SuccessClap

Best Coaching for UPSC MATHEMATICS

UPSC CSE 2021 Mathematics Optional Paper 1 – Authenic Solution

S.No	UPSC Question	Topic	SuccessClap Question Bank Source
1	1d	Calculus	SC B08 Qn 2
2	2a	Analytic Geometry	SC C07 Qn 30
3	3b	Analytic Geometry	SC C05 Qn9
4	3c(i)	Linear Algebra	SC A05 Qn 32
5	4b	Calculus	SC B15 Qn 22
6	5d	Dynamics	SC E04 Qn 34
7	8a(i)	ODE	SC D02 Qn 8
8	8a(ii)	ODE	SC D05 Qn 18
9	8c	Vector Analysis	SC F02 Qn 16

SuccessClap - Paper 1

① If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Show $A^2 = A^{-1}$

Note

Donot do A^2, A^3
↳ use Cayley Hamilton

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 1 \\ 2 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$

$(1-\lambda)(1+\lambda)\lambda + 1(-2\lambda) + 1(1+\lambda) = 0$

$\lambda - \lambda^3 - 2\lambda + 1 + \lambda = 0$

$\lambda^3 = 1 \Rightarrow A^3 = I$

Also $|A| = 1 \neq 0$ So A^{-1} exists

Pre multiply A^{-1} on both sides

$A^{-1} A^3 = A^{-1} . I$

$A^2 = A^{-1}$

SuccessClap · Paper I

(1b) Find matrix for $T(a,b,c) = (a+b, a-b, 2c)$ with basis $B = \{(0,1,1)\ (1,0,1)\ (1,1,0)\}$

Soln: Approach : we want

$$T(0,1,1) = a_1(0,1,1) + a_2(1,0,1) + a_3(1,1,0)$$
$$T(1,0,1) = b_1(0,1,1) + b_2(1,0,1) + b_3(1,1,0)$$
$$T(1,1,0) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$$

and Soln is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$T(a,b,c) = (a+b, a-b, 2c)$$
$$= \alpha(0,1,1) + \beta(1,0,1) + \gamma(1,1,0)$$
$$= (\beta+\gamma, \alpha+\gamma, \alpha+\beta)$$

$$\left.\begin{aligned} a+b &= \beta+\gamma \\ a-b &= \alpha+\gamma \\ 2c &= \alpha+\beta \end{aligned}\right\}$$

] subtract

$$\alpha - \beta = -2b$$
$$\alpha + \beta = 2c$$

$$2\alpha = 2c - 2b$$
$$\alpha = c - b$$

$\beta = 2c - \alpha = 2c - c + b = c + b$

$\gamma = a + b - \beta = a + b - c - b = a - c$

$T(a,b,c) = (c-b)(0,1,1) + (c+b)(1,0,1) + (a-c)(1,1,0)$

$T(0,1,1) = 0(0,1,1) + 2(1,0,1) - 1(1,1,0)$

$T(1,0,1) = 1(0,1,1) + 1(1,0,1) + 0(1,1,0)$

$T(1,1,0) = -1(0,1,1) + 1(1,0,1) + 1(1,1,0)$

$$T = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

✓

SuccessClap - Paper

① Given $\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

Find $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\Delta(x)}{x}$

Source of this question

→ Differential Calculus for IITJEE - Arihant

↳ SuccessClap Question Bank

$\Delta(0) = 0$ So $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\Delta(x)}{x}$ $\dfrac{0}{0}$ So L-Hospital rule

$= \underset{x\to 0}{\mathrm{Lt}} \dfrac{\Delta(x)}{x} = \underset{x\to 0}{\mathrm{Lt}} \dfrac{\Delta'(x)}{1} = \Delta'(0)$ ✓

$\Delta'(x) = \begin{vmatrix} f'(x+\alpha) & f'(x+2\alpha) & f'(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$ - Row 1 differed

$+ \begin{vmatrix} f(\alpha+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ 0 & 0 & 0 \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix} + \begin{vmatrix} f(\alpha+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ 0 & 0 & 0 \end{vmatrix}$

Row 2 differential Row 3 differen

Put $x=0$ Row1 = Row 3 So determinant 0

$\Delta'(0) = 0 + 0 + 0 = 0$

$\underset{x\to 0}{\mathrm{Lt}} \dfrac{\Delta(x)}{x} = \Delta'(0) = 0$

(d) Show b/w any two roots $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$

[10]

SuccessClap Question Bank
Mean Value Theorem SC-B08 Qn 2

Let α, β be two disting roots of $e^x \cos x = 1$

$\therefore \; e^\alpha \cos\alpha = 1, \; e^\beta \cos\beta = 1$

Define $f(x) = e^{-x} - \cos x$

↳ f is continous & differentiable

$f'(x) = -e^{-x} + \sin x$

$f(\alpha) = e^{-\alpha} - \cos\alpha$
$= \dfrac{1 - e^\alpha \cos\alpha}{e^\alpha} = 0$

Similarly
$f(\beta) = \dfrac{e^{-\beta} - \cos\beta}{e^\beta} = 0$

By Rolle : $f(\alpha) = 0$,
$f(\beta) = 0$

There exists γ .s.t
$\alpha < \gamma < \beta$ & $f'(\gamma) = 0$

So one root is γ $\quad f'(\gamma) = -e^{-\gamma} + \sin\gamma = 0$
$e^\gamma \sin\gamma - 1 = 0$

So one $\exists$ one root γ for the equation $e^x \sin x - 1 = 0$

(1e) Find equation of cylinder whose generators are parallel to line $x = \frac{-y}{2} = \frac{z}{3}$ and whose guiding curve is $x^2+2y^2=1$, $z=0$

Let $P(x_1, y_1, z_1)$ on cylinder

↳ Generator thru P $\quad \frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$

↓ meets $z=0$ $\quad \frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{0-z_1}{3}$

↳ pt is $\left[x_1 - \frac{z_1}{3},\ y_1 + \frac{2z_1}{3},\ 0\right]$

↓ Pass thru $x^2+y^2 \cdot 2 = 1$

$$\left(x_1 - \frac{z_1}{3}\right)^2 + 2\left(y_1 + \frac{2z_1}{3}\right)^2 = 1$$

Locus $x_1 \to x \quad y_1 \to y \quad z_1 \to z$

$$\left(x - \frac{z}{3}\right)^2 + 2\left(y + \frac{2z}{3}\right)^2 = 1$$

$$(3x-z)^2 + 2(3y+2z)^2 = 9$$

$$9x^2 - 6xz + z^2 + 2(9y^2 + 4z^2 + 12yz) = 9$$

$$9x^2 + 18y^2 + 9z^2 - 6xz + 24yz = 9$$

$$\boxed{3x^2 + 6y^2 + 3z^2 - 2xz + 8yz - 3 = 0}$$ Ans

(2a) Show that the planes which cut the cone $ax^2+by^2+cz^2=0$ in perpendicular generators touch cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$

SuccessClap Question Bank - SC-07-Qn-No 30

Cone : $ax^2+by^2+cz^2=1$

Let eqn of plane thru vertex $(0,0,0)$ of axis

$ux+uy+wz=0$

↳ plane cuts the cone in ⊥ generators ⇒

$(b+c)u^2+(c+a)v^2+(a+b)w^2=0$

↓

Normal to plane thru origin $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$ lies on cone $(b+c)x^2+(c+a)y^2+(a+b)z^2=0$ which is plane is a tangent plane to cone which is reciprocal of cone.

