Gamma, Zeta and Prime Numbers

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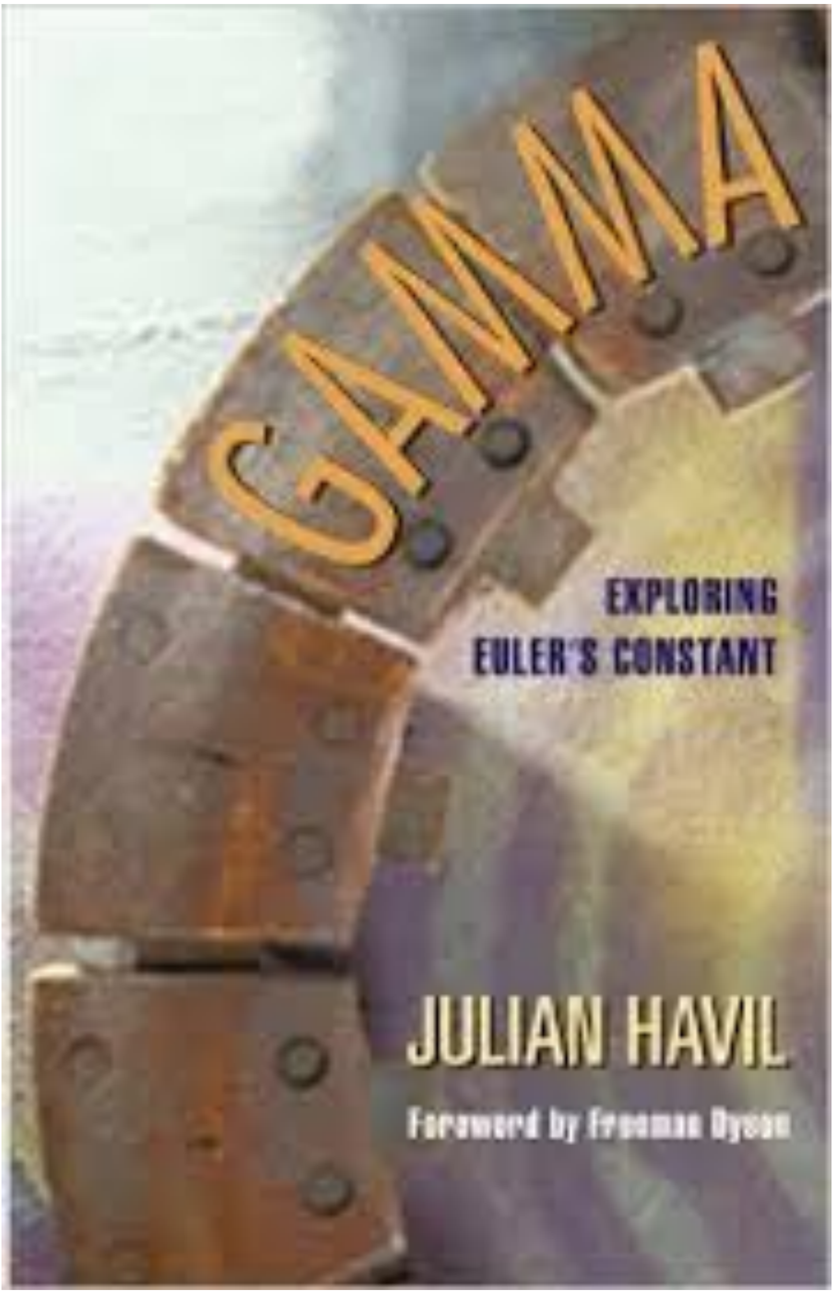
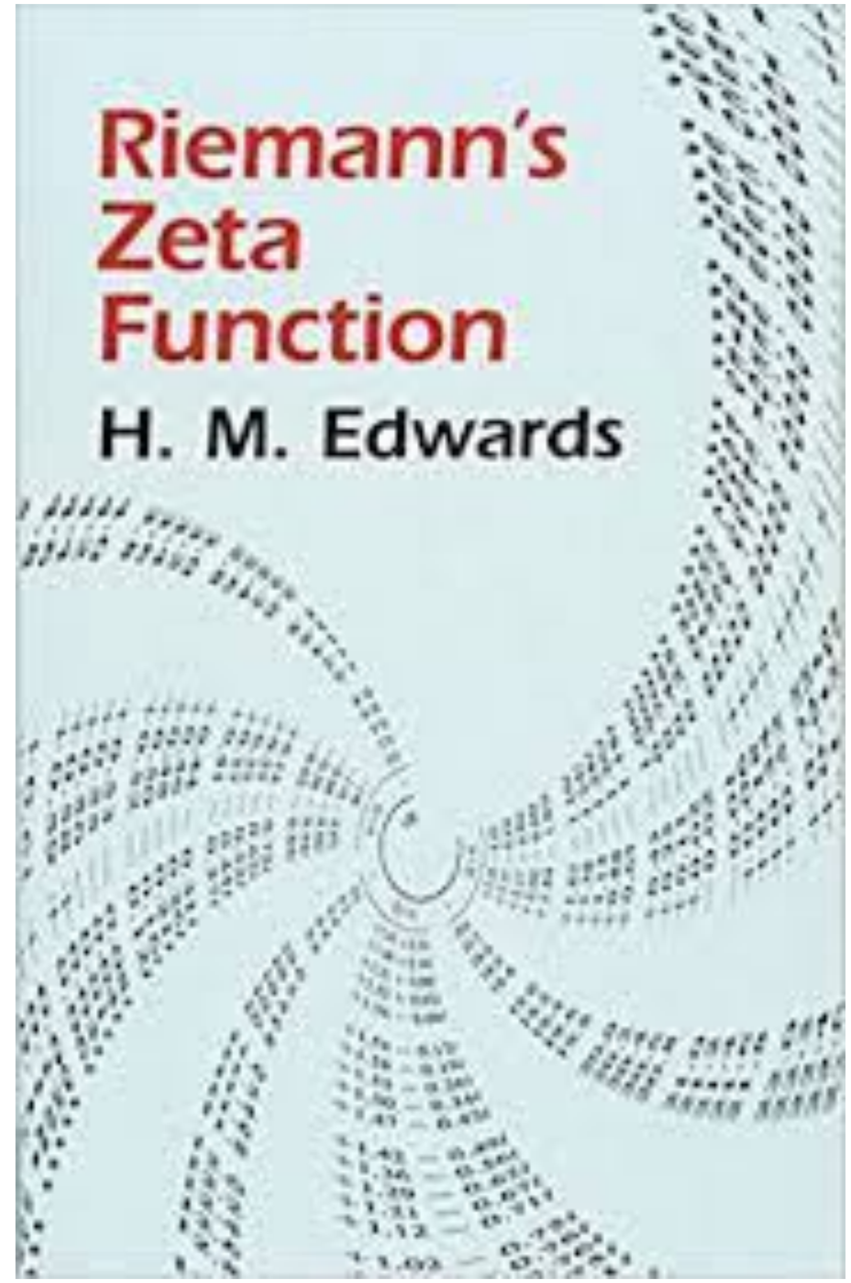

GAMMA
EXPLORING
EULER'S CONSTANT
JULIAN HAVIL
Foreword by Freeman Dyson


Riemann's
Zeta
Function
H. M. Edwards

Leonard Euler

Euler's Gamma

Definition:

$$\gamma = \lim_{n \to \infty} \left(H_n - \ln n \right)$$

Harmonic Numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Harmonic Series

$$H_\infty = \sum_{r=1}^{\infty} \frac{1}{r}$$

And its three fundamental properties:

- H_∞ diverges;
- H_n is non-integral, apart from $n = 1$;
- H_n is almost always a non-terminating decimal ($H_1 = 1$, $H_2 = 1.5$ and $H_6 = 2.45$).

