TEST-2 Paper-1

Wilsonpandu[redacted]

(1a) $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$Z_5 = \{0, 1, 2, 3, 4, 5\}$

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Consider $(5 +_7 5 +_7 5 +_7 5) = \frac{20}{7} = 6 \cong (20 \bmod 7)$

→ Solve all problems from Question bank

$\therefore 6 \notin Z_5$

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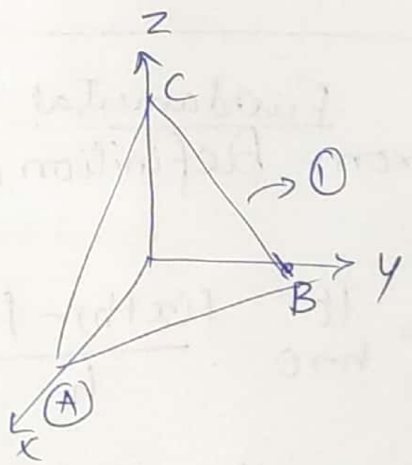
$\therefore$ Since $Z_7 \not\subset Z_5$

Z_7 is not a vector space over Z_5

(1b) Given plane: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \rightarrow (1)$



⇒ let (α, β, γ) P be the req. locus point on (1)

$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1 \rightarrow (2)$

Planes ~~tangent~~ Equation passes (α, β, γ) & have dr's (α, β, γ) is

$\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma) = 0$

$\alpha x + \beta y + \gamma z = (\alpha^2 + \beta^2 + \gamma^2) \rightarrow (3)$

this plane meets co-ordinate axis

at $x_1 = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$ $y_1 = \frac{\alpha^2+\beta^2+\gamma^2}{\beta}$ $z_1 = \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$.

(2)

Considering LHS $\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2} = \frac{\alpha^2}{(\alpha^2+\beta^2+\gamma^2)^2} + \frac{\beta^2}{(\alpha^2+\beta^2+\gamma^2)^2} + \frac{\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$

$$= \frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2} = \frac{1}{\alpha^2+\beta^2+\gamma^2}$$

→ Similarly RHS $\frac{1}{ax_1}+\frac{1}{by_1}+\frac{1}{cz_1} = \frac{(\alpha/a)}{\alpha^2+\beta^2+\gamma^2} + \frac{\beta/b}{\alpha^2+\beta^2+\gamma^2} + \frac{(\gamma/c)}{\alpha^2+\beta^2+\gamma^2}$.

$\left(\because \frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \text{ from } (2)\right) = \frac{(\alpha/a)+\beta/b+\gamma/c}{\alpha^2+\beta^2+\gamma^2} = \frac{1}{\alpha^2+\beta^2+\gamma^2}$

LHS = RHS Hence proved

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(1c) Fundamental

From Definition of derivation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

given $f(x+y) = f(x)\cdot f(y)$

$$= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h} = f(x)\cdot \lim_{h\to 0}\frac{f(h)-1}{h}$$

& $f(x) = 1+xg(x) \Rightarrow f(h) = 1+hg(h)$

$$= f(x) \lim_{h\to 0} \frac{\cancel{1}+hg(h)-\cancel{1}}{h}$$

$$\boxed{f'(x) = f(x)\cdot 1}$$

$$\boxed{\therefore f'(x) = f(x) \;\forall x}$$

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1d) $\lim_{x\to 0} \frac{\log(\log(1-x^2))}{\log\log\cos x}$ $\frac{\infty}{\infty}$ form

L-Hospital's rule

$$\lim_{x\to 0} \frac{\frac{1}{\log(1-x^2)}\cdot\frac{1}{1-x^2}\cdot(-2x)}{\frac{1}{\log\cos x}\cdot\frac{1}{\cos x}\cdot\sin x}$$

$$= \underbrace{\lim_{x\to 0} \frac{\log\cos x}{\log(1-x^2)}}_{\frac{0}{0}\text{ form}} \lim_{x\to 0} \frac{\cos x}{1-x^2} \left(\lim_{x\to 0} \frac{-2x}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot\sin x}{\frac{1}{(1-x^2)}2x} \quad 1 \times (-2) \qquad \because \lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$= \lim_{x\to 0} \frac{\tan x\,(1-x^2)}{2x} \quad (-2) \quad \frac{0}{0}\text{ (form)}$$

