DOES THE COSMOLOGICAL CONSTANT STAY HIDDEN?

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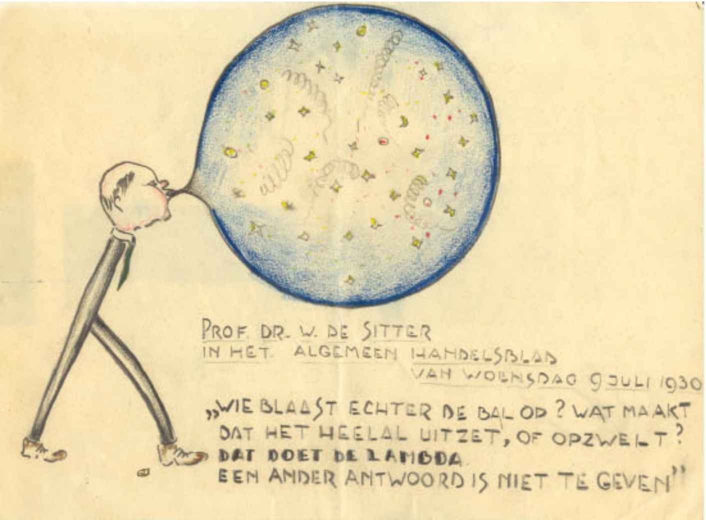
Outline

The cosmological constant



Some references on the cosmological constant

- A. Einstein, *Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie*, Sitzungs. König. Preuss. Akad. (1917) 142-152
- S. Weinberg, *The cosmological constant problem*, Reviews of Modern Physics, Vol. 61, No. 1, January 1989
- T. Padmanabhan, *Cosmological constant-the weight of the vacuum*, Physics Reports 380 (2003) 235-320
- J. Martin, *Everything you always wanted to know about the cosmological constant problem (but were afraid to ask)*, C. R. Physique 13 (2012) 566
- I. L. Buchbinder and I. L. Shapiro, *Introduction to Quantum Field Theory with Applications to Quantum Gravity*, OUP Oxford (2021)

General Relativity

Sort of "inevitability" [*S. Weinberg, Photons and gravitons in perturbation theory: Derivation of Maxwell's and Einstein's equations, Phys. Rev.* **138** *(1965), B988-B1002*], [*D. Lovelock, The Einstein Tensor and Its Generalizations, Journal of Mathematical Physics (1971) 12 (3) 498*]

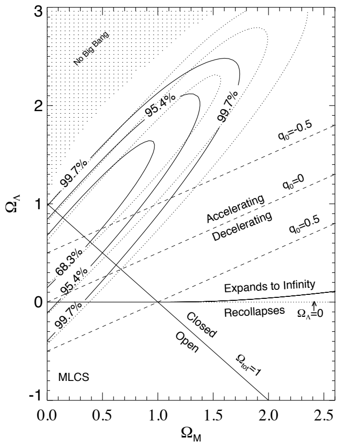
$$S = \frac{c^4}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{-g}\,(R - 2\Lambda) + S_{\mathrm{matter}}\left[g_{\mu\nu}, \Psi\right] . \qquad (1)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} . \qquad (2)$$

Λ was introduced by Einstein [*A. Einstein, Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie, Sitzungs. König. Preuss. Akad. (1917) 142-152*].
"Theoretically unsatisfactory" to Einstein, it revives today as the most successful model for the accelerated expansion of the universe.

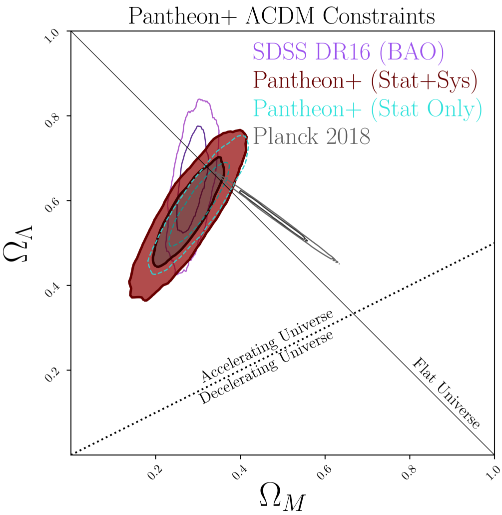
Accelerated expansion of the universe

A. G. Riess et al. *[Supernova Search Team], Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, Astron. J.* **116** *(1998) 1009*



Latest Supernovae type Ia data

D. Brout, D. Scolnic, B. Popovic, A. G. Riess, J. Zuntz, R. Kessler, A. Carr, T. M. Davis, S. Hinton and D. Jones, et al., The Pantheon+ Analysis: Cosmological Constraints,[arXiv:2202.04077 [astro-ph.CO]].