Divergence of the Harmonic Series

Nicole Oresme (1325-1382)

- Development of the French language;
- Taxation Theory;
- He taught heliocentric theory over 100 years before Copernicus;
- He suggested graphing equations nearly 200 years before Descartes;
- Treatise *De Moneta*;
- Probably the first to use “+”;
- In *Algorismus Proportionum* he introduces fractional and negative powers

Oresme's proof

$$
\begin{aligned}
H_\infty = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
+ \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \cdots \\
> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\
+ \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \cdots \\
= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\end{aligned}
$$

Another, very elegant, proof

$$
\begin{aligned}
H_\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots &= \frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \cdots \\
= \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots \\
< \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots \\
= H_\infty
\end{aligned}
$$

Provide your own proof!

How fast the Harmonic Series diverges?

Use Mathematica `HarmonicNumber[]`

$$H_{1000} = 7.486\ldots$$
$$H_{10^6} = 14.3927\ldots$$
$$H_{10^7} = 16.6953\ldots$$

Para somar os primeiros 10 milhões de termos Mathematica demora 15 segundos.

In order to have $H_n > 100$ one must have $n > 1.509 \times 10^{43}$.

Using the area below the function $1/x$, we can estimate how fast the Harmonic Series diverges:

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\int_1^\infty \frac{dx}{x}<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}\,.$$

hence

$$H_n-1<\ln n<H_n-\frac{1}{n}\,,$$

or, equivalently

$$\ln n+\frac{1}{n}<H_n<\ln n+1\,.$$

Therefore:

$$\frac{1}{n}<H_n-\ln n<1\,.$$

The limit $n\to\infty$ exists and gives:

$$\gamma=0.577218\ldots\ .$$

Euler called it C and considered it "worthy of serious consideration". Still today it is not known if γ is irrational, let alone transcendental.

Sub-Harmonic Series

These are obtained by culling specific terms from the Harmonic Series. The idea is that if we cull enough terms we could force convergence.

Culling odd denominators:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)\ ,$$

or even ones:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots>1+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$$

$$=1+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)\ ,$$

is clearly not enough!

Sub-Harmonic Series of Primes

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots$$

There are infinite terms because there are infinite primes. See e.g.:

Euclid's proof
Erdös's proof

Euler showed that the series of primes diverges!

Use Mathematica `Prime[n]` in order to get the n-th prime. The 100000 prime is 1299709.

```
N[Sum[1/Prime[i], {i, 1, 100000}]]
```

gives 2.90615

Euclid's proof using *Reductio ad absurdum*

Suppose that the number of primes is finite and N is the biggest of them. Multiply all the primes and add 1:
$P = 1 + 2 \times 3 \times 5 \times 7 \times \cdots \times N$.
Either P is prime, but since it is bigger than N this would contradict our initial hypothesis.
Therefore P is not prime. Yet, when divided by any of the primes up to N there is a remainder of 1.

Erdös's proof

Let N be any positive integer and $p_1, p_2, p_3, \ldots p_n$ the complete set of primes $\leq N$. Any positive integer less than or equal to N can be written in the form:

$$p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_n^{e_n} \times m^2 \ ,$$

with $e_i \in 0, 1$ and $m \leq \sqrt{N}$.
Thus, the integers less than or equal to N can be chosen in at most

$$2^n \times \sqrt{N} \ ,$$

and therefore

$$N \leq 2^n \times \sqrt{N} \ ,$$

from which one gets

$$n \geq \frac{1}{2} \log_2 N \ .$$

Since N is unbounded, so must the number of primes be!

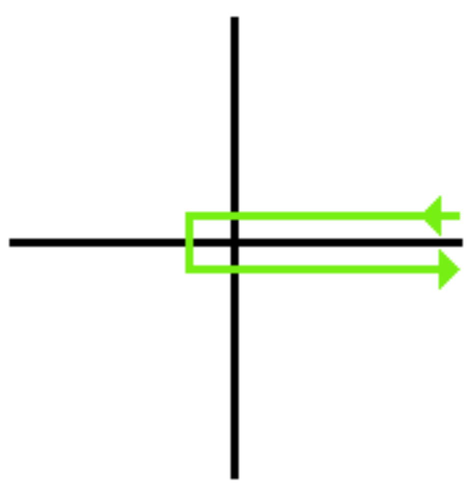
Riemann's Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\ , \ \text{for } s > 1$$

Long-known for $s > 1$ integer, extended by Riemann to complex s.

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i \oint_H \frac{(-x)^{s-1}}{e^x - 1} dx$$

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$



For non-positive integer:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \ ,$$

where the Bernoulli numbers are defined by:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \ .$$

This allows us to calculate $\zeta(0) = -1/2$ and to find the **trivial zeroes** of ζ, which are the ones corresponding to even negative integer (the odd Bernoulli numbers are all vanishing, except for the first).

The Basel problem

The case of

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

shows us once more Euler's genius. He considers the standard Taylor expansion of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and interpret $\sin x$ as a "polynomial of infinite degree" with roots $0, \pm\pi, \pm 2\pi, \ldots$:

$$\sin x = x(x^2 - \pi^2)(x^2 - 4\pi^2)(x^2 - 9\pi^2)\cdots$$

which can be rewritten as

$$\sin x = Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

with $A = 1$, since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Now equate the infinite product for $\sin x$ with its Taylor expansion:

$$x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

and equate the coefficient of x^3:

$$-\frac{1}{3!} = -\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\cdots$$

or

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$$

Euler’s wonderful identity

Since for any positive integer r we can write:

$$r = 2^{r_1} 3^{r_2} 5^{r_3} \cdots ,$$

then for $x > 1$:

$$\zeta(x) = \sum_{r=1}^{\infty} \frac{1}{r^x} = \sum_{r_1, r_2, r_3, \cdots \geq 0} \frac{1}{2^{x r_1} 3^{x r_2} 5^{x r_3} \ldots}$$

$$= \left(\sum_{r_1 \geq 0} \frac{1}{2^{x r_1}} \right) \left(\sum_{r_1 \geq 0} \frac{1}{3^{x r_1}} \right) \left(\sum_{r_1 \geq 0} \frac{1}{5^{x r_1}} \right) \cdots$$

$$= \prod_{p \text{ prime}} \left(\sum_{\alpha=0}^{\infty} \frac{1}{p^{x\alpha}} \right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-x}} .$$

For $x > 1$:

$$\boxed{\zeta(x) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-x}}}$$

Note the following:

- For $x \to 1$, ζ becomes the Harmonic Series, which diverges and thus the number of primes must be infinite.
- As you remember:

$$\zeta(2) = \frac{\pi^2}{6} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-2}}\,.$$

If the number of primes was finite, the right hand side would be rational. However, π^2 is irrational (proved by Legendre in 1796), therefore there are infinite primes.

