that these calculations are valid only in the weak-field regime. In the actual merger, fields are very strong and thus numerical General Relativity is needed.

Hulse-Taylor binary

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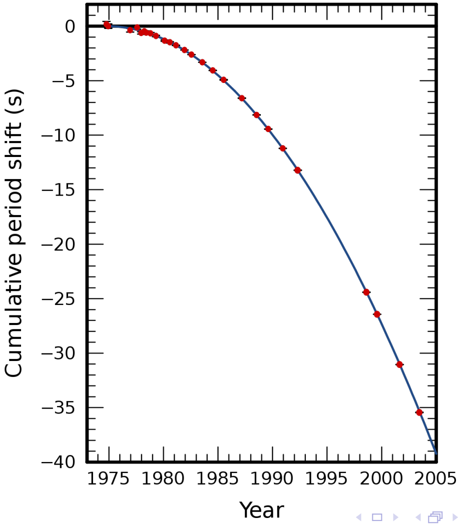

Chirp Mass

https://arxiv.org/pdf/gr-qc/9402014.pdf

The **Chirp Mass** is the combination:

$$\mathcal{M} \equiv \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \,, \tag{143}$$

which is relevant in the time-evolution of a GW waveform, at the leading Post-Newtonian order:

$$\frac{df}{dt} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f^{11/3} \,, \tag{144}$$

where f is the frequency of the GW.
Measuring thus how the frequency evolves with time, we are able to draw information on the mass of the binary system.

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