The reciprocal eqn of cone is

$$\frac{x^2}{b+c}+\frac{y^2}{a+c}+\frac{z^2}{a+b}=0$$

So if plane $ux+uy+wz=0$ cuts cone in perpen then it touches the cone

2b) Given $f(x,y) = |x^2 - y^4|$

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$ and show $f_{xy}(0,0) = f_{yx}(0,0)$

$$f_{xy}(0,0) = \underset{h\to 0}{Lt} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$f_{yx}(0,0) = \underset{k\to 0}{Lt} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$f_x(0,0) = \underset{h\to 0}{Lt} \frac{f(0+h,0) - f(0,0)}{h} = \underset{h\to 0}{Lt} \frac{|h^2 - 0| - 0}{h} = 0$$

$$f_x(0,k) = \underset{h\to 0}{Lt} \frac{f(0+h,k) - f(0,k)}{h} = \underset{h\to 0}{Lt} \frac{|h^2-k^4| - |-k^4|}{h}$$

As $h \to 0 \Rightarrow h$ becomes smaller $h^2 < k^2$

$|h^2 - k^4| = -(h^2 - k^4) = k^2 - h^2$; $|-k^4| = k^2$

$$= \underset{h\to 0}{Lt} \frac{k^2 - h^2 - k^2}{h^2} = \underset{h\to 0}{Lt} -h = 0$$

$$f_y(0,0) = \underset{k\to 0}{Lt} \frac{f(0,0+k) - f(0,0)}{k} = \underset{k\to 0}{Lt} \frac{|-k^4| - 0}{k} = 0$$

$$f_y(h,0) = \underset{k\to 0}{Lt} \frac{f(h,0+k) - f(h,0)}{k} = \underset{k\to 0}{Lt} \frac{|h^2-k^4| - |h^4|}{k}$$

$k \to 0$ $h^2 > k^2 \Rightarrow |h^2 - k^4| = h^2 - k^2$ $= \dfrac{h^2 - k^2 - h^2}{k}$

$= 0$

Put values

$$f_{xy}(0,0) = \underset{h \to 0}{Lt} \frac{0-0}{h} = 0$$

$$f_{yx}(0,0) = \underset{k \to 0}{Lt} \frac{0-0}{k} = 0$$

$$f_{xy}(0,0) = f_{yx}(0,0)$$

(2c) Show $S = \{(x, 2y, 3x) : x, y \text{ are real}\}$ is subspace of $R^3(R)$. Find two bases. Also find dimension of S

To prove subspace : $\alpha, \beta \in S$, $a, b \in R$
$\exists\ a\alpha + b\beta \in S$

Let $\alpha = (x_1, 2y_1, 3x_1)$ $\quad \beta = (x_2, 2y_2, 3x_2)$

$$a\alpha + b\beta = \left[\underbrace{ax_1 + bx_2}, \ 2(ay_1 + by_2), \ 3(ax_1 + bx_2)\right]$$

If $ax_1 + bx_2 = x_3$, $ay_1 + by_2 = y_3$

then $a\alpha + b\beta = [x_3, 2y_3, 3x_3]$

↳ Form of subspace ✓

Bases

$$\begin{pmatrix} x \\ 2y \\ 3x \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Bases

Bases $= (1,0,3), (0,2,0)$ & dimension $= 2$

(3a) If $u = x^2 + y^2 \quad v = x^2 - y^2 \quad x = r\cos\theta \quad y = r\sin\theta$

Find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ [7]

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} =$$

$$= \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-8xy)(r)$$

$$= -8xy\sqrt{x^2+y^2}$$

or

$$= -4r^3\sin 2\theta$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$
$\sin 2\theta = 2\sin\theta\cos\theta$

3a(ii) If $\int_0^x f(t)dt = x + \int_x^1 t f(t)dt$. Find $f(1)$ [5]

$$I(x) = \int_{\varphi(x)}^{\psi(x)} f(t)dt$$

$$\frac{d}{dx}[I(x)] = f\{\psi(x)\}\left\{\frac{d}{dx}\psi(x)\right\} - f(\varphi(x))\left\{\frac{d}{dx}\varphi(x)\right\}$$

differentiate w.r.t x both sides

$$f(x)\frac{dx}{dx} - f(0)\frac{d0}{dx} = 1 + \left\{1f(1)\frac{d(1)}{dx} - xf(x)\frac{dx}{dx}\right\}$$

$$f(x) = 1 - xf(x)$$

$$(1+x)f(x) = 1 \qquad f(x) = \frac{1}{1+x}$$

$$f(1) = \frac{1}{2}$$

3b) Express $\int_a^b (x-a)^m (b-x)^n dx$ in terms of Beta function [8]

Soln : we have $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ $m>0$, $n>0$

$I = \int_a^b (x-a)^m (b-x)^n dx$

Put $x = a + (b-a)t$ $\quad dx = (b-a)dt$

$x - a = (b-a)t$

$b - x = b - a - (b-a)t = (b-a)[1-t]$

$x = a \Rightarrow t = 0$ $\quad x = b \Rightarrow t = 1$

$$I = \int_{t=0}^{t=1} \underline{(b-a)^m t^m} \, \underline{(b-a)^n (1-t)^n} \, \underline{(b-a)dt}$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^{(m+1)-1} (1-t)^{(n+1)-1} dt$$

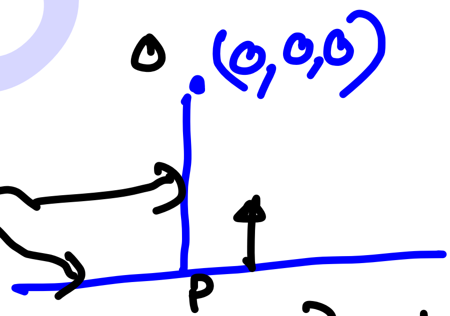
$$= (b-a)^{m+n+1} B(m+1, n+1)$$

(3b) A sphere of constant radius r passes through origin O and cuts the axes at points A, B & C. Find, the locus of the foot of the perpendicular drawn from O to the plane ABC.

Let $(a,0,0)$ $(0,b,0)$, $(0,0,c)$ be A, B, C.

Sphere thru OABC is $x^2+y^2+z^2-ax-by-cz=0$

plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$



$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

DRs $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$

Line OP is

$$\frac{x-0}{1/a}=\frac{y-0}{1/b}=\frac{z-0}{1/c}=\lambda$$

Let P be (α,β,γ)

$\frac{\lambda}{a}=\alpha$, $\frac{\lambda}{b}=\beta$ $\frac{\lambda}{c}=\gamma$

$a=\lambda/\alpha$, $b=\lambda/\beta$ $c=\lambda/\gamma$

Foot P pass thru $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow \frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1$

Put a, b, c value $\Rightarrow$

$$\frac{1}{\lambda}\left(\alpha^2+\beta^2+\gamma^2\right)=1 \quad \text{—(1)}$$

Radius of sphere $=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{c}\right)^2+\left(\frac{c}{2}\right)^2}=r$

$4r^2 = a^2+b^2+c^2$

$= \lambda^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)$ — Put value — ②

Eliminate λ from ① & ②

$(\alpha^2+\beta^2+\gamma^2)^2(\alpha^{-2}+\beta^{-2}+\gamma^{-2}) = 4r^2$

Locus of foot (α,β,γ) is

$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2}) = 4r^2$

(3ci) Prove that eigen vectors, corresponding to two distinct eigen values of a real symmetric matrix are orthogonal [8]

SuccessClap Question Bank SC_A05 Qn 32

$x_1 \to \lambda_1$ Eigen vectors, values
$x_2 \to \lambda_2$ Given $\lambda_1 \neq \lambda_2$

To show x_1, x_2 are orthogonal : $x_2^T x_2 = 0$

$$A x_1 = \lambda_1 x_1$$
$$A x_2 = \lambda_2 x_2$$

$$(\lambda_1 - \lambda_2) x_2^T x_1 = \lambda_1 x_2^T x_1 - \lambda_2 x_2^T x_1$$
$$= x_2^T (\lambda_1 x_1) - (\lambda_2 x_2)^T x_1$$
$$= x_2^T A x_1 - (A x_2)^T x_1$$
$$= x_2^T A^T x_1 - (A x_2)^T x_1$$
$$= (A x_2)^T x_1 - (A x_2)^T x_1$$
$$= 0$$

$A = A^T$

As $\lambda_1 \neq \lambda_2$ only possibility $x_2^T x_1 = 0$
x_1, x_2 are orthogonal