(Note that π is transcendental, but we don't know if π^2 also is.)

From Euler's formula analytic number theory came into being. Using it, one can also establish that:

$$\lim_{n \to 0} \left(\sum_{p<n, p \text{ prime}} \frac{1}{p} - \ln \ln n \right) = \gamma + \sum_{p \text{ prime}} \left[\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] = 0.2614972128\ldots\,.$$

Problems with primes

Three fundamental questions about primes:

- Is a given number prime?
- How many primes are there less than or equal to a given number x?
- What is the xth prime?

Prime counting function:

$$\boxed{\pi(x)}$$

It gives the number of primes less than o equal to x.

A useful formula which determines $\pi(x)$ is based on a theorem by John Wilson (1741-1793), which however was proved by Lagrange in 1773. It states that:

$$\boxed{\text{If } p \text{ is prime, then } p \text{ divides } (p-1)!+1}$$

Therefore, in 1964 C. P. Willans proposed:

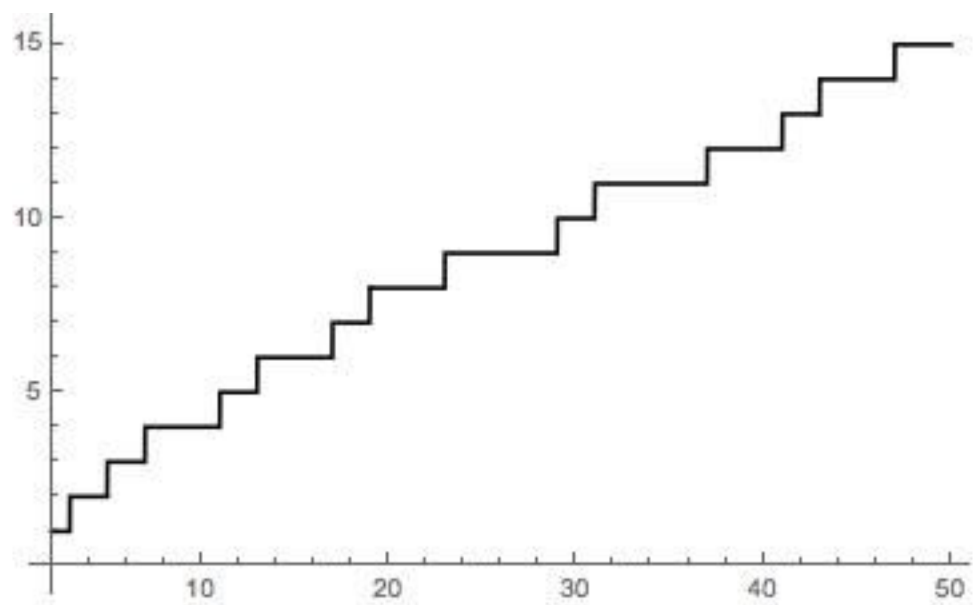
$$\pi(x) = -1 + \sum_{n=1}^{x} \left\lfloor \cos^2 \pi \frac{(n-1)!+1}{n} \right\rfloor \ .$$

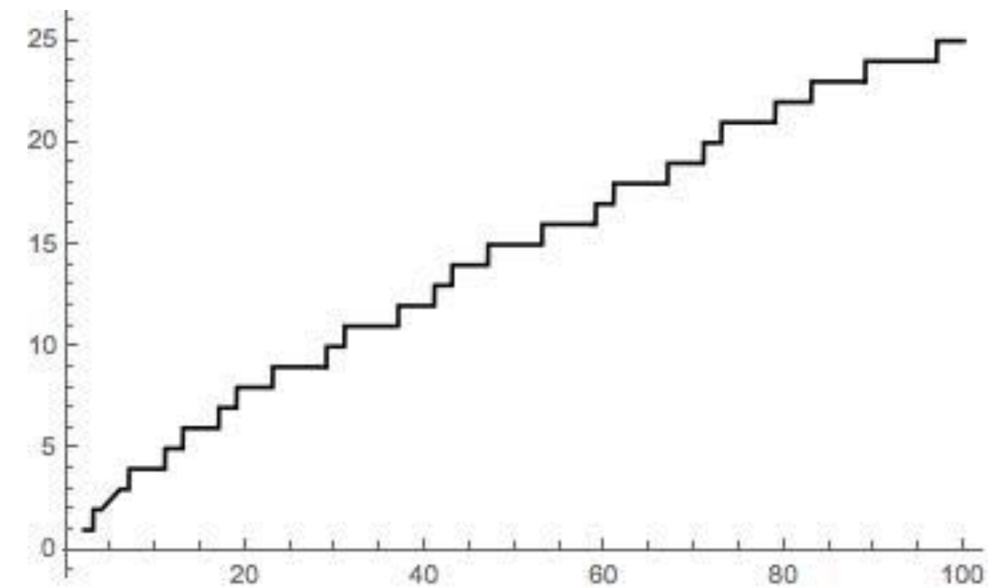
However, we are rather more interested in an approximation