$$= \frac{-\cancel{2}}{\cancel{2}} \lim_{x\to 0} \frac{\sec^2 x(1-x^2)+\tan x(-2x)}{1}$$

$$= -1 \times 1 = +1$$

✓ 8

(1c) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

$|A - \lambda I| = 0$

Characteristic Eq

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\big((1-\lambda)(2-\lambda)\big) - 1(0) + 1(-1)(1-\lambda) = 0$$

$$(1-\lambda)\left((2-\lambda)^2 - 1\right) = 0$$

$$(1-\lambda)\left(4 + \lambda^2 - 4\lambda - 1\right) = 0$$

$$(1-\lambda)(\lambda^2 - 4\lambda + 3) = 0 \Rightarrow (\lambda^2 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

From Caley Hamilton theorem Every sq matrix satisfies its char equation

$$\therefore A^3 - 5A^2 + 7A - 3I = 0$$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3A) + A + I$$

$$= A^5(0) + A(0) + A + I$$

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$(A^2 + A + I)$

$$\therefore d = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

(2a) Given Tr. matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$ w.r.t standard basis

& Basis = (0,1,-1) (1,-1,1) (-1,1,0)

writing in terms of standard basis

$(0,1,-1) = 0e_1 + e_2 - e_3$

$(1,-1,1) = e_1 - e_2 + e_3$

$(-1,1,0) = -e_1 + e_2 + 0e_3$

Change of Basis matrix let P

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

New transformation matrix w.r.t new basis

Obtained by $A = PTQ^{-1}$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & -3 \end{pmatrix}$$ which is req Transformation Matrix

2b) $ux+vy+wz=0 \rightarrow (1)$

$ax^2+by^2+cz^2=0 \rightarrow (2)$

let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ = be a gen passing thug vertex

from (1) & (2)

$ul+vm+wn=0 \quad \underbrace{al^2+bm^2+cn^2=0}$

Solving these 2 eq eliminating n

$n=-\frac{(ul+vm)}{w}$

$al^2+bm^2+c\frac{(ul+vm)^2}{w^2}=0$

$w^2(al^2+bm^2)+c(u^2l^2+v^2m^2+2uvlm)=0$

$(w^2a+cu^2)\frac{l^2}{m^2}+2\frac{l}{m}uvc+(w^2b+cv^2)=0$

$\frac{l_1l_2}{m_1m_2}=\frac{w^2b+cv^2}{w^2a+cu^2}$

$\frac{l_1l_2}{w^2b+cv^2}=\frac{m_1m_2}{aw^2+cu^2}$

Similarly $\frac{n_1n_2}{av^2+bu^2}$

for perpendicular condition $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow w^2a+cu^2+w^2b+cv^2+av^2+bu^2=0$

$$\boxed{u^2(b+c)+v^2(a+c)+(a+b)w^2=0}$$

Required condition

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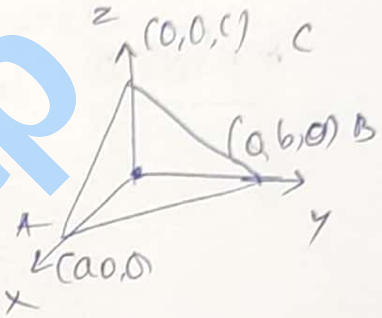
4(a) Sphere of const rad. passes origin

$\therefore x^2+y^2+z^2+2ux+2vy+2wz=0$ as $\underline{d=0}$ Cont

meets co oridinate axis

at $x=a \Rightarrow a^2+2ua=0 \Rightarrow 2u=-a$

Similarly for $y=b$, $z=c$: $2v=-b$, $2w=\underline{-c}$



Equation of planes through (A, B, C) is of form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

here a, b, c are $\underline{-2u, -2v, -2w}$.

$\therefore \frac{x}{u}+\frac{y}{v}+\frac{z}{w}=-2$

$\therefore$ let (x_1, y_1, z_1) be a point on plane

$O(0,0,0) \Rightarrow$ dr's $\Rightarrow (0-x_1, 0-y_1, 0-z_1)$.