3c(ii) For two square matrices A and B of order 2, show trace AB = trace BA. Hence show that $AB - BA \neq I_2$ [7]

I_2 is identity matrix of order 2

$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$AB = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix}$

$BA = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} b_1a_1 + b_2a_3 & b_1a_2 + b_2a_4 \\ b_3a_1 + b_4a_3 & b_3a_2 + b_4a_4 \end{bmatrix}$

Trace $AB = a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4$

Trace $BA = b_1a_1 + b_2a_3 + b_3a_2 + b_4a_4$

Clearly Trace AB = Trace BA

Let $C = AB - BA$

Trace C = Trace$(AB - BA)$

= Trace AB − Trace BA

$= 0$

$\Rightarrow C \neq I_2$ → If $C = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Trace $C = 2$

$\neq 0$

(4a i) Reduce to row-reduced echelon form and find rank. [10]

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 - 3R_1$ $\quad R_2 \to R_2/2$

$$\begin{vmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{vmatrix}$$

$R_3 \to R_3 + R_2$

$R_4 \to R_4 + 5R_1$

$$\begin{vmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{vmatrix}$$

$R_4 \to \frac{R}{4}$ $\quad R_3 \leftrightarrow R_4$

$$\begin{vmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \to \underline{\text{Rank is } 3}$$

(4a ii) Find eigen values and vectors of $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ over complex field [10]

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + i^2 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$\underline{\lambda = -1}$: $(A-\lambda I)X = 0$ $\quad \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{array}{l} x - iy = 0 \\ ix + y = 0 \end{array}\right] \text{ Both eqn are same}$$

$x - iy = 0$ Let $y = \alpha$ $\quad x = i\alpha$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} i\alpha \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}$$

E. vector $\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\underline{\lambda = 1}$: $(A-\lambda I)X = 0$ $\quad \begin{vmatrix} -1 & -i \\ i & -1 \end{vmatrix}\begin{vmatrix} x \\ y \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{array}{l} x + iy = 0 \\ ix - y = 0 \end{array}\right] \text{ Both are same} : \text{Let } y = \beta$$

$$\Rightarrow x = -i\beta$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -i\beta \\ \beta \end{bmatrix} = \beta\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

E. vector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

(4b) Show entire area of Astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is $\frac{3}{8}\pi a^2$ [15]

SucessClap Question Bank : Areas On 22

$x=a\cos^3\theta$

$y=a\sin^3\theta$

$\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$

$$\text{Area}=4\int_0^a y\,dx$$

$$=4\int_{\theta=\pi/2}^{\theta=0}(a\cos^3\theta)(3a\cos^2\theta)(-\sin\theta)\,d\theta$$

$$=12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$4+2=6$

$$(12a^2)\frac{3.1.1}{6.4.2}\times\frac{\pi}{2}=\frac{3}{8}\pi a^2$$

(4c) Find equation of plane containing lines

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} \quad \& \quad \frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$$

Also find pt of intersection of given lines

[15]

Eqn of plane containing lines

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

$$x-2y+z=0$$

Any pt on Line 1 is $(3r_1-1, 5r_1-3, 7r_1-5)$

Any pt on Line 2 is $(r_2+2, 3r_2+4, 5r_2+6)$

It intersects $3r_1-1=r_2+2 \Rightarrow 3r_1-r_2-3=0$

$5r_1-3=3r_2+4 \Rightarrow 5r_1-3r_2-7=0$

$7r_1-5=5r_2+6 \Rightarrow 7r_1-5r_2-11=0$

$$\frac{r_1}{7-9} = \frac{r_2}{-15+21} = \frac{1}{-9+5} \Rightarrow r_1=\frac{1}{2} \quad r_2=\frac{-3}{2}$$

$\Rightarrow \left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ is pt of intersection

Section B

(5a) Solve $\frac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x\cos 2x$

AE: $D^2+2=0$ $\quad D=\pm i\sqrt{2}$

$CF = C_1\cos x\sqrt{2} + C_2\sin x\sqrt{2}$

$PI_1 = \frac{1}{D^2+2}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2+2}x^2 = e^{3x}\frac{x^2}{D^2+6D+11}$

$= e^{3x}\frac{1}{11\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)}x^2 = \frac{e^{3x}}{11}\left[1+\left(\frac{6D+D^2}{11}\right)\right]^{-1}x^2$

$= \frac{e^{3x}}{11}\left\{1-\left(\frac{6D+D^2}{11}\right)+\left(\frac{6D+D^2}{11}\right)^2+\dots\right]x^2$

$= \frac{e^{3x}}{11}\left\{1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+(\,)D^3+(\,)D^4\right]x^2$

$Dx^2 = 2x \quad D^2x^2 = 2 \quad D^3x^2 = 0$

$= \frac{e^{3x}}{11}\left\{x^2-\frac{12x}{11}-\frac{2}{11}+\frac{72}{121}\right\}$

$= \frac{e^{3x}}{11}\left\{x^2-\frac{12x}{11}+\frac{50}{121}\right\}$

$36 \times 2 = 72$

$72 - 22 = 50$

$$PI_2 = \frac{1}{D^2+2} e^x \cos 2x$$

$$= e^x \frac{1}{(D+1)^2+2} \cos 2x = e^x \frac{\cos 2x}{D^2+2D+3}$$

$D^2 = -2^2$

$-4+2D+3$
$= 2D-1$

D

$$= e^x \frac{1}{2D-1} \cos 2x$$

$$= e^x \frac{2D+1}{4D^2-1} \cos 2x$$

$$= e^x \frac{(2D+1)\cos 2x}{(-17)}$$

$$= \left(\frac{-e^x}{17}\right)(-4\sin 2x + \cos 2x)$$

$$= \frac{e^x}{17}(4\sin 2x - \cos 2x)$$

(5b) Solve using Laplace

$$\frac{d^2y}{dx^2}+4y = e^{-2x}\sin 2x \qquad y(0)=y'(0)=0$$

$$L(y'') + 4L(y) = L(e^{-2x}\sin 2x)$$

$$s^2L(y) - sy(0) - y'(0) + 4L(y) = \frac{2}{(s+2)^2+4}$$

$$(s^2+4)L(y) = \frac{2}{s^2+4s+8}$$

$$L(y) = \frac{2}{(s^2+4s+8)(s^2+4)} = 2\left[\frac{1-s}{20(s^2+4)} + \frac{s+3}{20(s^2+4s+8)}\right]$$

$$= \frac{1}{10}\left\{\frac{1-s}{s^2+4} + \frac{s+3}{s^2+4s+8}\right\} \qquad s+3 = s+2+1$$

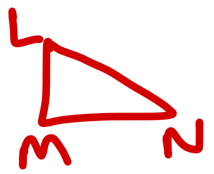
$$= \frac{1}{10}\left\{\frac{1}{s^2+4} - \frac{s}{s^2+4} + \frac{s+2}{(s+2)^2+2^2} + \frac{1}{2}\frac{2}{(s+2)^2+2^2}\right\}$$