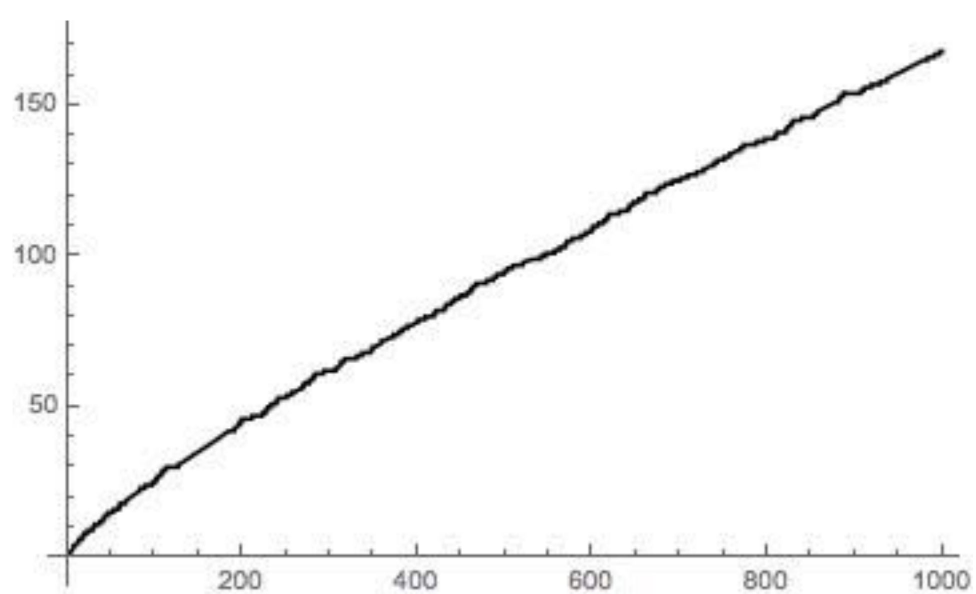
$$\pi(x) = f(x) + \epsilon_x \ ,$$

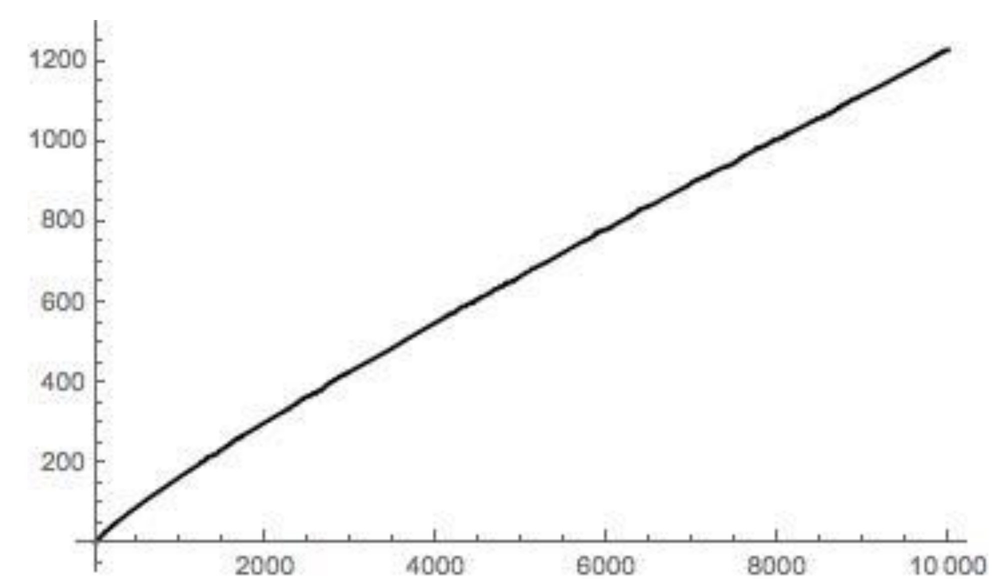
for some easily computable function $f(x)$ and an error term $\epsilon_x \to 0$ for large x. This is analytic number theory and we are seeking a pattern into the distribution of primes.

Mathematica has the built-in `PrimePi[x]` function. Let's try to do some plots.



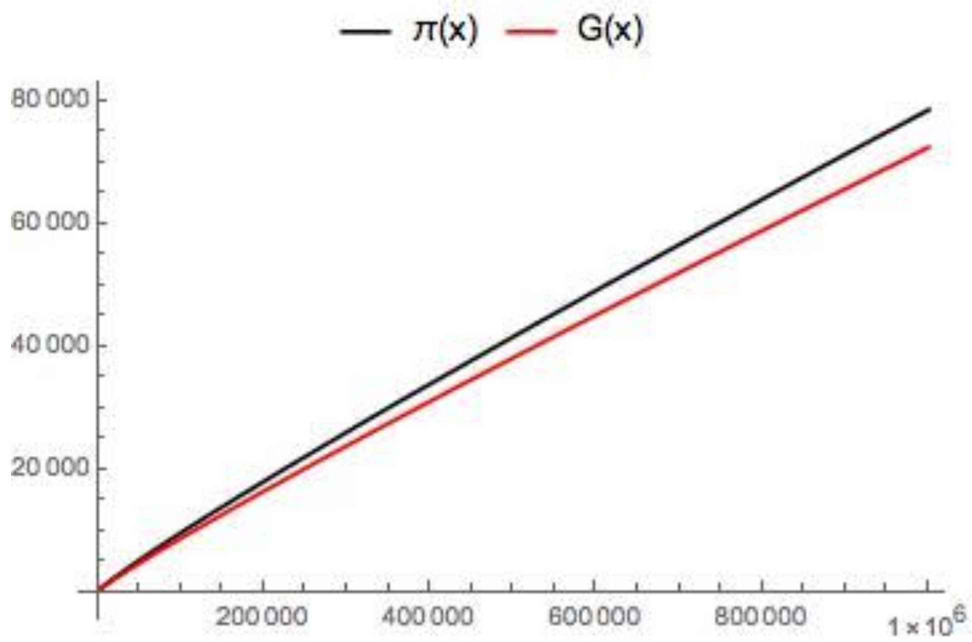


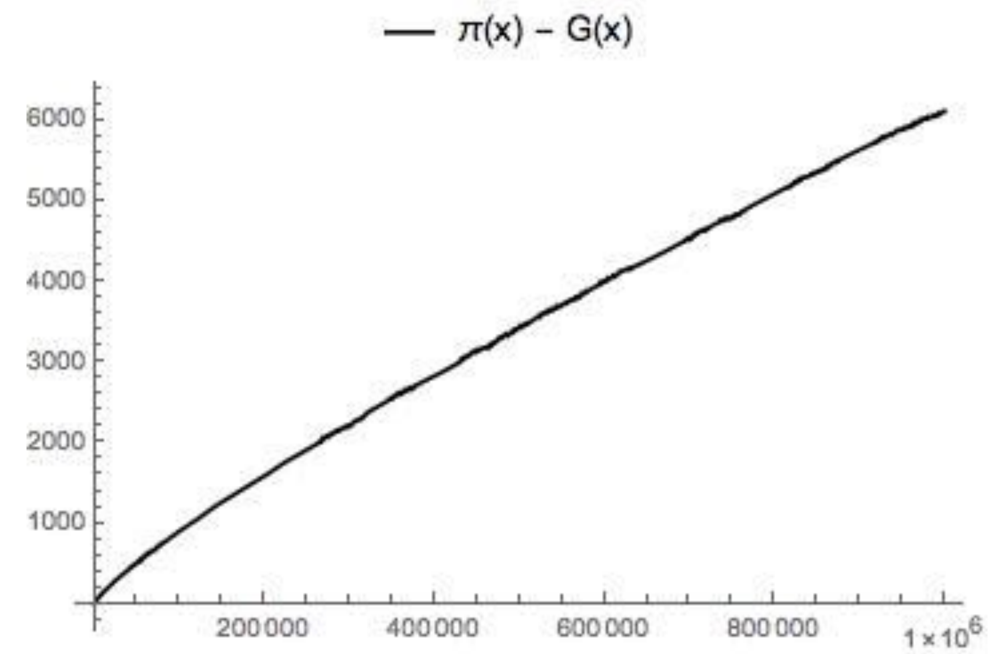


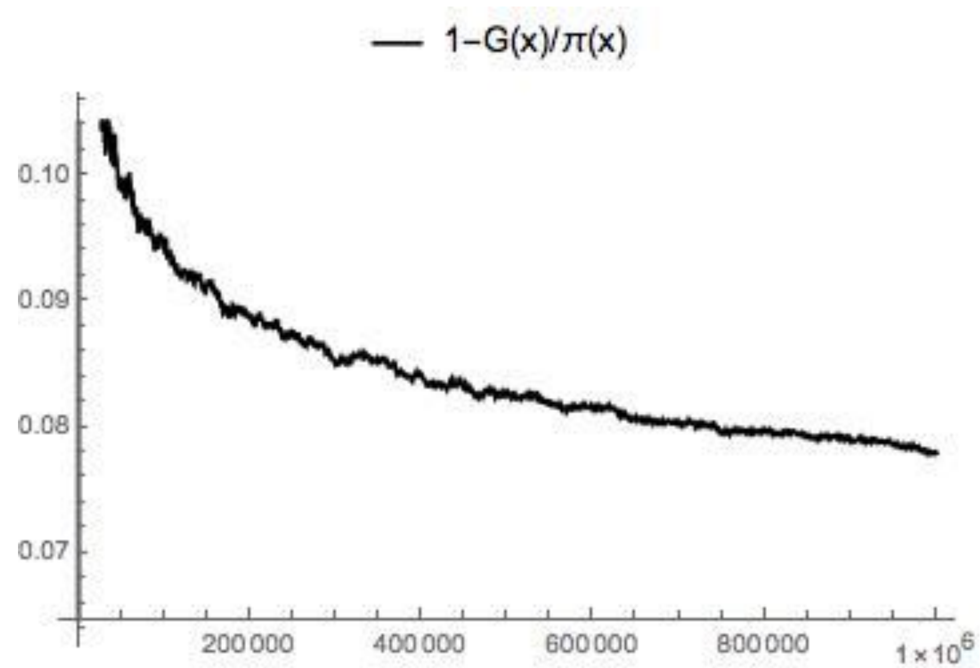


There are various proposals for $\pi(x)$. Gauss's original estimate:

$$\pi(x) = \frac{x}{\ln x} + \epsilon_x \equiv G(x) + \epsilon_x$$

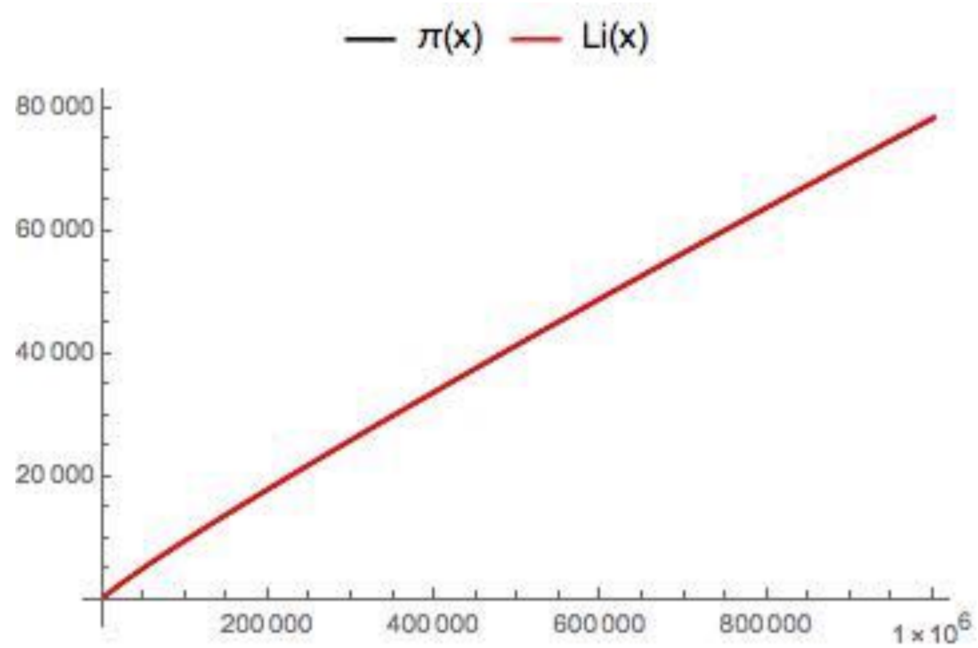


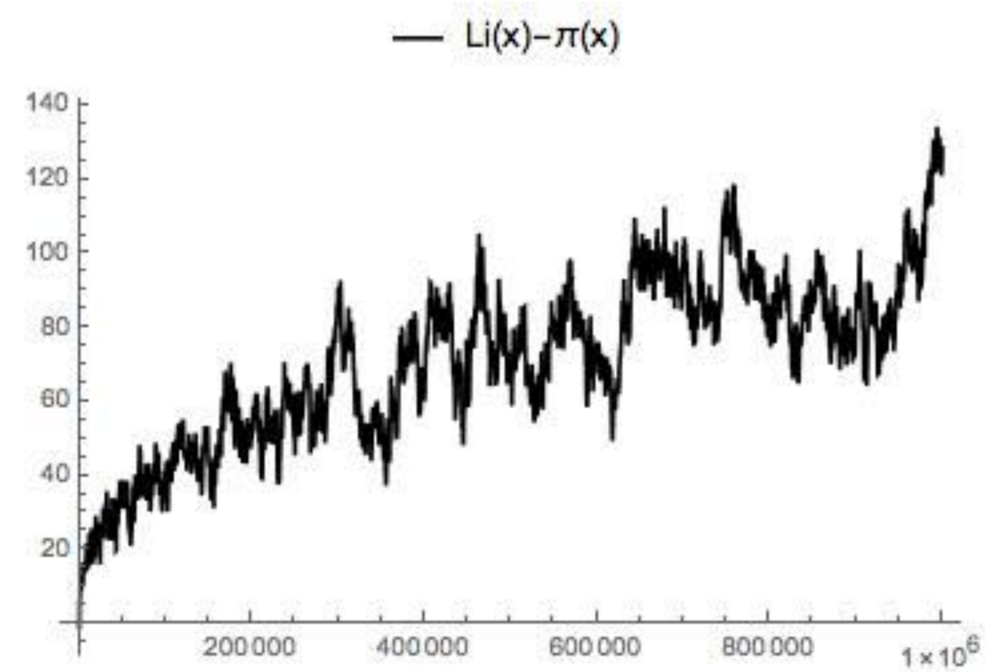


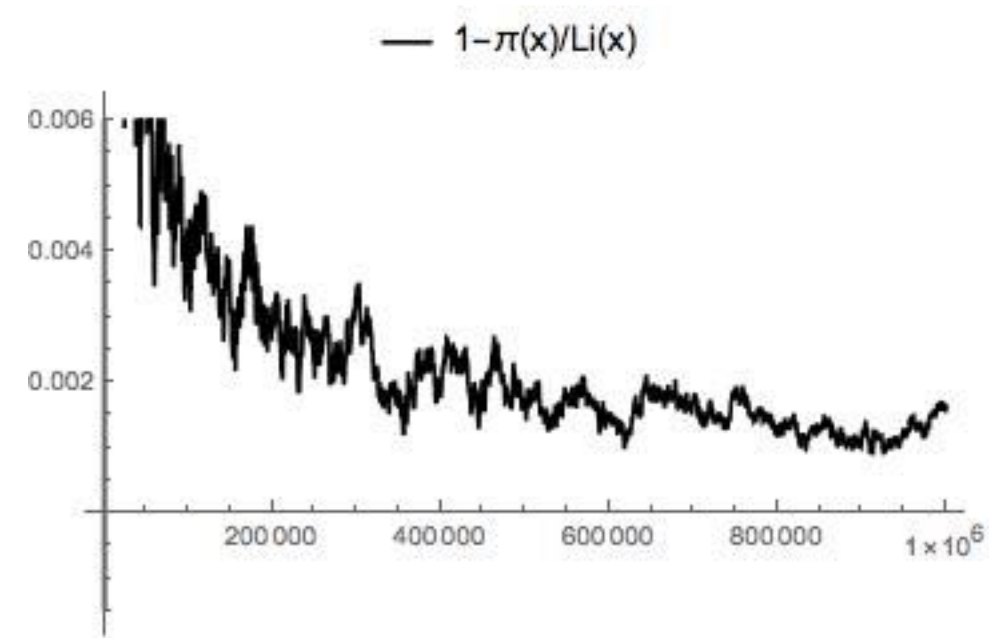


Gauss's refined estimate:

$$\pi(x) = \int_2^x \frac{du}{\ln u} + \epsilon_x \equiv Li(x) + \epsilon_x$$



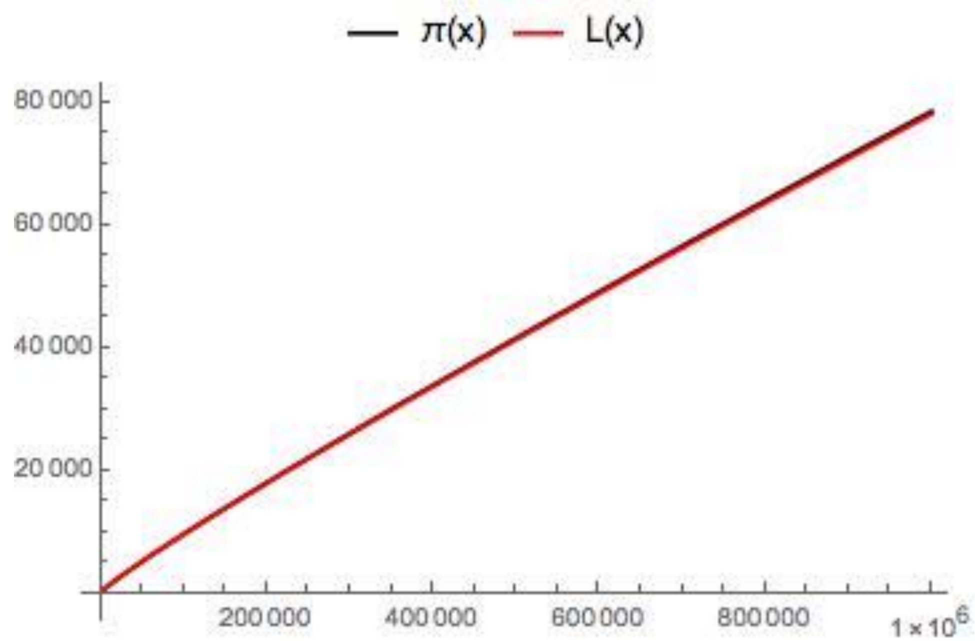


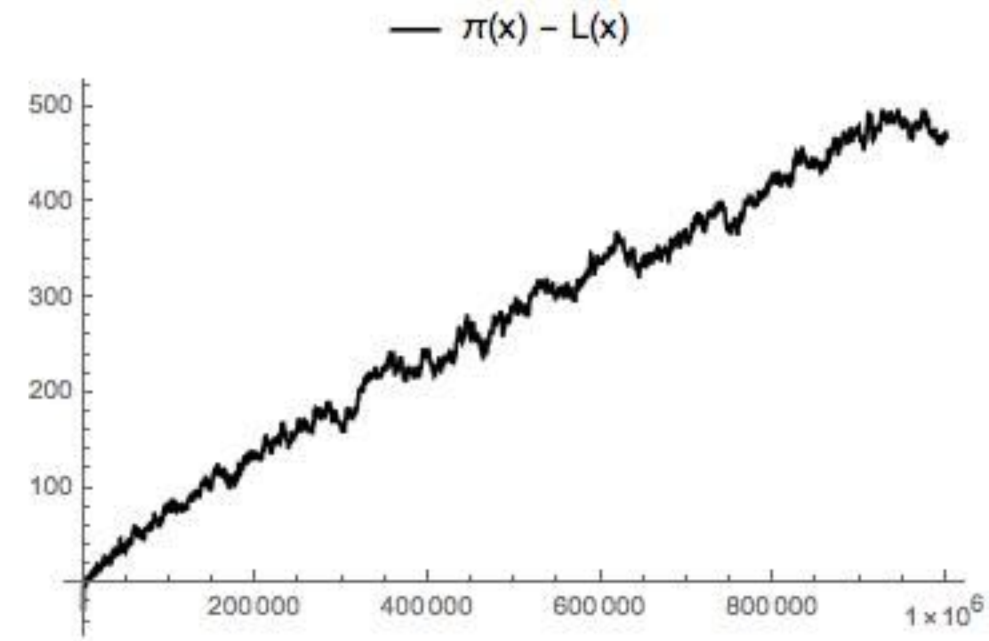


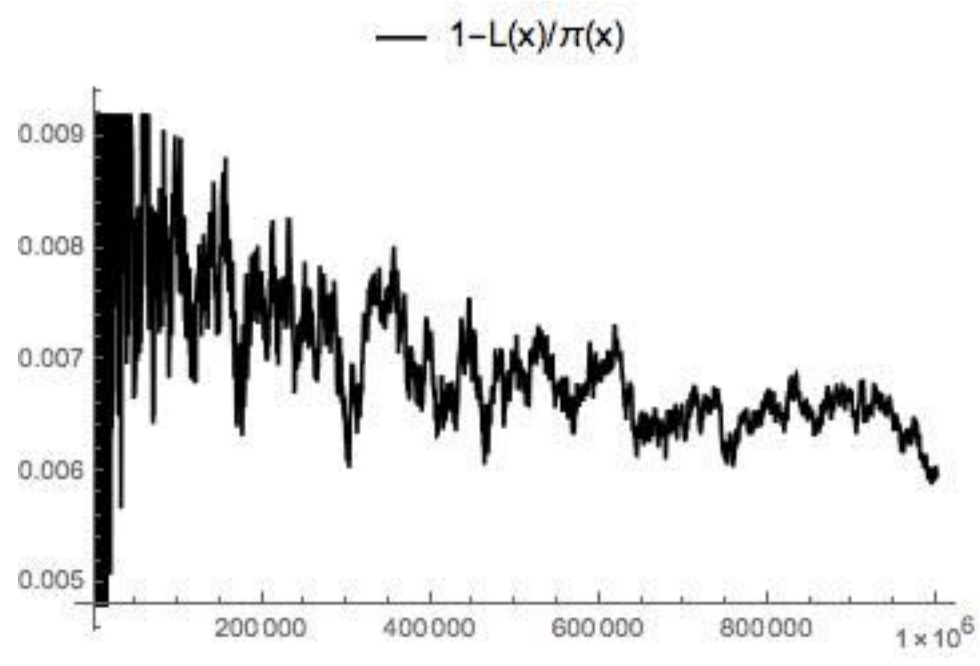
Legendre's estimate:

$$\pi(x) = \frac{x}{\ln x - A(x)} + \epsilon_x \equiv L(x) + \epsilon_x \ ,$$

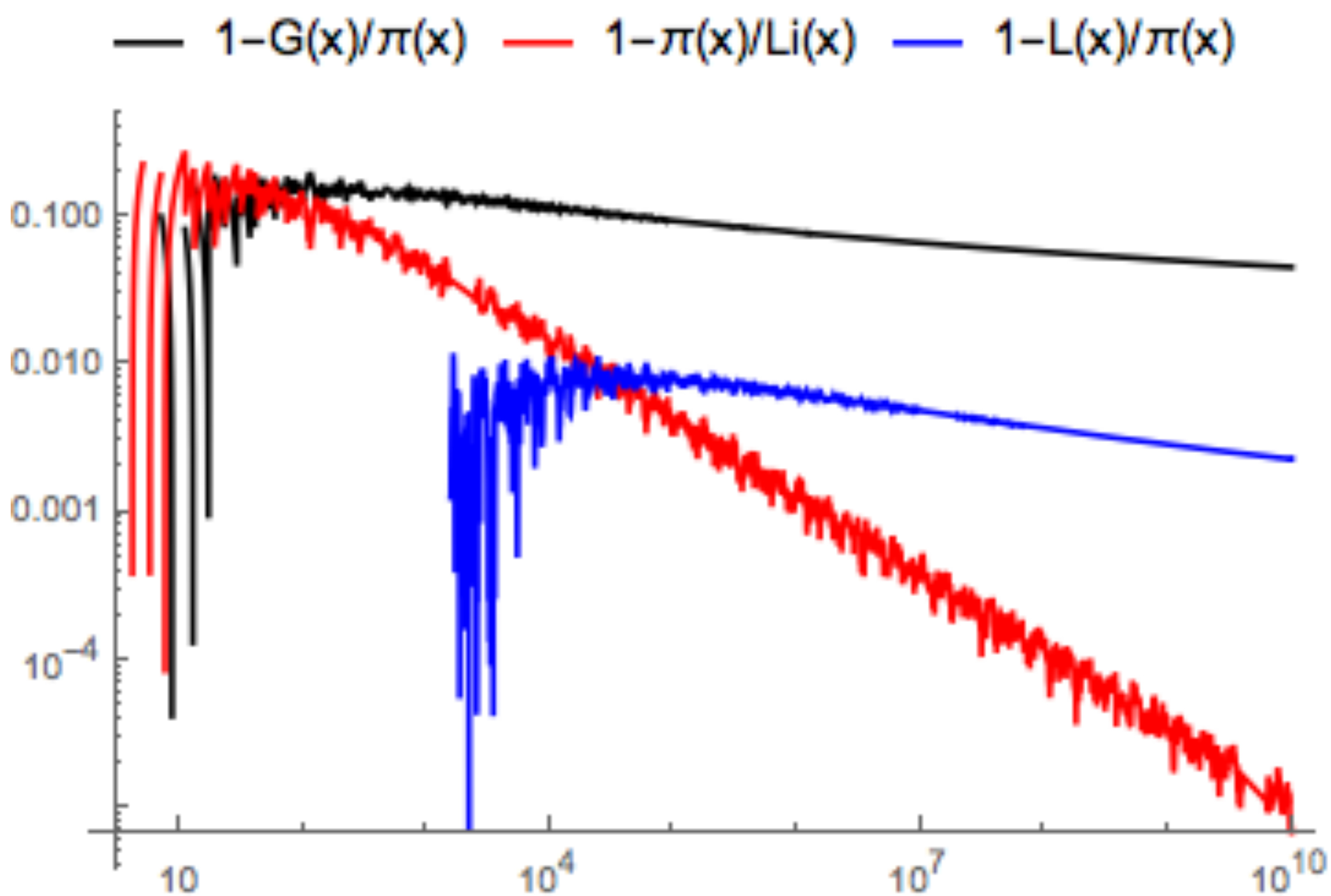
with $A(x) \approx 1.08366$.







Comparison among the relative errors:



It is not difficult to show that the three estimates have the same asymptotic, i.e.

$$G(x) \sim Li(x) \sim L(x) \ , \ \text{for } x \to \infty \ ,$$

and then we are in the position of stating the:

$$\boxed{\text{Prime Number Theorem: } \pi(x) \sim G(x) \sim Li(x) \sim L(x)}$$

Chebychev managed to show in 1854 that:

$$A_1 < \frac{\pi(x)}{x/\ln x} < A_2 \ ,$$

with $0.922\cdots < A_1 < 1$ and $1 < A_2 < 1.105\cdots$. However, the proof had to wait for Riemann, Von Mangoldt, De la Vallèe Poussin and Jacques Hadamard. The last two proved the Prime Number Theorem in 1896.

Riemann proposed a new, weighted prime counting function:

$$\Pi(x) = \sum_{p^r < x,\ p \text{ prime}} \frac{1}{r}$$

Let's see how it works:

$$\Pi(20) = \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right) .$$

$$\Pi(30) = \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{1}+\frac{1}{2}\right)$$
$$+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right)+\left(\frac{1}{1}\right) .$$

One can establish:

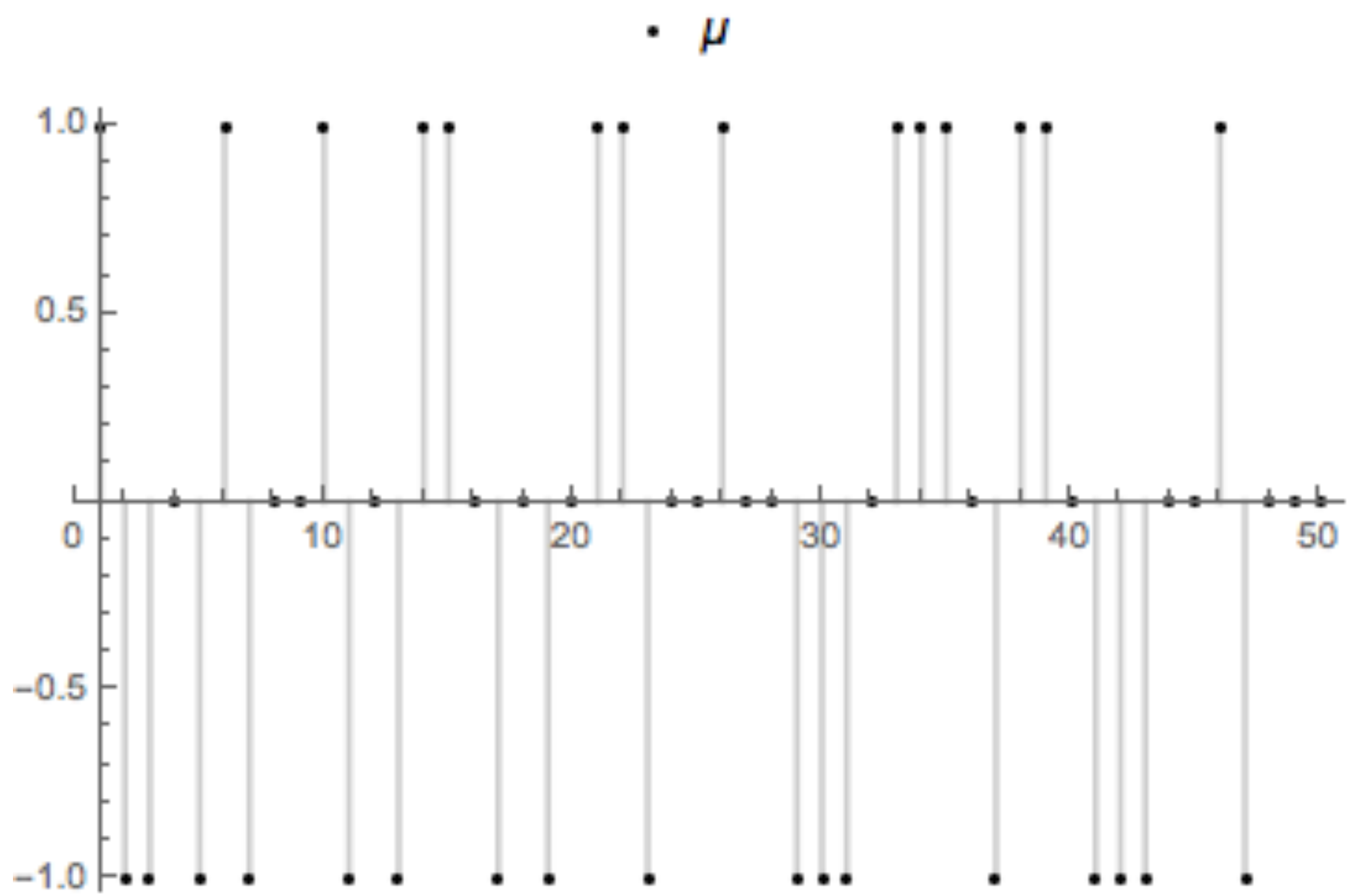
$$\Pi(x) = \sum_{r=0}^{\infty} \frac{1}{r}\pi(x^{1/r}) ,$$