$\therefore$ ler condition $\underline{l_1l_2+m_1m_2+n_1n_2=0}$

$\therefore \frac{-x_1}{u}+\frac{-y_1}{v}+\frac{-z_1}{w}=0$

$\Rightarrow \boxed{\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=0} \rightarrow$ Required $\underline{\text{locus}}$

$\underline{\text{where } (u, v, w)}$ is centre of Sphere

4b

Let $F(a) = \int_0^1 \log\left(\frac{1+ax}{1-ax}\right) \frac{dx}{x\sqrt{1-x^2}}$

~~Let~~ $F'(a)$ with x Cont $= \int_0^1 \frac{(1-ax)}{(1+ax)} \frac{(1-ax)a + a(1+ax)}{(1-ax)^2} \frac{dx}{x\sqrt{1-x^2}}$

$F'(a) = \int_0^1 \frac{2a}{(1-a^2x^2)} \cdot \frac{dx}{x\sqrt{1-x^2}}$

& $F'(0)=0$

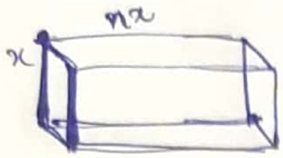
$= 2a \int_0^1 \frac{dx}{x\sqrt{1-x^2}\,(1-a^2x^2)}$

$= \int_0^1 a\left(\frac{1}{1+ax} + \frac{1}{1-ax}\right) \frac{dx}{x\sqrt{1-x^2}}$

$= a \int_0^1 \frac{dx}{(1+ax).x\sqrt{1-x^2}} + \int_0^1 \frac{dx}{(1-ax)x\sqrt{1-x^2}}$

4c

4c) closed Rect box.



volume $\to V$ given

$V = nx^2 h$ let ~~long~~ hieght & ~~width~~

$V = nx^2 h \to (1)$ hieght $= (h)$ ~~same~~ !!

$\Rightarrow$ ~~Sl = echoone~~

Surface area $S = 2(nx^2) + 2xh + 2nxh \to (2)$ ✓

$\therefore$ from (1) $h = \dfrac{V}{nx^2}$

$$\Rightarrow S = 2nx^2 + 2x \cdot \frac{V}{nx^2} + 2nx \cdot \frac{V}{nx^2}$$

$$S = 2nx^2 + \frac{2V}{nx} + \frac{2V}{x} \to (3)$$

$$\frac{ds}{dx} = 4nx - \frac{2V}{n}\frac{1}{x^2} - 2V\frac{1}{x^2} = 0$$

$$2\cancel{4}nx^3 = \frac{\cancel{2}V}{n} + 2V$$

$$\boxed{x^3 = \frac{V}{2n}\left(1+\frac{1}{n}\right)}$$

$$\frac{d^2s}{dx^2} = 4n + \frac{2V}{n}\left(\frac{2}{x^3}\right) + 2V\left(\frac{2}{x^3}\right)$$

$$= 4n + \left(\frac{4V}{n} + 4V\right)\left(\frac{1}{x^3}\right)$$

$$= 4n + 4V\left(1+\frac{1}{n}\right)\frac{2n}{V}\left(\frac{1}{1+1/n}\right) = 4n + 8n$$

$$= 12n > 0$$

$\dfrac{d^2s}{dx^2} > 0$ $\therefore$ vehi

$\therefore$ least surface at $x = \left(\frac{V}{2n}\left(1+\frac{1}{n}\right)\right)^{1/3}$

from Eq (3)

$$S = 2n\left(\frac{V}{2n}\left(1+\frac{1}{n}\right)^{1/3}\right)^2 + \left(\frac{2V}{n} + 2V\right)\frac{1}{x}$$