The value of Λ from cosmology

There is strong observational evidence of $q_0 < 0$. Is it Λ that cause the acceleration of the expansion?
If so, the amount of Λ energy density in the universe is about 70% percent, hence:

$$\rho_\Lambda \approx \Omega_\Lambda \rho_{\rm cr} \approx 10^{-47}\ \mathrm{GeV}^4 \approx 10^{-52}\ \mathrm{m}^{-2}\ . \tag{3}$$

What is the problem with Λ?

- Huge discrepancy with the predictions coming from quantum field theory (old cosmological constant problem);
- Why ρ_Λ has the above tiny value? (new cosmological constant problem).

The first question was tackled already by Zel'dovich and Sakharov [*Y. B. Zel'dovich, JETP letters 6 (1967), 316-317*; *A. D. Sakharov, Dokl. Akad. Nauk SSSR (1967) 177, 70-71*].

Vacuum fluctuations

In Minkowski space we have that

$$\langle T_{\mu\nu} \rangle \propto \eta_{\mu\nu} \ , \tag{4}$$

hence by the equivalence principle, in curved space one has:

$$\langle T_{\mu\nu} \rangle = -\rho_{\rm vac}(x) g_{\mu\nu} \ , \tag{5}$$

and because of Bianchi identities $\rho_{\rm vac}$ has to be a constant. So, the field equations become:

$$G_{\mu\nu} + \Lambda_{\rm eff} g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ , \tag{6}$$

with

$$\Lambda_{\rm eff} = \Lambda + 8\pi G_N \rho_{\rm vac} \ , \tag{7}$$

the effective cosmological constant (whose measured value is 10^{-47} GeV^4).

CLASSICAL CONTRIBUTIONS

These come from fields which settle at the minimum of the potential to which they are subject. Consider a simple example:

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi + V(\Phi) \right] . \tag{8}$$

If the field rolls down to a minimum of its potential:

$$\langle T_{\mu\nu} \rangle = -V(\Phi_{\rm min}) g_{\mu\nu} . \tag{9}$$

If we can set $V(\Phi_{\rm min}) = 0$ there is no contribution to $\Lambda_{\rm eff}$. Phase transitions are therefore quite problematic because the position of the minimum is shifted and $V(\Phi_{\rm min}) = 0$ can be realised only before or after the phase transition.

Electroweak phase transition

Realistic cases are the electroweak phase transition and the QCD transition. In the electroweak case (after the transition) we have the potential ($\lambda \simeq 0.1$ is a coupling):

$$V(H) = -\frac{\lambda v^4}{4} + \frac{1}{2}\lambda v^2 H^2 + \frac{\lambda}{2}\frac{v}{\sqrt{2}}H^3 + \frac{\lambda}{16}H^4 \,, \qquad (10)$$

with $m_H^2 = \lambda v^2$ being the Higgs mass and $v = \langle H \rangle$. Then:

$$\rho_{\rm vac} = -\frac{1}{4}m_H^2 v^2 \,, \qquad v^2 = \frac{\sqrt{2}}{4G_F^2} \,, \qquad (11)$$

lead to:

$$\rho_{\rm vac} = -\frac{\sqrt{2}}{16}\frac{m_H^2}{G_F^2} \approx -1.2 \times 10^8 \ \mathrm{GeV}^4 \,. \qquad (12)$$

$G_F \simeq 1.16 \times 10^{-5}$ GeV^{-2} is Fermi's constant and $m_H \approx 125$ GeV.

Electroweak phase transition

Higgs Potential (plots taken from Martin's review)

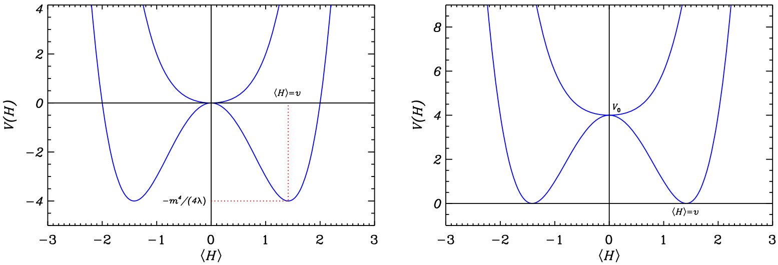


Fig. 2. Effective potential of the Higgs boson before and after the electroweak phase transition. The left panel corresponds to a situation where the vacuum energy vanishes at high temperature. As a consequence $\rho_{\rm vac}$ is negative at temperature smaller than the critical temperature. This is the situation treated in the text where the quantity $-m^4/(4\lambda)$ is explicitly calculated. On the right panel, the off-set parameter V_0 is chosen such that the vacuum energy is zero after the transition. As a consequence, it does not vanish at high temperatures.