Take lap-inverse

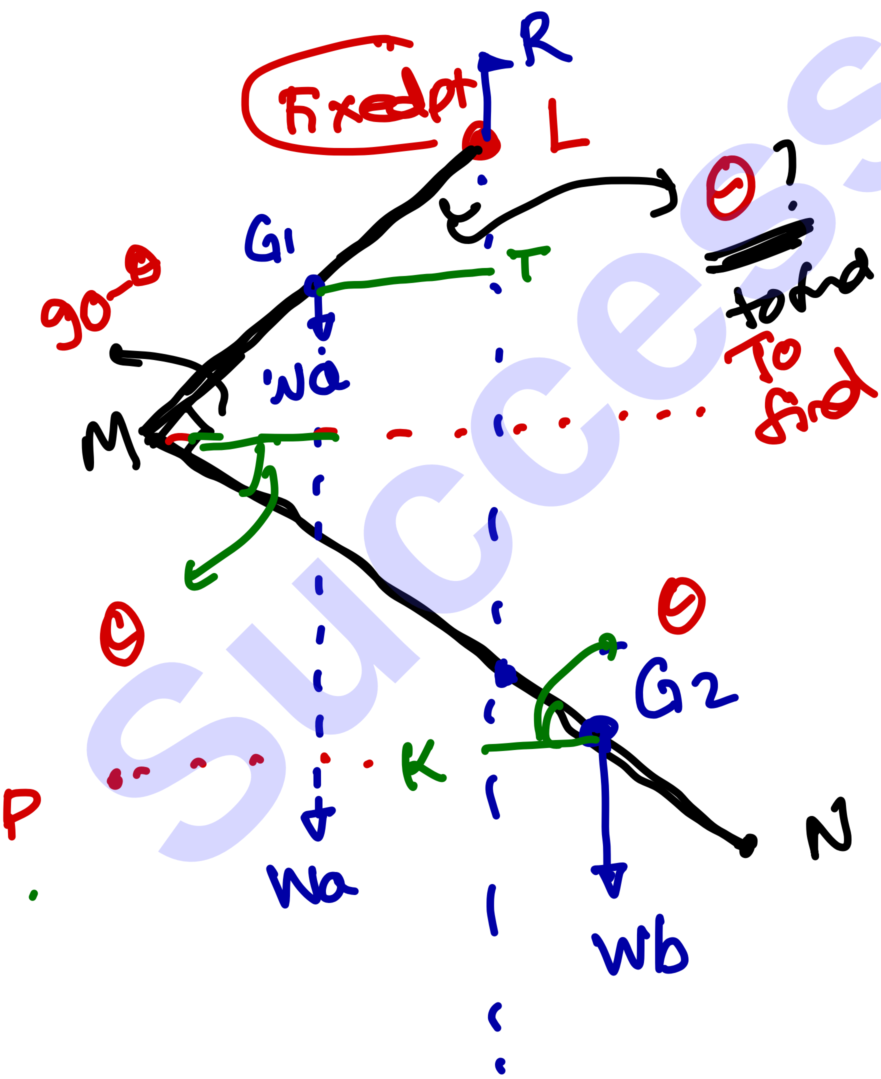
$$y = \frac{1}{10}\left\{\frac{\sin 2t}{2} - \cos 2t + e^{-2t}\cos 2t + \frac{e^{-2t}\sin 2t}{2}\right\}$$

$$= \frac{1}{20}\left(\sin 2t - 2\cos 2t + 2e^{-2t}\cos 2t + e^{-2t}\sin 2t\right)$$

(5c) Two rods LM and MN are jointly rigidly at the point M such that $LM^2 + MN^2 = LN^2$ and they are hanged freely in equilibrium from a fixed point L. Let ω be the weight per unit Length of both rods which are uniform. Determine angle, which rod LM makes with vertical direction, in terms of length of rods

$LM^2 + MN^2 = LN^2 \Rightarrow$ 

Rods LM] joined
Rod MN] at O



length LM = a
length MN = b
G_1, G_2, centre
↳ because uniform
w → weigth per unit length

Reaction at R
L → fixed

Let θ]
Angle: rod LM makes with vertical

wa → Make clockwise rotation

$W_b \rightarrow$ Anti clockwise rotation

Take Moment at L

$$(W_a)(G_1T) = (W_b)(G_2K)$$

$$G_1T = \frac{a}{2}\sin\theta$$

$$G_2P = \frac{b}{2}\cos\theta$$

$$PK = LM\cos(90-\theta)$$

$$= a\sin\theta$$

$$G_2K = G_2P - PK$$

$$= \frac{b}{2}\cos\theta - a\sin\theta$$

$$W_a \cdot \frac{a}{2}\sin\theta = W_b\left(\frac{b}{2}\cos\theta - a\sin\theta\right)$$

$$a^2\sin\theta = b^2\cos\theta - ab\sin\theta \cdot 2$$

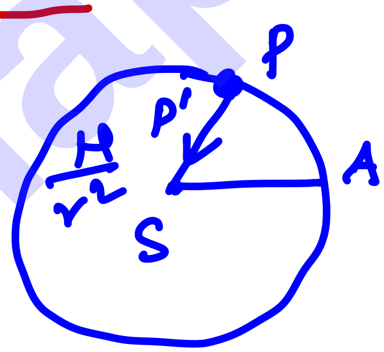
$$(a^2+2ab)\sin\theta = b^2\cos\theta$$

$$\tan\theta = \frac{b^2}{a^2+2ab}$$

5d) If a planet, which revolves around Sun in circular, is suddenly stopped in its orbit, then find the time in which it would fall into Sun. Also, find the ratio of its falling time to the period of revolution of the planet

SuccessClap Question Bank SCE04 Qn 34

acc $= \frac{\mu}{r^2}$ along $P'S$ ($r \downarrow$ as $t \uparrow$)



$$v\frac{dv}{dr} = -\frac{\mu}{r^2} \qquad 2v\,dv = -\frac{2\mu}{r^2}dr$$

$$v^2 = \frac{2\mu}{r} + A \qquad \rightarrow \text{Initially at } P \; r=a \; v=0$$
$$\Rightarrow A = -\mu/a$$

$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) \qquad v = \frac{dr}{dt}$$

$$\left(\frac{dr}{dt}\right)^2 = -\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-r}{r}} \qquad -\text{ve} \rightarrow r \downarrow \text{as } t \uparrow$$

$$\int_0^{T'} dt = -\sqrt{\frac{a}{2\mu}}\int_a^0 \sqrt{\frac{r}{a-r}}\,dr$$

$$r = a\cos^2\theta$$
$$dr = 2a\cos\theta\sin\theta$$

$$T' = 2a\sqrt{\frac{a}{2\mu}}\int_0^{\pi}\cos^2\theta\, d\theta$$

$$= 2a\sqrt{\frac{a}{2\mu}}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^{3/2}}{2\sqrt{2}\mu^{1/2}}$$

T = Time period of planet revolution

$$= \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$\frac{T'}{T} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$$

$$\boxed{T' = \frac{\sqrt{2}}{8}T}$$

SuccessClap - Paper I

5e) Show $\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r}\right)\right] = \frac{2}{r^4}$ $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

In Books $\nabla^2\left[\nabla\cdot\frac{\vec{r}}{r^2}\right] = \frac{2}{r^4}$ (Krisha Series & SuccessClap Question Bank)

UPSC forgot to add 2 while lifting question.

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}(\nabla\cdot\vec{r}) + \vec{r}\cdot\left(\nabla\frac{1}{r}\right)$$

$\nabla\cdot(\phi A) = \phi(\nabla\cdot A) + A\cdot(\nabla\phi)$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}$$

$\nabla\cdot r = 3$

$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r} = -\frac{\vec{r}}{r^3}$

$$\nabla^2\left(\frac{2}{r}\right) = \nabla\cdot\nabla\left(\frac{2}{r}\right)$$

$$= (-2)\,\nabla\cdot\left(\frac{\vec{r}}{r^3}\right)$$

$\nabla\left(\frac{2}{r}\right) = \frac{-2}{r^2}\hat{r} = \frac{-2\vec{r}}{r^3}$

$$= (-2)\left[\frac{1}{r^3}\nabla\cdot\vec{r} + \vec{r}\cdot\nabla\left(\frac{1}{r^3}\right)\right]$$

$\nabla\left(\frac{1}{r^3}\right) = \frac{-3}{r^4}\hat{r} = \frac{-3\vec{r}}{r^5}$

$$= (-2)\left[\frac{3}{r^3} - \frac{3}{r^3}\right]$$

$$= 0$$

Answer is zero

(6a) A heavy string, non uniform density, is hung from two points. Let T_1, T_2, T_3 be tensions at A, B, C where its inclination to horizontal are in Arithmetic progression with common difference β. Let w_1, w_2 weights of parts AB, BC of string.