4(d)

$$S = 2n\left(\frac{V}{2n}\left(\frac{n+1}{n}\right)\right)^{2/3} + 2V\left(1+\frac{1}{n}\right)\frac{1}{x}$$

$$+ 2V\left(1+\frac{1}{n}\right)\frac{1}{\frac{(2V)^{1/3}}{(n)^{1/3}}\left(1+1/n\right)^{1/3}}$$

$$S = 2n\left(\frac{V}{2n}\right)^{2/3}\left(1+\frac{1}{n}\right)^{2/3} + n^{-1/3}\left(1+\frac{1}{n}\right)^{2/3}$$

$$S = (2n)^{1-2/3} V^{2/3}\left(1+\frac{1}{n}\right)^{2/3} + (2V)^{1-1/3} n^{1/3}\left(1+\frac{1}{n}\right)^{2/3}$$

$$S = 2^{1/3} n^{1/3} V^{2/3}\left(1+\frac{1}{n}\right)^{2/3} + 2^{2/3} V^{2/3} n^{1/3}\left(1+\frac{1}{n}\right)^{2/3}$$

$$S = (2^{1/3}+2^{2/3})(n)^{1/3}(V)^{2/3}\left(1+\frac{1}{n}\right)^{2/3}$$

Both side cubic

$$S^3 = (2^{1/3}+2^{2/3})^3 (n)\frac{(n+1)^2}{n^2} V^2$$

$$nS^3 = (2^{1/3}+2^{2/3})^3 (n+1)^2 V^2$$

$$nS^3 = 54(n+1)^2 V^2$$

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4d) 'S' is n rowed real skew-symmetric.

$\therefore$ $\boxed{S^T = -S}$ & also $\bar{S}^T = S^\theta = -S$

we have $(I+S)(I-S) = (I-S)(I+S)$

(i) $(I-S)$ Consider $|S-\lambda I| = 0$ Since $S \to$ real

$\bar{S} \to S$

for skew hermetian λ img or zero $\bar{S}^T = -S \Rightarrow$ $\boxed{\text{Skew-Hermitian}}$

$\boxed{\therefore \lambda \neq 1}$

$|S - \lambda I| \neq 0$

$\Rightarrow (I-S)$ is non singular

(ii)

$A = (I+S)(I-S)^{-1}$

For orthogonal $(AA^T = A^T A)$

$\Rightarrow$ let $A^T = ((I+S)(I-S)^{-1})^T$

we have $(I+S)(I-S) = (I-S)(I+S) \to (1)$

$A^T = ((I-S)^{-1})^T (I^T + S^T)$

$A^T = (I+S^{-1})(I-S)$

$AA^T = (I+S)(I-S)^{-1}(I+S)^{-1}(I-S)$ from (1)

$I \cdot I$

$= I$

$\therefore AA^T = I$ orthogonal

(iii)

$(I+S)(I-S) = (I-S)(I+S)$

pre & post multiply $(I-S)^{-1}$

$(I-S)^{-1}(I+S)(I-S)(I-S)^{-1} = (I-S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$\therefore (I-S)^{-1}(I+S) = (I+S)(I-S)^{-1}$

$= A$

(iv) Given $X \rightarrow$ Eigen vector

$$SX = \lambda X$$

$$\cancel{X} + SX = \cancel{X} + \lambda X$$

$$(I+S)X = (1+\lambda)X \rightarrow (1)$$

Form also $-SX = -\lambda X$

$$(I-S)X = (1-\lambda)X$$

$$\underbrace{(I-S)^{-1}(I-S)}X = (I-S)^{-1}(1-\lambda)X$$

$$IX = (1-\lambda)(I-S)^{-1}X \rightarrow (2)$$

$$(I+S)(I-S)^{-1}(1-\lambda)X = (1+\lambda)X \quad (\text{from } (1))$$

$$(I+S)(I-S)^{-1}X = \frac{(1+\lambda)}{(1-\lambda)} X$$

$$\boxed{AX = \left(\frac{1+\lambda}{1-\lambda}\right)X}$$

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5a)

$$(y^2+2x^2y)dx + (2x^3 - xy)dy = 0 \rightarrow (1)$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y + 2x^2 \qquad \frac{\partial N}{\partial x} = 6x^2 - y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (4x^2 - 3y)$$