Hiding the cosmological constant

By Marsyas (2007), CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=471575

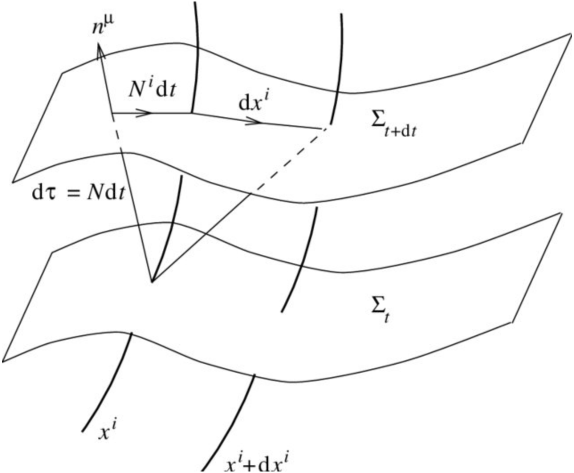
Hiding the cosmological constant
[S. Carlip, PRL 123, 131302 (2019)]

The author describes a mechanism by which the effect of a huge cosmological constant might average out on macroscopic scales.

I show that if one does not assume homogeneity and an arrow of time at the Planck scale, a very large class of general relativistic initial data exhibit expansions, shears, and curvatures that are enormous at small scales, but quickly average to zero macroscopically. Subsequent evolution is more complex, but I argue that quantum fluctuations may preserve these properties. The resulting picture is a version of Wheeler's "spacetime foam", in which the cosmological constant produces high curvature at the Planck scale but is nearly invisible at observable scales.

The initial value formulation of General Relativity

3 + 1 foliation of the spacetime



D. L. Wiltshire, "An Introduction to quantum cosmology," [arXiv:gr-qc/0101003 [gr-qc]]

The initial value formulation of General Relativity

Metric and constraint equations

Adopting a 3+1 foliation of spacetime, with Σ a Cauchy 3-hypersurface, where to establish our initial conditions, a metric can be written as:

$$ds^2 = -(N^2 - N_i N^j)dt^2 + 2N_i dx^i dt + g_{ij} dx^i dx^j \ . \qquad (13)$$

The usual Einstein equations in vacuum and in presence of a cosmological constant are equivalent to two constraints:

$$R - K^i{}_j K^j{}_i + K^2 - 2\Lambda = 0 \ , \qquad (14)$$

$$D_i(K^i{}_j - \delta^i_j K) = 0 \ , \qquad (15)$$

and to two evolution equations:

The initial value formulation of General Relativity
Dynamic equations

(Gauss-Codazzi equations)

$$\frac{1}{N}\partial_t g_{ij} = 2K_{ij} + \frac{1}{N}(D_j N_i + D_i N_j) \,, \tag{16}$$

$$\begin{aligned}\frac{1}{N}\partial_t K^i{}_j &= -R^i{}_j - KK^i{}_j + \delta^i_j \Lambda \\ &\quad + \frac{D^i D_j N}{N} + \frac{1}{N}(K^i{}_k D_j N^k - K^k{}_j D_k N^i + N^k D_k K^i{}_j) \,,\end{aligned} \tag{17}$$

where K_{ij} is the extrinsic curvature.

Back to [S. Carlip, PRL 123, 131302 (2019)]

Some assumptions

- We want to use the above equations on very small scales, where $\Lambda \simeq 1/\ell^2$, with the length scale ℓ possibly being the Planck scale;
- The initial values of g_{ij} and $K^i{}_j$ are random variables, of quantum origin;
- However, the above equations are classical. We would need to incorporate into them quantum effects, but we do not have a quantum theory of gravity at our disposal;
- As an approximation, we could consider instead those evolution equations valid for *averaged* quantities, assuming a sort of gravitational Ehrenfest's theorem.