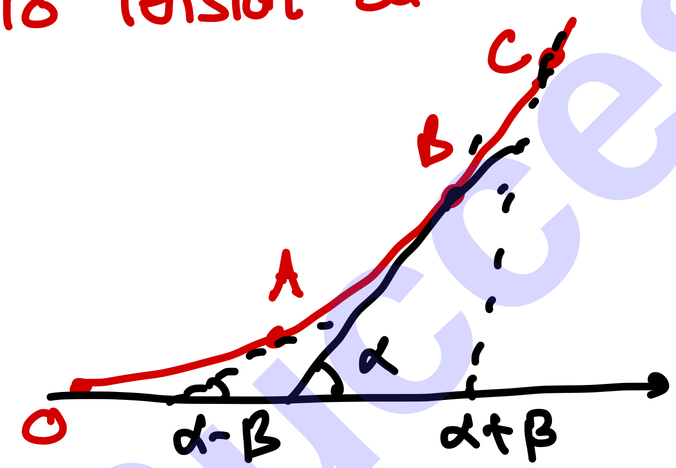
Prove (i) Harmonic mean of $T_1, T_2, T_3 = \frac{3T_2}{1+2\cos\beta}$

(ii) $\frac{T_1}{T_3} = \frac{w_1}{w_2}$ [20]

Let O be lowest pt

T_1, T_2, T_3 Tension at A, B, C

T_0 Tension at O

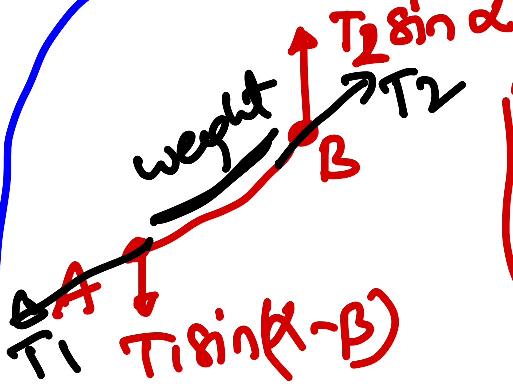


If angle at B is α
↳ Angle at A is α-β } why? given
↳ Angle at C is α+β } Arithmetic progression with difference β.

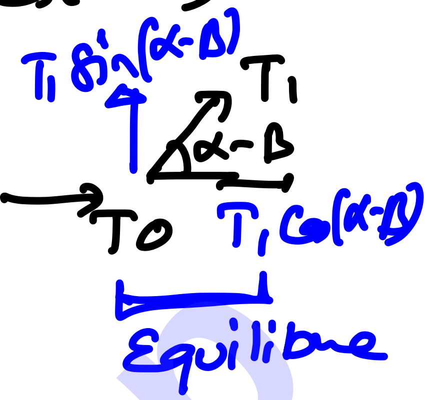
So $\angle A = \alpha-\beta$ $\angle B = \alpha$ $\angle C = \alpha+\beta$

Equilibrium

$$T_0 = T_1\cos(\alpha-\beta) = T_2\cos\alpha = T_2\cos(\alpha+\beta)$$



$$\boxed{T_2\sin\alpha - T_1\sin(\alpha-\beta) = w_1}$$



Similarly $\boxed{T_3\sin(\alpha+\beta) - T_2\sin\alpha = w_2}$

$$\frac{1}{T_1}+\frac{1}{T_3} = \frac{\cos(\alpha-\beta)}{T_0}+\frac{\cos(\alpha+\beta)}{T_0}$$

$$= \frac{1}{T_0}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$= \frac{1}{T_0}2\cos\alpha\cos\beta = \frac{2\cos\beta}{T_2}$$

Similarly

$$\frac{T_0}{\cos\alpha} = T_2$$

Harmonic mean of T_1, T_2, T_3 is $\dfrac{3}{\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}}$

$$\underbrace{\frac{1}{T_1}+\frac{1}{T_3}}+\frac{1}{T_2} = \frac{2\cos\beta}{T_2}+\frac{1}{T_2} = \frac{2\cos\beta+1}{T_2}$$

$$= \frac{3}{\frac{2\cos\beta+1}{T_2}} = \frac{3T_2}{1+2\cos\beta}$$

(ii) $\omega_1 T_3 - \omega_2 T_1 = T_2 T_3 \sin\alpha - T_1 T_3 \sin(\alpha-\beta) - T_1 T_3 + T_1 T_2 \sin\alpha \sin(\alpha+\beta)$

$\omega_1 = T_2 \sin\alpha - T_1 \sin(\alpha-\beta)$

$\omega_2 = T_3 \sin(\alpha+\beta) - T_2 \sin\alpha$

$T_0 = T_1 \cos(\alpha-\beta) = T_2 \cos\alpha = T_3 \cos(\alpha+\beta)$

Get $T_1 T_3$ format

Put $T_2 = \dfrac{T_1 \cos(\alpha-\beta)}{\cos\alpha}$ in 1st $\quad T_2 = \dfrac{T_3 \cos(\alpha+\beta)}{\cos\alpha}$ in 4th

$$\omega_1 T_3 - \omega_3 T_1 = T_1 T_3 \left\{ \frac{\sin\alpha \cos(\alpha-\beta)}{\cos\alpha} - \sin(\alpha-\beta) - \sin(\alpha+\beta) + \frac{\sin\alpha \cos(\alpha+\beta)}{\cos\alpha} \right\}$$

$$= T_1 T_3 \left\{ \frac{\sin\alpha}{\cancel{\cos\alpha}} \left[\cancel{2\cos} A \cos B \right] - 2 \sin A \cos B \right]$$

$$\cos(A-B) + \cos(A+B) = \cos A \cos B + \sin A \sin B + \cos A \cos B - \sin A \sin B = 2\cos A \cos B$$

$$\sin(A+B) + \sin(A-B) = \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B = 2 \sin A \cos B$$

$\omega_1 T_3 - \omega_3 T_1 = 0$

$$\Rightarrow \frac{T_1}{\omega_1} = \frac{T_2}{\omega_2}$$

(65) Solve $\frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2 x=\cos^4 x$

change to independent variable $z=f(x)$

$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1=R_1$ $\quad P_1=\dfrac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$

$Q_1=\dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ $\quad R_1=\dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

choose $\left(\frac{dz}{dx}\right)^2=\cos^2 x$ $\quad$ s.t $Q_1=\frac{2\cos^2 x}{\cos^2 x}=1$

$\frac{dz}{dx}=\cos x$ $\quad z=\sin x$

$\frac{d^2z}{dx^2}=-\sin x \Rightarrow P_1=\dfrac{-\sin x+(\tan x-3\cos x)\cos x}{\cos^2 x}$

$=\dfrac{-\sin x+\sin x-3\cos^2 x}{\cos^2 x}=-3$

$R_1=\frac{\cos^4 x}{\cos^2 x}=$

$=\cos^2 x$

$=1-\sin^2 x$

$=1-z^2$

$\frac{d^2y}{dz^2}-3\frac{dy}{dz}+2=1-z^2$

$m^2-3m+2=0 \Rightarrow (m-1)(m-2)=0$

$m=1,2$

$y_c(z)=Ae^{z}+Be^{2z}$

$y_c=Ae^{\sin x}+Be^{2\sin x}$

$$u_p = \frac{1}{D^2-3D+2}(1-z^2) = \frac{1}{2\left(1-\frac{3D}{2}+\frac{D^2}{2}\right)}(1-z^2)$$

$$= \frac{1}{2}\left[1-\left(\frac{3D}{2}-\frac{D^2}{2}\right)\right]^{-1}(1-z^2)$$

$$= \frac{1}{2}\left\{1+\left(\frac{3D}{2}-\frac{D^2}{2}\right)+\left(\frac{3D}{2}-\frac{D^2}{2}\right)^2+\cdots\right](1-z^2)$$

$$= \frac{1}{2}\left\{1+\frac{3D}{2}-\frac{D^2}{2}+\frac{9D^2}{4}+\frac{D^4}{4}-\frac{3D^3}{2}\right]$$