let ~~Re~~ $(y+2x^2)\,y\,dx + (2x^2-y)\,x\,dy = 0$

form $f_1\, y\,dx + f_2\, x\,dy = 0$

$$\Rightarrow \underline{Mx - Ny} = xy^2 + 2x^3y - 2x^3y + xy^2$$

$$= \cancel{\quad} = \underline{2xy^2}$$

Int. factor $\frac{1}{(Mx-Ny)} \times (1) \Rightarrow \frac{1}{2xy^2}(y^2+2x^2y)dx + \frac{(2x^3-xy)}{\cancel{\quad}}dy = 0$

$$\cancel{\frac{1}{xy(2y)}\, 2x^2(ydx+xdy) + y^2dx \ldots}$$

$$\frac{1}{2xy^2}(y^2+2x^2y)\,dx + \left(\frac{2x^3}{2xy^2} - \frac{xy}{2xy^2}\right)dy = 0$$

$$\underset{y\ \text{cont}}{\int}\left(\frac{1}{2x} + x\right)dx + \underset{\substack{\text{not Contain}\\ x}}{\int} \frac{-1}{2y}\,dy = 0$$

$$\frac{1}{2}\log x + \frac{x^2}{2} - \frac{1}{2}\log y = \log c'$$

$$\log\frac{x}{y} + x^2 = \log c$$

$$\log\frac{cy}{x} = x^2$$

$$\boxed{cy = xe^{x^2}}$$

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(5c) given Eqn of surface.

$f(x,y,z): 2xz^2 - 3xy - 4x = 7$

Direction obtained by $\nabla f \Rightarrow (2z^2-3y-4)\hat{i} - 3x\hat{j} + 4xz\hat{k}$

at $(1,-1,2) \Rightarrow (7)\hat{i} - 3\hat{j} + 8\hat{k} = \vec{n}$

$\therefore$ tangen Eq. $\frac{\partial f}{\partial x}(x-x_1) + \frac{\partial f}{\partial y}(y-y_1) + \frac{\partial f}{\partial z}(z-z_1) = 0$

$7(x-1) + (-3)(y+1) + 8(z-2) = 0$

$x - 3y + 8z = 26$

Normal Eqn. $\frac{x-x_1}{\partial f/\partial x} = \frac{y-y_1}{\partial f/\partial y} = \frac{z-z_1}{\partial f/\partial z}$

$\Rightarrow \frac{x-1}{7} = \frac{y+1}{-3} = \frac{z-2}{8}$

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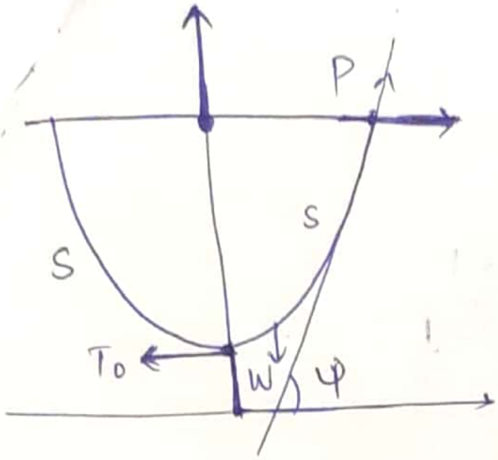

$T\cos\psi = T_0 = wc$ $\qquad$ $\underline{T = wy}$

$T\sin\psi = W = sw$

$\underline{s = c\tan\psi}$

$\downarrow$

$y^2 + s^2 = c^2$ $\qquad$ $\boxed{s = b}$ maximum tension

$\underline{\frac{b}{c} = \tan\psi}$

$\sec^2\psi - 1 =$

(5e)

$$(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$$
$$=\frac{1}{3}(3x+2)^2-\frac{1}{3}$$