and through a trick named Möbius inversion:

Corr: r starts from 1

$$\pi(x) = \sum_{r=0}^{\infty} \frac{\mu(r)}{r}\Pi(x^{1/r}) ,$$

where $\mu(r)$ is the Möbius function. It is 0 if r has a squared prime factor, 1 if it has and even number of prime factors, -1 if it has an odd number of prime factors.



μ
1.0
0.5
−0.5
−1.0
0
10
20
30
40
50

Riemann-Von Mangoldt formulae

$$\psi(x) = \sum_{p^k \leq x} \ln p$$

Chebyshev function

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$$

Prime Number Theorem

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \ln(2\pi)$$

The ρ's represent the zeroes of the Riemann zeta function.

One can deal directly with the trivial zeroes -2, -4, -6, …

$$\sum_{n=1}^{\infty} \frac{1}{2nx^{2n}} = -\frac{1}{2}\ln(1-x^{-2})$$

Now the zeroes are just the non-trivial ones.

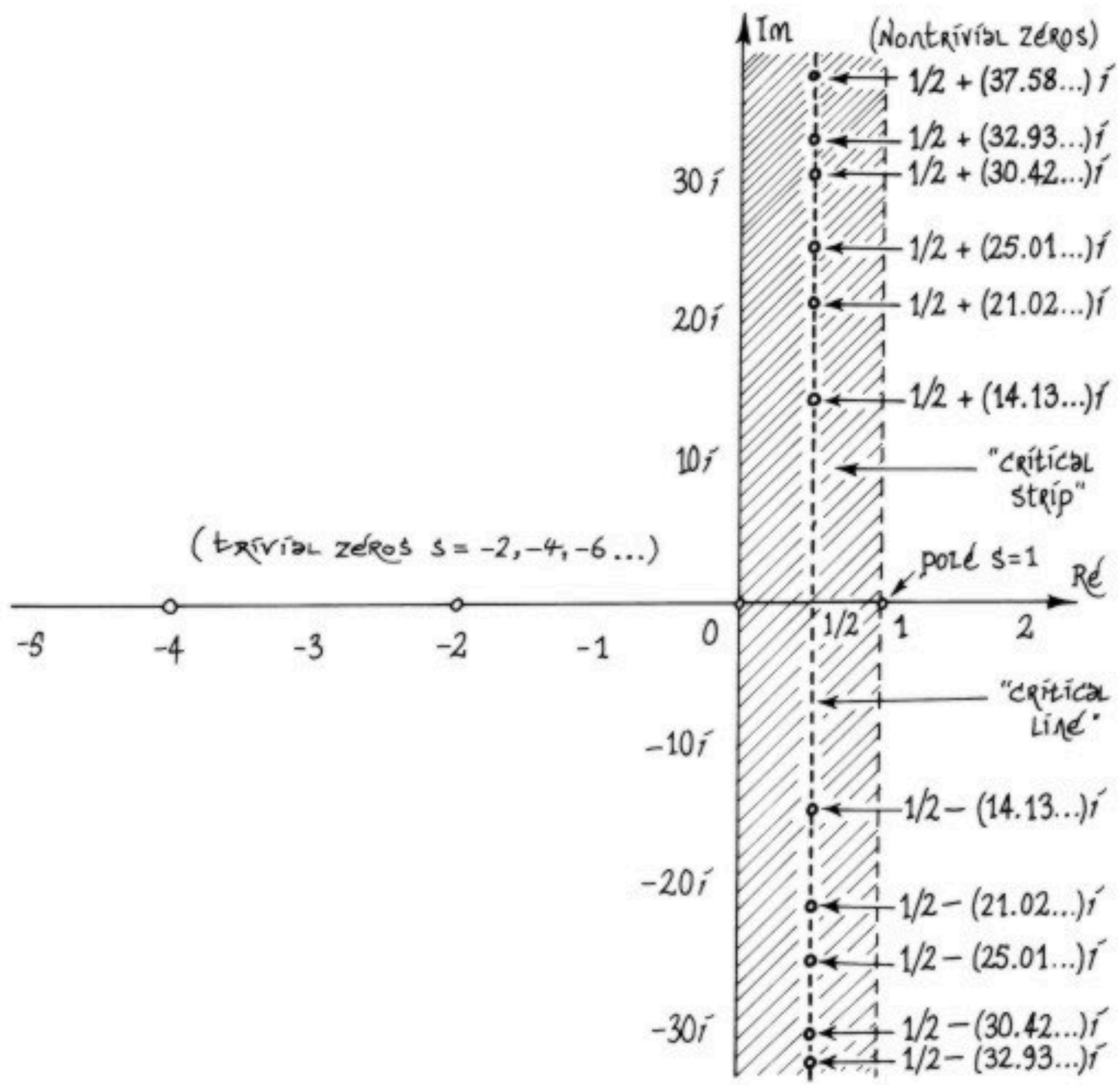
$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \ln(2\pi) - \frac{1}{2}\ln(1-x^{-2})$$

If:

$$\Re(\rho) < 1$$

then the dominant term is x.
This is how the PNT was proved.

Zeta zeroes and Riemann hypothesis: the critical strip



Implications of Riemann's hypothesis

$$|\pi(x) - \mathrm{Li}(x)| < \frac{1}{8\pi}\sqrt{x}\log(x), \qquad \text{for all } x \geq 2657,$$

Von Koch (1901)

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2(x), \qquad \text{for all } x \geq 73.2,$$

Schoenfeld (1976)

$$x < p \leq x + \frac{4}{\pi}\sqrt{x}\log x$$

Dudek (2015)

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Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

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Riemann Hypothesis

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called *prime* numbers, and they play an important role, both in pure mathematics and its applications. The distribution of such prime numbers among all natural numbers does not follow any regular pattern. However, the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function

$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \ldots$

called the *Riemann Zeta function*. The Riemann hypothesis asserts that all *interesting* solutions of the equation

$\zeta(s) = 0$

lie on a certain vertical straight line.

This has been checked for the first 10,000,000,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

This problem is:	Unsolved

Rules:
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