$D(1-z^2) = -2z$

$D^2(1-z^2) = -2$ $\quad D^3 \to 0 \quad D^4 \to 0$

$$\frac{9D^2}{4} - \frac{D^2}{2}\times\frac{2}{2}$$

$$\frac{7}{4}D^2$$

$$u_p = \frac{1}{2}\left\{(1-z^2)-3z-\frac{7}{2}\right)$$

$\frac{3}{2}D$

$\frac{3}{2}(-2z) \quad \frac{7}{4}(-2)$

$-3z$

$$= -\frac{1}{2}\left\{z^2-3z+\frac{5}{2}\right)$$

$$= -\frac{1}{4}(2z^2-6z+5)$$

$$1-\frac{7}{2} = \frac{-5}{2} \quad -\frac{7}{2}$$

$$u_p = -\frac{1}{4}(2\sin^2 x - 6\sin x + 5)$$

(6c) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, C is arbit closed curve in xy-plane and $\vec{F} = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ [15]

Note: Curve not given → Origin lie inside Curve
→ origin lie outside Curve

$d\vec{r} = dx\hat{i}+dy\hat{j}$

$x = r\cos\theta$

$dx = -r\sin\theta\, d\theta + \cos\theta\, dr$

$y = r\sin\theta$

$dy = r\cos\theta\, d\theta + \sin\theta\, dr$

$$\vec{F}\cdot d\vec{r} = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

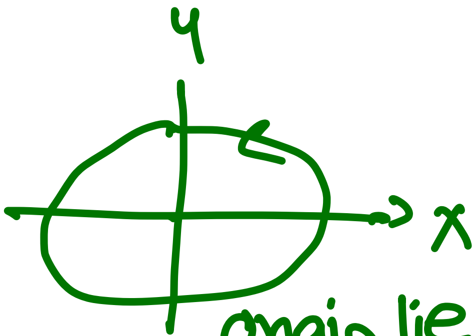
$$= \frac{r^2\sin^2\theta\, d\theta - r\sin\theta\cos\theta\, dr + r^2\cos^2\theta\, d\theta + r\sin\theta\cos\theta\, dr}{r^2}$$

$$= \frac{r^2}{r^2}(\sin^2\theta+\cos^2\theta)\,d\theta = d\theta$$

$$I = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi \quad \text{(lie inside)}$$

$$= \int_{\theta_0}^{\theta_0} d\theta = 0 \quad \text{(lie outside)}$$



origin lie inside

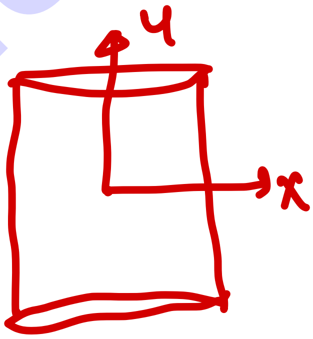
origin lie outside

(7a) Verify Gauss divergence Thm for $F = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ taken over first octant $y^2+z^2=9$, $x=2$ [20]

$$\int_S F.n\,ds = \int_V \nabla.F\,dV$$

$$\nabla.F = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8xz$$



$$I = \int_V \nabla.F\,dV = \int_{z=0}^{z=3}\int_{y=0}^{y=\sqrt{9-z^2}}\int_{x=0}^{x=2}\underbrace{\left(4xy-2y+8xz\right)dx}\,dy\,dz$$

$$= \int_0^3\int_0^{\sqrt{9-z^2}} \left.\frac{4yx^2}{2} - 2xy + 8z\frac{x^2}{2}\right|_0^2 dy\,dz$$

$$(8y - 4y + 16z)\,dy\,dz$$

$$(4y+16z)\,dy\,dz$$

$$\left.\frac{4y^2}{2} + 16zy\right|_0^{\sqrt{9-z^2}} dz$$

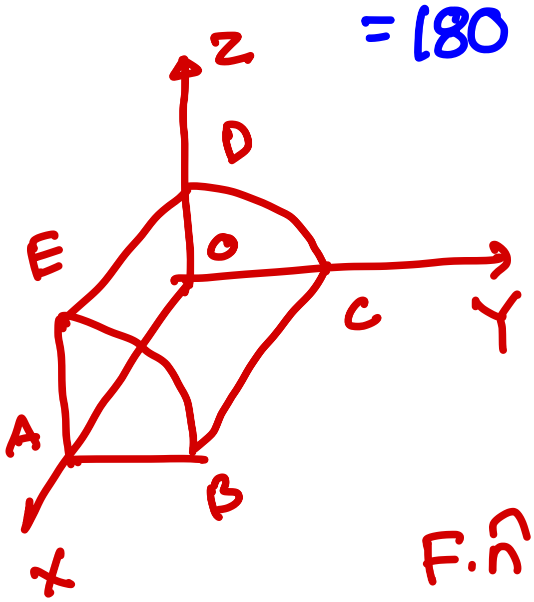
$$\int_0^3 \left[2(9-z^2) + 16.z.\sqrt{9-z^2}\right]dz$$

$$18 - 2z^2 + 16z\sqrt{9-z^2}$$

$$18z - \frac{2z^3}{3} - 8\cdot\frac{2}{3}(9-z^2)^{3/2}\Big|_0^3$$

$$18.3 - \frac{2}{3}.27 - \frac{16}{3}(-27)$$

$= 180$ (By divergace



$S_1 : OABC$

$\int_S F.n\,dS$ $\quad \hat{n} = -\hat{k}$

$dS = dx\,dy$

$z = 0$

$F.\hat{n} = -4xz^2$

$z = 0 \Rightarrow F.n = 0 \Rightarrow \int_S F.n\,dS = 0$

$S_2 = OAED \quad : \quad \hat{n} = -\hat{j} \quad y = 0$

$F.n = y^2$ But $y = 0 \Rightarrow \int_S F.n\,dS = 0$

$= 0$

$S_3 : ABE \quad \hat{n} = \hat{i} \quad dS = dy\,dz \quad F.n = 2x^2y$

$x = 2$ $\qquad = 8y$

$$\int_S F.n\,dS = \int_S 8y\,dy\,dz$$

$$= \int_{z=0}^{z=3}\int_{y=0}^{\sqrt{9-z^2}} 8y\,dy\,dz$$

$$= \int_0^3 \frac{8y^2}{2} \Big|_0^{\sqrt{9-z^2}} dz$$

$$= \int_0^3 4(9-z^2)\, dz = 4\left[9z - \frac{z^3}{3}\right]_0^3$$

$$= 4[27-9] = 72$$

S_4: ODK $\quad x=0 \quad \hat{n} = -\hat{i} \quad ds = dy\,dz$

$$F.\hat{n} = -2x^2y \quad x=0 \Rightarrow F.n = 0$$

$$\int_S F.n\,ds = 0$$

S_5: BCDE Curved

$y^2+z^2-9=0 \quad \vec{n} = \nabla(y^2+z^2-9) = 2y\hat{j} + 2z\hat{k}$

$$\hat{n} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4y^2+4z^2}} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4.9}} = \frac{y\hat{j}+z\hat{k}}{3}$$

$$\hat{n}.\hat{k} = \frac{z}{3}$$

If $dS = \dfrac{dx\,dy}{|\hat{n}.\hat{k}|}$ $\quad \displaystyle\int F.\hat{n} \frac{dx\,dy}{|\hat{n}.\hat{k}|}$

$$F.n = \frac{-y^3+4xz^3}{3}$$

$$\int_S F.n\,ds = \int_S \frac{-y^3+4xz^3}{3} \cdot \frac{dx\,dy}{z/3}$$

$$= \int \left[\frac{-y^3}{2} + 4xz^2\right] dx\,dy$$

$$= \int\int \left[\frac{-y^3}{\sqrt{9-y^2}} + 4x(9-y^2)\right] dx\,dy$$

$$= \int_{y=0}^{y=3} \left[\int_{x=0}^{x=2} \frac{-y^3}{\sqrt{9-y^2}} + 4x(9-y^2)\right] dx\,dy$$