$\Rightarrow$ Let $(3x+2)=e^z \Rightarrow z=\log(3x+2)$

$(3x+2)^2D^2 = 3^2\cdot D_1(D_1-1)$

$$(9D_1(D_1-1)+3(3)D_1-36)y=\frac{1}{3}(e^z)^2-\frac{1}{3}$$

$$(9D_1^2-\cancel{9D_1}+\cancel{9D_1}-36)y=\frac{1}{3}(e^{2z}-1)$$

$$9(D_1^2-4)y=\frac{1}{3}(e^{2z}-1)$$

$$(D_1^2-4)y=\frac{1}{27}(e^{2z}-1)$$

y_c : $m^2-4=0 \quad m=\pm 2$

$$y_c = C_1e^{2z}+C_2e^{-2z} = C_1(3x+2)^2+C_2\frac{1}{(3x+2)^2} \rightarrow (1)$$

$$\rightarrow y_p=\frac{1}{27}\frac{1}{(D_1^2-4)}e^{2z}-\frac{1}{27}\frac{1}{(D_1^2-4)}e^0$$

$$=\frac{z\,e^{2z}}{(4)27}+\frac{1}{4(27)}$$

$$y_p=\frac{1}{108}(ze^{2z}+1)=\frac{1}{108}((3x+2)^2\log(3x+2)+1) \rightarrow (2)$$

$\therefore y=y_c+y_p$ (from (1) & (2))

(8)

1a

$$(px^2+y^2)(px+y) = (p+1)^2$$

let $v = x^2+y^2$ $u = x+y$

$dv = 2(x+y\frac{dy}{dx})$ $du = dx+dy$

$$\frac{dv}{du} = \frac{2(x\,dx+y\,dy)}{dx+dy}$$

$$p' = \frac{2(x+yp)}{(1+p)}$$

$$(1+p)p' = 2(x+yp)$$

$$p(p'-2y) = (2x-p')$$

$$p = \frac{(2x-p')}{(p'-2y)}$$

$$\frac{(2x-p')}{(p'-2y)}(x^2)+y^2) = \frac{2}{p'}$$

$$\cancel{(2xp'-p')(x^2+y^2) = 2p'-4y}$$

$$(2x-p')x^2+y^2(p'-2y) = \frac{2(p'-2y)}{p'}$$

let, $v = xy$

$u = x+y$

$dv = x\,dy + y\,dx$

$du = dx+dy$

$$\frac{dv}{du} = \frac{xp+y}{(p+1)} = p'$$

$$p'p+p' = xp+y$$

$$p = \frac{y-p'}{(p'-x)}$$

$$\left(\frac{y-p'}{p'-x}x^2+y^2\right) = (p+1)\left(\frac{p+1}{px+y}\right)$$

$$\frac{yx^2-p'x^2+p'y^2-xy^2}{(p'-x)} = \left(\frac{y-p'}{p'-x}+1\right)\frac{1}{p'}$$

$$xy(x-y)+p'(y^2-x^2) = \frac{1}{p'}(y-x)$$

$$xy(x-y) = p'(x^2-y^2) + \frac{1}{p'}(y-x)$$

$$xy = p'(x+y) - \frac{1}{p'}$$

$v = p'u - \frac{1}{p'}$ it is of Clairaut's form

$\therefore$ Solution is $\boxed{p'=c}$

$$xy = (x+y)c - \frac{1}{c}$$

$$\boxed{c^2(x+y) - cxy - 1 = 0}$$

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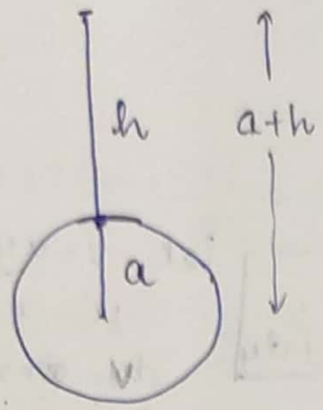
7b



Given $\frac{d^2x}{dt^2} \propto$ $F \propto \frac{1}{x^2}$ let F: attraction

$$m\frac{d^2x}{dt^2} = -\frac{\mu'}{x^2}$$

-ve in x reducing

$\Rightarrow$ multiply $2\left(\frac{dx}{dt}\right)$ & integrate

$$\frac{2\left(\frac{dx}{dt}\right)^2}{2} = \int -\frac{2\mu}{x^2}\,dx \qquad \left(\text{let} : \frac{\mu'}{m} = \mu \text{ since } m : \text{const}\right)$$

$$= -2\mu\left(-\frac{1}{x}\right) + c$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + c$$

