Dt: 18/11/21

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SuccessClap

Best Coaching for UPSC MATHEMATICS

Test Copy of Mr. Pandu Wilson, AIR 603, CSE 2021

TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2021

FULL LENGTH TEST -2 PAPER 2

Time Allowed: Three Hours Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions:

There are **EIGHT** questions divided in **TWO SECTIONS**

Candidate must attempt **FIVE** questions in all.

Question Nos. **1** and **5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from one section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question – cum – Answer (QCA) Booklet in the space provided. No marks will be given for answers written in a medium other than the authorized one.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless and otherwise indicated, symbols and notations carry their usual standard meaning.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question – cum – Answer Booklet must be clearly struck off.

Scan the Answer sheet to pdf and email to SuccessClap@gmail.com

Section A

1(a) The function f (z) has a double pole at z = 0 with residue 2, a simple pole at z = 1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If f (2) = 5 and f (-1) = 2, find f (z). (10)

Given f(z) has double pole & z = 1 single pole

$\therefore f(z) = \dfrac{\phi(z)}{(z-0)^2(z-1)}$

$= \dfrac{\phi(z)}{z^2(z-1)}$

Residue at z = 0 $\Rightarrow \dfrac{d}{dz}\dfrac{\phi(z)}{(z-1)} = \dfrac{\phi'(z)(z-1) - \phi(z)}{(z-1)^2}$

at $z = 0 \Rightarrow \phi'(0) - \phi(0) = 2$

at z = 1

$f(z) = \lim_{z\to 1} (z-1)\dfrac{\phi(z)}{z^2(z-1)} = 2$

$\phi(1) = 2$

Since $|z| \to \infty$ bounded

$\therefore \phi(z)$ degree ≤ 3

let $\phi(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$

given $f(2) = 5 \Rightarrow \dfrac{\phi(2)}{(4)(1)} = 5 \Rightarrow \phi(2) = 20$

$f(-1) = \dfrac{\phi(-1)}{(1)(-2)} = 2 \Rightarrow \phi(-1) = -4$

$a_0 + 2a_1 + 4a_2 + a_3 8 = 20$

$a_0 - a_1 + a_2 - a_3 = -4$

$a_0 + a_1 + a_2 + a_3 = 2$

$-a_0 + a_1 = 2$

solving

$\phi(z) = -\frac{4}{3} + \frac{2}{3}z + \frac{1}{3}z^2 + \frac{7}{3}z^3$

$\therefore f(z) = \dfrac{1}{3}\dfrac{(7z^3 + z^2 + 2z - 4)}{z^2(z-1)}$

wrong approach

Given simple pole @ z = 1, residue is 2 means f(z) has $\dfrac{2}{z-1}$

double pole at z = 0, residue 2 mean f(z) has $\dfrac{2}{z} + \dfrac{b}{z^2}$

So f(z) must be $\sum a_n z^n + \dfrac{2}{z-1} + \dfrac{2}{z} + \dfrac{b}{z^2}$

1(b) Let N be a normal subgroup of G. Show that G/N is abelian iff $xyx^{-1}y^{-1} \in N$ *for all* $x, y \in G$. (10)

N be a NSG $\therefore$ $xNx^{-1} \in N$ where $x \in G$

& Similar $yNy^{-1} \in N$ $y \in G$

By def. ~~let~~ $\frac{G}{N}$ of Quotient grp.

$$\frac{G}{N} = \{ Nx \mid x \in G \}$$

Let $\frac{G}{N}$ abelian $\therefore Nx \cdot Ny = Ny \cdot Nx$

Since $Nx \cdot Ny = Nxy$.

$\therefore Nxy = Nyx$

$Nxyx^{-1} = Ny$

$Nxyx^{-1}y^{-1} = N \Rightarrow xyx^{-1}y^{-1} \in N$

Let consider :

$xyx^{-1}y^{-1} \in N$

~~Let~~ $e \in G/N$ (identity elt) & ~~let~~

let $xyx^{-1}y^{-1} = e$ [Since $e, e \in \frac{G}{N}$ $\therefore e \in N$]

$xyx^{-1} = y$

$xy = yx$

$\therefore \frac{G}{N}$ is abelian ..