$$= \int_0^3 \frac{-y^3}{\sqrt{9-y^2}}x + \frac{4x^2}{2}(9-y^2)\Big|_0^2 dy$$

$$\int_0^3 \left[\frac{-2y^3}{\sqrt{9-y^2}} + 8(9-y^2)\right] dy$$

$$9-y^2 = t^2 \qquad -2y\,dy = 2t\,dt$$

$$\int_3^0 \frac{(9-t^2)\,2t\,dt}{t} = \int_3^0 18 - 2t^2 = 18t - \frac{2t^3}{3}\Big|_3^0$$

$$= -18\times 3 + 2\times\frac{3^3}{3} = -54 + 18 = -36$$

$$\begin{array}{r} 18 \\ \times\ 3 \\ \hline 54 \\ -18 \\ \hline 36 \end{array}$$

$$\int_0^3 8(9-y^2)dy = 8\left(9y - \frac{y^3}{3}\right)\Big|_0^3$$

$$= 8[27-9] = 144$$

$$\begin{array}{r} 27 \\ -9 \\ \hline 18 \\ \times\ 8 \\ \hline 144 \end{array}$$

$$\int_{S_5} F.n\,dS = -36 + 144 = 108$$

$$\int_{S_1+S_2+S_3+S_4+S_5} F.n\,dS = 0+0+72+0+108$$

$$= 180$$

Hence verified : Same value 180 by both methods

(76) Find possible solutions of

$$y^2 \log y = xy\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

[15M]

Let $v = \log y$ $\qquad \frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$ $\qquad p = \frac{dy}{dx}$

$p = yP$ $\qquad P = \frac{dv}{dx}$

$$y^2 \log y = (xy)yP + y^2P^2$$

$$\log y = xP + P^2$$

$$v = xP + P^2$$ Clairaut

$\downarrow$ soln replace P by c

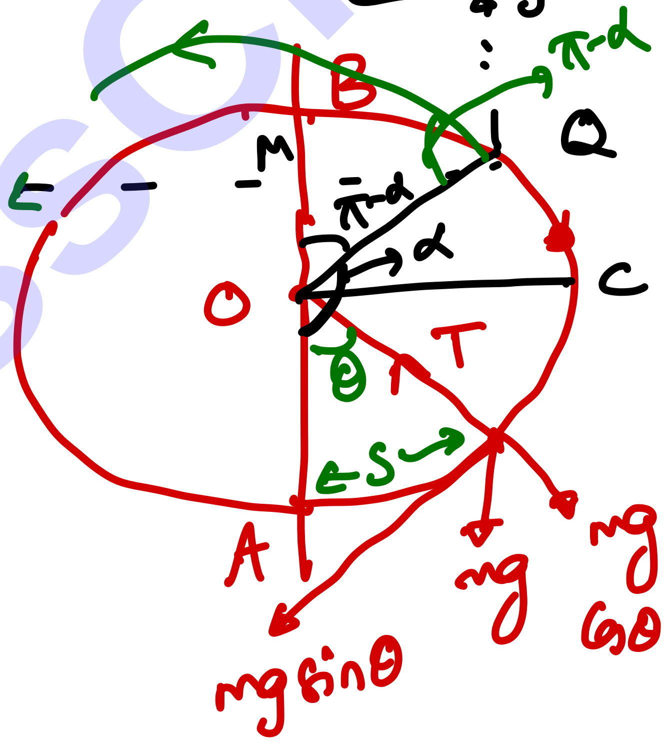
$$v = cx + c^2$$

$$\log y = cx + c^2$$

(7c) A heavy particle hangs by inextensible string of length a from fixed pt and is projected horizontally with velocity $\sqrt{2gh}$. If $\frac{5a}{2} > h > a$, prove the circular motion ceases when particle reached height $\frac{1}{3}(a+2h)$ from pt of projection. Also prove greatest height ever reached by particle above pt of projection is

$$\frac{(4a-h)(a+2h)^2}{27a^2}$$

[15]



Particle projected at A with $\sqrt{2gh}$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$\frac{mv^2}{a} = T - mg\cos\theta$$

$$s = a\theta$$

$\ddot{s} = a\ddot{\theta}$ $\quad ma\ddot{\theta} = -mg\sin\theta$

$a\ddot{\theta} = -g\sin\theta$

$\ddot{\theta} = \left(-\frac{g}{a}\right)\sin\theta$

$\dot{\theta}\,d\dot{\theta} = \left(-\frac{g}{a}\right)\sin\theta\, d\theta$

Integrate

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d}{dt}\dot{\theta}$$

$$\frac{\dot\theta^2}{2} = \frac{g}{a}\cos\theta + c$$

$$= \frac{d\dot\theta}{d\theta}\frac{d\theta}{dt}$$

$$\ddot\theta = \dot\theta\frac{d\dot\theta}{d\theta}$$

$$\dot\theta^2 = \frac{2g}{a}\cos\theta + c'$$

$$v = a\dot\theta \Rightarrow \dot\theta = \frac{v}{a}$$

$$v^2 = 2ag\cos\theta + c''$$

$\theta = 0 \quad v = \sqrt{2gh}$ Initial $\Rightarrow c'' = 2gh - 2ag$

$$\boxed{v^2 = 2ag\cos\theta + 2gh - 2ag}$$

$$T = \frac{m}{a}v^2 + mg\cos\theta$$

$$= \frac{m}{a}(2ag\cos\theta + 2gh - 2ag) + mg\cos\theta$$

$$= \frac{m}{a}(3ag\cos\theta + 2gh - 2ag)$$

↳ Leaves circular path when $T = 0$ at $\underline{\underline{Q}}$

↳ Let at $\theta = \alpha$

$$\Rightarrow 0 = \frac{m}{a}(3ag\cos\alpha + 2gh - 2ag)$$

$$\cos\alpha = -\left(\frac{2h-2a}{3}\right) = -\frac{2}{3}(h-a)$$

Given $h > a \Rightarrow \cos\alpha$ is negative & $|\cos\alpha| < 1$

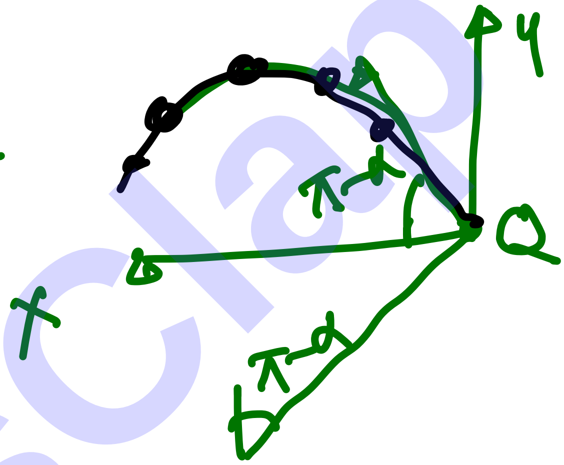
$\Rightarrow \alpha$ lies b/w $\frac{\pi}{2}$ & π ie $\frac{\pi}{2} < \alpha < \pi$

$$v^2 = 2gh - 2ga + 2ga\cos\alpha = 2g(h-a) - 2ga\frac{2}{3}(h-a)$$
$$= \frac{2g(h-a)}{3}$$

Height of Q above A $= AO + OM = a + a\cos(\pi - \alpha)$
$= a - a\cos\alpha \quad = a + \frac{1}{3}(2h - 2a) = \frac{a+2h}{3}$

Part 2:

↳ Particle leave at Q



Max height reached above Q is

$$H = \frac{v^2\sin^2\theta}{2g} = \frac{v^2\sin^2(\pi-\alpha)}{2g} \qquad \theta = \pi - \alpha$$

$$= \frac{v^2(1-\cos^2\alpha)}{2g} = \frac{2g(h-a)}{6g}\left\{1 - \frac{(2h-2a)^2}{9a^2}\right\}$$

$$= \frac{1}{27a^2}(h-a)(5a^2 + 8ah - 4h^2)$$
$$= \frac{(h-a)(a+2h)(5a-2h)}{27a^2}$$