Back to [S. Carlip, PRL 123, 131302 (2019)]

Two important properties on the random initial conditions

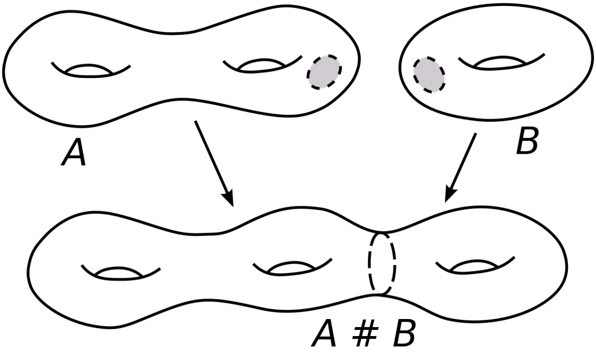
Since:

1. The equations are time reversal invariant, i.e. if (g, K) is allowed initial data for a manifold Σ, so is $(g, -K)$
2. Two manifolds Σ_1 and Σ_2 with initial data (g_1, K_1) and (g_2, K_2) can be glued to form a manifold $\Sigma_1 \# \Sigma_2$ (connected sum) for which the initial data are unchanged outside arbitrarily small neighborhoods of the points where the gluing is performed. [P. T. Chrusciel, J. Isenberg, and D. Pollack, Phys. Rev. Lett. 93, 081101 (2004), Commun. Math. Phys. 257, 29 (2005)]

then it is natural that, averaging over a sufficiently large volume:

$$\langle K^i{}_j \rangle \approx 0 \,. \tag{18}$$

Connected sum of manifolds



Generality

The result is quite general because any orientable 3-manifold is isomorphic to a connected sum of prime manifolds and this sum is unique. [J. Milnor, Am. J. Math. 84, 1 (1962); D. Giulini, Int. J. Theor. Phys. 33, 913 (1994)]

Is the above property preserved with time?

Initial value of the first derivative of $\langle K \rangle$

Define the average [T. Buchert, Gen. Relativ. Gravit. 32, 105 (2000)]:

$$\langle X \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} X \sqrt{g} d^3 x \, , \qquad V_{\mathcal{D}} = \int_{\mathcal{D}} \sqrt{g} d^3 x \, . \tag{19}$$

Use $\langle K \rangle = 0$ as initial condition. *On the initial-value hypersurface*:

$$\langle K \rangle^{\bullet} = \frac{1}{V} \int N(-R + 3\Lambda) \sqrt{g} d^3 x \, . \tag{20}$$

Since $N > 0$, there are infinite possible choices of it such that $\langle K \rangle^{\bullet} = 0$, as long as $-R + 3\Lambda$ has not a definite sign.

Is the above property preserved with time?

Initial value of the second derivative of $\langle K\rangle$

On the initial-value hypersurface:

$$\langle K\rangle^{\bullet\bullet} = \frac{1}{V}\int [(\dot{N}+NK)(-R+3\Lambda)+2N^2K^{ij}R_{ij}]\sqrt{g}\,d^3x\,. \quad (21)$$

Assuming N invariant under $(g,K)\to(g,-K)$, and being $\dot{N}$ specifiable independently, we can make the above derivative to vanish.

A similar argument goes on for higher derivatives of $\langle K\rangle$. Essentially, we exploit the freedom of choice of higher derivatives of N on the initial-value hypersurface.

Small spatial curvature

Note that we also need, in order to match our universe, that on sufficiently large scales the spatial curvature vanishes. From the constraint:

$$\langle R \rangle = \langle K^i{}_j K^j{}_i \rangle - \langle K^2 \rangle + 2\Lambda \, . \tag{22}$$

Even if Λ is very large, it might be "compensated" (or, again, "hidden") by the other terms (e.g. $\langle K^2 \rangle$ if $\Lambda > 0$).

Open question: can we choose a foliation such that $\langle K \rangle$ and $\langle R \rangle$ remain simultaneously small?

Does Λ stay hidden?