$$v^2 = \frac{2\mu}{x} + c$$

at $x = (a+h)$ $v = 0$

$$\therefore 0 = \frac{2\mu}{a+h} + c = 0 \Rightarrow c = -\frac{2\mu}{a+h}$$

$$\therefore v^2 = \frac{2\mu}{x} - \frac{2\mu}{(a+h)}$$

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{a+h-x}{(a+h)x}\right)$$

$$\left(\frac{dx}{dt}\right) = \sqrt{2\mu}\sqrt{\frac{(a+h)-x}{(a+h)x}}$$

$\therefore$ let $x = (a+h)\sin^2\theta$

$$dx = (a+h)2\sin\theta\cos\theta\, d\theta$$

$$\sqrt{\frac{a+h}{2\mu}} \int \frac{\sqrt{x}}{\sqrt{a+h-x}}\, dx = dt$$

$$t = \sqrt{\frac{a+h}{2\mu}} \int_{a+h}^{h} -\frac{\sqrt{x}\, dx}{\sqrt{a+h-x}} \qquad \text{(-ve as } x \text{ dec.)}$$

$$= \sqrt{\frac{a+h}{2\mu}} \int_{h}^{a+h} \frac{\sqrt{x}\, dx}{\sqrt{(a+h)-x}}$$

at $x = a+h$ $\boxed{\theta_2 = \pi/2}$

at $x = h$ $\boxed{\theta_1 = \sin^{-1}\left(\sqrt{\frac{h}{a+h}}\right)}$

$$t = \sqrt{\frac{a+h}{2\mu}} \int_{\theta_1}^{\theta_2} \frac{\sqrt{a+h}\, \sin\theta\, (2)(a+h)\sin\theta\cos\theta\, d\theta}{\sqrt{(a+h)}\sqrt{(1-\sin^2\theta)}}$$

$$= \sqrt{\frac{a+h}{2\mu}} \int_{\theta_1}^{\theta_2} 2\sin^2\theta\, d\theta$$

$$= \sqrt{\frac{a+h}{2\mu}}\, (a+h) \int_{\theta_1}^{\theta_2} (1-\cos 2\theta)\, d\theta$$

$$= \sqrt{\frac{a+h}{2\mu}}\, (a+h) \left(\theta \Big]_{\theta_1}^{\theta_2} - \frac{\sin 2\theta}{2} \Big]_{\theta_1}^{\theta_2} \right)$$

$$= \sqrt{\frac{a+h}{2\mu}}\, (a+h) \left(\frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{h}{a+h}}\right) - \frac{1}{2}\left(\frac{\pi}{2} - \sin\left(\sin^{-1}\left(\sqrt{\frac{h}{a+h}}\right)\right)\right)\right)$$

$$= \sqrt{\frac{a+h}{2\mu}} \left(\sqrt{\frac{h}{(a+h)}}\sqrt{(h+a)} + (a+h)\sin^{-1}\left(\frac{h}{a+h}\right)^{1/2} \right)$$

$\therefore$ we have $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \Rightarrow -g = -\frac{\mu}{a^2} \qquad \mu = ga^2$

$$\therefore \sqrt{\frac{a+h}{2ga^2}} \left(\sqrt{h} + (a+h)\sin^{-1}\left(\frac{h}{a+h}\right)^{1/2} \right)$$

$$t = \sqrt{\frac{a+h}{2g}} \left(\sqrt{\frac{h}{a^2}} + \left(\frac{a+h}{a}\right)\sin^{-1}\left(\frac{h}{a+h}\right)^{1/2} \right)$$

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7(c) $L\left(\int_0^t \frac{\sin x}{x}\,dx + te^{-t}\cos^2 2t\right)$

∴ ~~From property~~

2nd part $te^{-t}\cos^2 2t$

$= te^{-t}\frac{(1+\cos 4t)}{2}$

$= \frac{te^{-t}}{2}(1+\cos 4t) = \frac{1}{2}te^{-t} + te^{-t}\cos 4t$

$= \frac{1}{2}\frac{1}{(s+1)^2} + \left(-\frac{d}{ds}\right)\frac{s}{(s+1)^2+4^2}$

$= \frac{1}{2}\left(\frac{1}{(s+1)^2} - \frac{(s+1)^2+4^2 - s(2(s+1))}{((s+1)^2+4^2)^2}\right)$