Greatest height $= AM + H$

$$= \left(\frac{a+2h}{3}\right) + \frac{(h-a)(a+2h)(5a-2h)}{27a^2}$$
$$= \frac{1}{27a^2}(a+2h)(4a^2 + 7ah - 2h^2)$$
$$= \frac{1}{27a^2}(4a-h)(a+2h)^2$$

(8a i) Find orthogonal trajectory of

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad a>b>0, \ \lambda \text{ is parameter}$$

Show the given family is self orthogonal. [10]

SuccessClap Question Bank SC202-Qn.8

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \xrightarrow{\text{differentiate}}$$

$$\frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0 \Rightarrow \lambda=-\frac{\{b^2x+a^2y\,dy/dx\}}{x+y\,dy/dx}$$

$$a^2+\lambda=\frac{(a^2-b^2)x}{x+y\,dy/dx} \qquad b^2+\lambda=-\frac{(a^2-b^2)\,y\,dy/dx}{x+y\,dy/dx}$$

Puting values we get

$$\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=a^2-b^2 \rightarrow \text{DE of family}$$

↓ replace $\frac{dy}{dx} \rightarrow -\frac{dx}{dy}$

we again get

$$\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dy}{dx}\right)=a^2-b^2$$

→ Both eqn same. So self orthogonal

(8aii) Find General Soln of

$$x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=0$$

↳ Solve $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$ by method of variation of parameters [10]

SuccessClap Qn Bank SC-D05 Qn18 (Partial)

$$\frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=0$$

$P=-\frac{2(1+x)}{x}$

$Q=\frac{2(1+x)}{x^2}$

$R=0$

$P+Qx=0 \Rightarrow u=x$

General soln $y=vu=vx$

$u=x$

$\frac{du}{dx}=1$

$$\Rightarrow \frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=0$$

$$\frac{d^2v}{dx^2}+\left[-\frac{2(1+x)}{x}+\frac{2}{x}\right]\frac{dv}{dx}=0$$

$\frac{dv}{dx}=p$

$$\frac{dp}{dx}-2p=0$$

$$\frac{dp}{p}=2dx \Rightarrow \ln p=2x+c_1 \quad p=c_1'e^{2x}$$

$$\frac{dv}{dx}=p=c_1'e^{2x} \quad \Rightarrow v=\frac{c_1'}{2}e^{2x}+c_2$$

$$y=uv=c_2x+c_3xe^{2x}$$

To Solve $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$

by Variation of parameter method

$R=x$

$u=x \qquad v=xe^{2x}$

$$W=\begin{vmatrix} x & xe^{2x} \\ 1 & e^{2x}(1+2x) \end{vmatrix} = 2x^2e^{2x}$$

$y=Au+Bv$

$$A=-\int \frac{vR}{W}$$

$$=-\int \frac{xe^{2x}.x}{2x^2e^{2x}}dx+c_1 = \frac{-x}{2}+c_1$$

$$B=\int \frac{uR}{W} = \int \frac{x.x}{2x^2e^{2x}}dx+c_2 = \frac{1}{2}\int e^{-2x}+c_2$$

$$= \frac{e^{-2x}}{-4}+c_2$$

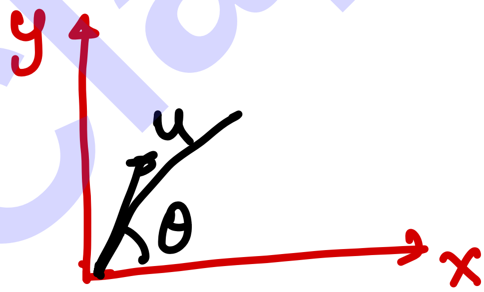
$$y=\left(-\frac{x}{2}+c_1\right)x+\left(-\frac{1}{4}e^{-2x}+c_2\right)xe^{2x}$$

$$=c_1x+c_2xe^{2x}-\frac{x^2}{4}-\frac{x}{4}$$

(8) Describe the motion & path of particle of mass m which is projected in a vertical plane through point of projection with velocity u in a direction making an angle θ with horizontal dr x.

Further if particle projected with $4\sqrt{g}$ velocity, determine the locus of vertices of paths.

Initial velocity = u



$m\ddot{x}=0$

$m\ddot{y}=-mg$ $\quad \ddot{y}=g$

$\quad \hookrightarrow \dot{y}=-gt+B$

$\dot{x}=u\cos\alpha$

$t=0 \quad \dot{y}=u\sin\alpha \Rightarrow B=u\sin\alpha$

$\dot{y}=u\sin\alpha-gt$

$\hookrightarrow$ Integrate $\quad y=u\sin\alpha\, t-\frac{gt^2}{2}+D$

$t=0 \; y=0 \Rightarrow D=0$

Integrate $\quad x=u\cos\alpha\, t$

$$\boxed{\begin{aligned} x&=u\cos\alpha\, t \\ y&=u\sin\alpha\, t-\frac{gt^2}{2} \end{aligned}}$$

Put $t=\frac{x}{u\cos\alpha}$

we get $y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \frac{g}{2}\left(\frac{x}{u\cos\alpha}\right)^2$

$$\boxed{y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}}$$ Path/Trajectory

↳ Parabola

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

Multiply both sides by $-\frac{2u^2\cos^2\alpha}{g}$

$$x^2 - \frac{2u^2x\sin\alpha\cos\alpha}{g} = -\frac{2u^2y\cos^2\alpha}{g}$$

Add $\left(\frac{u^2\sin\alpha\cos\alpha}{g}\right)^2$ on both sides

$$\left(x - \frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$$

↳ Parabola $(x-h)^2 = -4a(y-k)$

vertex is $\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$

Vertex is $\left(\frac{u^2\sin2\alpha}{2g}, \frac{u^2\sin^2\alpha}{2g}\right)$ latus rectum $= \frac{2u^2\cos^2\alpha}{g}$

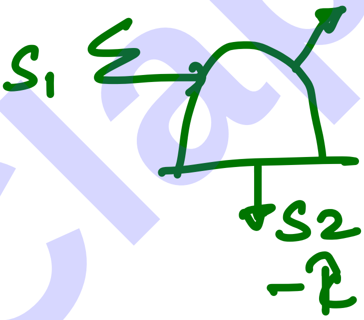
(ii) $u = 4\sqrt{g}$ vertex $(8\sin2\alpha, 8\sin^2\alpha)$

(8c) Use stoke to evaluate $\int_S (\nabla \times F) \cdot n \, dS$

$F = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xy + z^2)\hat{k}$

S is surface of paraboloid $z = 4 - (x^2 + y^2)$ above xy-plane [15]

SuccessClap Question Bank SC-F02 Qn 16



$$\int_{\substack{\text{Complete} \\ S_1 + S_2}} (\nabla \times F) \cdot dS = 0$$

$$= \int_{S_1} + \int_{S_2}.$$

we want

$$I = \int_{S_1} = -\int_{\substack{S_2 \\ -\hat{k}}} (\nabla \times F) \cdot n \, dS \qquad \hat{n} = -\hat{k}$$

$$= \int_{S_2} (\nabla \times F) \cdot \hat{k} \, ds$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x & x \end{vmatrix} = -2\hat{j} + (3y - 1)\hat{k}$$

$(\nabla \times F) \cdot \hat{k} = (3y - 1)$

$$I = \int_{S_2} (3y - 1) \, dS$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} (3r\sin\theta - 1) \, r \, d\theta \, dr$$

Polar
$y = r\sin\theta$
$dS = r\,dr\,d\theta$
$r = 4$

$$= \int_0^{2\pi}\int_0^4 3r^2 \sin\theta \, d\theta \, dr - \int_0^{2\pi}\int_0^4 r \, d\theta \, dr$$

$$= 0 - \int_0^{2\pi} \frac{r^2}{2}\Big|_0^4 d\theta \qquad \int_0^{2\pi} \sin\theta \, d\theta = 0$$

$$= -16\pi$$

$8 \times 2\pi$