[O. F. Piattella, "Does the cosmological constant stay hidden?," Phys. Rev. D **102** (2020) no.10, 104039]

Idea: instead of assessing the vanishing of the derivatives of $\langle K \rangle$, order by order, on the initial-value hypersurface, let us look for a closed differential equation for $\langle K \rangle$.
Take the average of the trace of the Gauss-Codazzi equation for $K^i{}_j$:

$$\langle K \rangle^\bullet = -\frac{\dot{V}}{V}\langle K \rangle + \frac{1}{V}\int [\dot{K} + K\dot{g}/(2g)]\sqrt{g}d^3x = -\frac{\dot{V}}{V}\langle K \rangle + \frac{1}{V}\int [N(-R + 3\Lambda) + D^k D_k N + N^k D_k K + K D_k N^k]\sqrt{g}d^3x \,. \quad (23)$$

The derivative of the volume is:

$$\frac{\dot{V}}{V} = \frac{1}{V}\int [\dot{g}/(2g)]\sqrt{g}d^3x = \frac{1}{V}\int (NK + D_k N^k)\sqrt{g}d^3x \,. \quad (24)$$

Evolution equation for $\langle K \rangle$

We then have:

$$\langle K \rangle^{\bullet} = -[\langle NK \rangle + \mathcal{B}_1]\langle K \rangle - \langle NR \rangle + 3\langle N \rangle \Lambda + \mathcal{B}_2 + \mathcal{K} \; , \qquad (25)$$

where $\mathcal{B}_{1,2}$ are the boundary terms depending only on the shift and lapse functions, respectively:

$$\mathcal{B}_1 := \frac{1}{V}\int D_k N^k \sqrt{g} d^3x \; , \quad \mathcal{B}_2 := \frac{1}{V}\int D^k D_k N \sqrt{g} d^3x \; , \qquad (26)$$

and $\mathcal{K}$ is a boundary term involving K itself:

$$\mathcal{K} := \frac{1}{V}\int D_k (K N^k) \sqrt{g} d^3x \; . \qquad (27)$$

We have to get rid of the term $\langle NK \rangle$ in order to find a closed equation for $\langle K \rangle$.

Evolution equation for $\langle K \rangle$

For a generic quantity Ψ, one has:

$$\langle \Psi \rangle^{\bullet} = \langle \dot{\Psi} \rangle + \langle NK\Psi \rangle - \langle NK \rangle \langle \Psi \rangle \, . \tag{28}$$

Therefore, choosing $\Psi = 1/N$, we obtain:

$$\langle 1/N \rangle^{\bullet} = -\langle \dot{N}/N^2 \rangle + \langle K \rangle - \langle NK \rangle \langle 1/N \rangle \, , \tag{29}$$

from which we can obtain:

$$\langle NK \rangle = \frac{\langle K \rangle}{\langle 1/N \rangle} - \frac{\langle 1/N \rangle^{\bullet} + \langle \dot{N}/N^2 \rangle}{\langle 1/N \rangle} \, . \tag{30}$$

Evolution equation for $\langle K \rangle$

We find a Riccati-type equation:

$$\langle K \rangle^{\bullet} = -f_2 \langle K \rangle^2 + f_1 \langle K \rangle + f_0 \ , \tag{31}$$

with:

$$f_2 := \frac{1}{\langle 1/N \rangle} \ , \quad f_1 := \frac{\langle 1/N \rangle^{\cdot} + \langle \dot{N}/N^2 \rangle}{\langle 1/N \rangle} - \mathcal{B}_1 \ , \tag{32}$$

$$f_0 = -\langle NR \rangle + 3\langle N \rangle \Lambda + \mathcal{B}_2 + \mathcal{K} \ . \tag{33}$$

Since:

$$f_1 = -\frac{\dot{f}_2}{f_2} + f_2 \langle \dot{N}/N^2 \rangle - \mathcal{B}_1 \ , \tag{34}$$

we can cast Eq. (31) as follows:

$$(f_2 \langle K \rangle)^{\cdot} = -(f_2 \langle K \rangle)^2 + (f_2 \langle \dot{N}/N^2 \rangle - \mathcal{B}_1)(f_2 \langle K \rangle) + f_2 f_0 \ , \tag{35}$$

adopting $f_2 \langle K \rangle$ as the unknown function.

Evolution equation for $\langle K \rangle$

Since $N > 0$, i.e. the lapse function is strictly positive because an arrow of time is established, then $f_2 = \frac{1}{\langle 1/N \rangle} > 0$, provided $\sqrt{g} > 0$.

If singularities develop, for example due to the gluing technique which allows us to choose a vanishing initial $\langle K \rangle$, $\sqrt{g}$ might diverge somewhere in the averaging region badly enough to make, despite the integration, $\langle 1/N \rangle$ diverging and thus f_2 to vanish. We do not consider this possibility here and simply assume $f_2 > 0$ from now on.