$= \frac{1}{2}\frac{1}{(s+1)^2} - \frac{s^2+2s+1+16-2s^2-2s}{((s+1)^2+4^2)}$

1st part

$= \frac{1}{2}\left(\frac{1}{(s+1)^2} + \frac{s^2-17}{(s+1)^2+4^2}\right) \rightarrow (1)$

$\int_0^t \frac{\sin x}{x}dx$

7C

(ii)

$$t\frac{d^2y}{dx^2}+2\frac{dy}{dx}+ty=\cos t$$

$$-\frac{d}{ds}\left(s^2y(s)-sy(0)-\cancel{y'(0)}^{\,0}\right)+2\left(sy(s)-\cancel{y'(0)}^{\,0}\right)+-\frac{d}{ds}(y(s))=\frac{s}{1+s^2}$$

$$-2sy(s)-s^2y'(s)+y(0)+2sy'(s)-2y'(s)-y'(s)$$

$$-(s^2+2s+1)y'(s)-(2s+2)y(s)+1=\frac{s}{1+s^2}$$

$$y'(s)+\frac{(2s+2)y}{(s^2+2s+1)}=\frac{1}{(s+1)^2}-\frac{s}{(s^2+1)(s+1)^2}$$

$\therefore$ 1st order ODE

$$I.F=e^{\int\frac{2s+2}{(s^2+2s+1)}ds}=e^{\ln(s+1)^2}$$

$$=(s+1)^2$$

$$(s+1)^2\,y(s)=\int\left(\frac{1}{(s+1)^2}-\frac{s}{(s^2+1)(s+1)^2}\right)(s+1)^2\,ds$$

$$y(s)(s+1)^2=s-\frac{1}{2}\ln(s^2+1)$$

$$y(s)=\frac{s}{(s+1)^2}-\frac{\ln\sqrt{1+s^2}}{(s+1)^2}$$

$$L^{-1}\{y(s)\}=$$

(7d) $(8p^3-27)x = 12p^2y \to ①$

$$y = \frac{8p^3-27}{12p^2}x$$

$$y = \frac{2}{3}px - \frac{9x}{4p^2}$$

$$\frac{dy}{dx} = \frac{2}{3}p + \frac{2}{3}x\frac{dp}{dx} - \frac{9}{4p^2} + \frac{9x(2)}{4p^3}\frac{dp}{dx}$$

$$p = \frac{2}{3}p + \frac{2}{3}x\frac{dp}{dx} + \frac{9}{2}\frac{x}{p^3}\frac{dp}{dx} - \frac{9}{4p^2}$$

$$0 = -\frac{9}{4p^2} - \frac{1}{3}p + \frac{2}{3}x\frac{dp}{dx} + \frac{9}{2}\frac{x}{p^3}\left(\frac{dp}{dx}\right)$$

$$\Rightarrow -p\left(\frac{1}{3} + \frac{9}{4p^3}\right) + 2x\frac{dp}{dx}\left(\frac{1}{3} + \frac{9}{4p^3}\right) = 0$$

$$\left(\frac{1}{3} + \frac{9}{4p^3}\right)\left(2x\frac{dp}{dx} - p\right) = 0$$

$$2x\frac{dp}{dx} = p \qquad \text{as } \left(\frac{1}{3} + \frac{9}{4p^3}\right) \neq 0$$

$$2\frac{dp}{p} = \frac{dx}{x}$$

$$2\ln p = \ln x + \ln c$$

from ①

$$\boxed{p^2 = cx} \to ②$$

$$(8(cx)^{3/2} - 27)x = 12\,cx\,y$$

$$8(cx)^{3/2} = (12cy + 27)$$

$$(8)^2(cx)^3 = (12cy+27)^2 \to \text{Required solution}$$

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Singular solution: $\frac{df(p)}{dp} \Rightarrow 24p^2x = 24py = 0$

$$p^2x - py = 0 \qquad B^2 - 4AC = 0$$

$$(-y)^2 = 0 \qquad y = 0$$

$y = 0 \Rightarrow p = 0$ does not satisfy eq ①

No singular solution