Existence of the $\langle K \rangle = 0$ solution

No boundary terms

Let us now focus on the simplest case, in which we neglect the boundary terms $\mathcal{B}_{1,2}$ and $\mathcal{K}$:[1]

$$(f_2\langle K\rangle)^{\cdot} = -(f_2\langle K\rangle)^2 + f_2\langle \dot{N}/N^2\rangle(f_2\langle K\rangle) + f_2(-\langle NR\rangle + 3\langle N\rangle\Lambda) \,. \tag{36}$$

The solution $\langle K\rangle = 0$ exists if:

$$\langle NR\rangle = 3\langle N\rangle\Lambda \,. \tag{37}$$

This is the same condition required by [S. Carlip, PRL 123, 131302 (2019)], but only on the initial values hypersurface. From our analysis here we see that we need it to hold true throughout the whole time evolution. This seems already somehow problematic.

[1] This special case is the one treated in [S. Carlip, PRL 123, 131302 (2019)].

Stability of the $\langle K \rangle = 0$ solution

No boundary terms

If $\langle NR \rangle = 3\langle N \rangle \Lambda$, then $\langle K \rangle = 0$ is a stable solution if:

$$f_2 \langle \dot{N} / N^2 \rangle < 0 \,. \tag{38}$$

This can be achieved only if $\dot{N} < 0$, provided again that no singularities for which $g = 0$ develop.

But since $N > 0$, N cannot arbitrarily decrease. At a certain time $\bar{t}$ we expect $\dot{N}$ to vanish and possibly to change sign. When this happens, $\langle K \rangle$ would start to grow away from $\langle K \rangle = 0$.

General solution
No boundary terms

A general solution, if we know a particular solution say $\mathcal{F}$, is:

$$f_2\langle K\rangle = \mathcal{F} + \frac{\Phi(t)}{C + \int^t dt'\Phi(t')} \,, \tag{39}$$

$$\Phi = \exp\left\{\int^t dt' \left[-2\mathcal{F}(t') + f_2\langle \dot{N}/N^2\rangle\right]\right\} \,, \tag{40}$$

where C is some integration constant. For $\langle K\rangle = 0$, whose existence we have assumed, the general solution becomes then:

$$\langle K\rangle = \frac{1}{f_2}\frac{\Phi(t)}{C + \int^t dt'\Phi(t')} \,, \qquad \Phi = \exp\left(\int^t dt' f_2\langle \dot{N}/N^2\rangle\right) \,. \tag{41}$$

INSTABILITY OF THE GENERAL SOLUTION
NO BOUNDARY TERMS

A reflection of the above mentioned instability is seen in the denominator of Eq. (41). Indeed, $1/C$ is the initial value of $f_2\langle K\rangle$. So, if $f_2\langle K\rangle$ is vanishingly small but not with fixed sign, then C is large and positive or negative. When $C<0$, the denominator:

$$C+\int^t dt'\Phi(t') \ , \qquad (42)$$

might diverge, because $\int^t dt'\Phi(t')$ is always positive. In particular, we expect this to happen if $f_2\langle \dot{N}/N^2\rangle > 0$, because in this case Φ is a growing function. If $C>0$ there is no such divergence, but still $\langle K\rangle$ should in principle grow away from $\langle K\rangle = 0$ if $f_2\langle \dot{N}/N^2\rangle > 0$, at least until the $-(f_2\langle K\rangle)^2$ term in the Riccati equation (36) dominates on the $f_2\langle \dot{N}/N^2\rangle(f_2\langle K\rangle)$ one, in which case $\langle K\rangle$ starts again to decrease.

Conclusions

- Through a simple definition of average, we have built an evolution equation for $\langle K\rangle$ and have analysed the stability of its $\langle K\rangle = 0$ solution, provided that this exists.
- Unfortunately, it seems that such solution is unstable, because a necessary condition for its stability is $\dot{N} < 0$, which is not admissible throughout the whole evolution since $N > 0$.
- The fact that N is required to decrease in order to have a stable solution might be an indication that singularities develop during the evolution, as already discussed.
- Therefore, at least for the very simple case considered, the hiding of Λ is not preserved in time.

Perspectives

There are some aspects which deserve to be considered in more detail.

- A rigorous proof of the existence of the $\langle K \rangle = 0$ solution. We simply assumed that it exists, and assessed its stability.
- Taking into account the quantum evolution. We simply neglected quantum fluctuations, assuming a sort of gravitational Ehrenfest theorem.
- We neglected the boundary terms, but are we really allowed to do that?
- We simply neglected the shift vector N^i, but it could be put to more use.
- Can we choose a foliation such that $\langle K \rangle$ and $\langle R \rangle$ remain simultaneously small throughout the evolution?