(8)

1(c) A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in table

Food type	Yield per unit			Cost per unit (₹)
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirements	800	200	700	

Formulate linear programming model for the problem. (10)

Given minimisation problem

∴ inequalities $\geq$

∴ let food types x_1, x_2, x_3, x_4

for protein → $3x_1 + x_2 + x_3 + x_4 \geq 800$

$\Rightarrow 3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$

for Fats $\Rightarrow 2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$

for carbs $\Rightarrow 6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$.

Objective function will be

min $Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$

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1(d) Prove that of all rectangular parallelepiped of the same volume, the cube has the least surface. (10)

Let $2x, 2y, 2z$ be the edges of parallelopiped.

Surface of parallelopipd : $S = 4(xy+yz+zx)$ $\to$ ①

volume = $V = 8xyz$: ϕ $\to$ ②

By lagranges multiplier meth

Let $f : S + \lambda\phi$

$4(xy+yz+zx) + \lambda(8xyz - V) = 0 \to$ ③

$f_x : 4(y+z) + \lambda(8yz) = 0$ $\quad 2\lambda yz = y+z = \frac{1}{y}+\frac{1}{z}$ $\to$ ④

$f_y : 4(x+z) + \lambda(8xz) = 0$ $\quad \lambda = \frac{1}{x}+\frac{1}{z}$ $\to$ ⑤

$f_z : 4(y+x) + \lambda(8yx) = 0 \Rightarrow \lambda = \frac{1}{x}+\frac{1}{y}$ $\to$ ⑥

from ④, ⑤ ⑥ $\frac{1}{x} = \frac{1}{y} = \frac{1}{z}$

$\Rightarrow x = y = z$

∴ By Lagranges method least surface volume if it is Cube.

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1(e) Test for convergence the integral (10)

$\int_0^\infty \sin x^2 dx$

Qns of type $\int_0^\infty \sin x^2$, $\int \sin x^p$, $\int \cos x^2$, $\int$ must prove by Dirichlet's Test

$$\int_0^\infty \sin x^2 \, dx$$

consider $g(x) = \frac{1}{x^2}$

let

$\Rightarrow \frac{f(x)}{g(x)} = \frac{\sin x^2}{x^2} \Rightarrow \int_0^\infty \frac{\sin x^2}{x^2} = \int_0^1 \frac{\sin x^2}{x^2} + \int_1^\infty \frac{\sin x^2}{x^2}$

Consider $\int_0^1 \frac{\sin x^2}{x^2}$ has pnts of discontinuity at $x=0$

$\therefore$ at $x = 0$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin x^2}{x^2} = 1$ (finite)

$\therefore$ f(x) & g(x) cg or diverge together

$\therefore g(x) = \frac{1}{x^2}$ ~~convergent~~ divergent as $2 \not< 1 \rightarrow$ (1)

for $\int_1^\infty \frac{\sin x^2}{x^2}$ discontinuity at $x = \infty$

$\lim_{x\to\infty} \frac{\sin x^2}{x^2} = 0$ from limit property

$\therefore$ f(x) & g(x) cnvg or dvg together

convergent since $2 > 1 \rightarrow$ (2)

But from (1) & (2) whole fun. $\boxed{\text{divergent}}$

~~as $\frac{1}{x^2}$ at $x=0$ dvgt.~~

Solve all problems from Question Bank $\rightarrow$ Improper Integrals

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2(a) If $f(x,y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, when $x \neq 0, y \neq 0$ and $f(x,y) = 0$, otherwise; show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. (10)

$$f_{xy} = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$(f_x) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \frac{h^2 \tan^{-1}(k/h) - k^2 \tan^{-1}(h/k) - 0}{h}$$

at $(0,k)$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}(k/h) - k^2 \tan^{-1}\left(\frac{h}{k}\right) - 0}{h} \quad [\because f(0,k) = 0]$$

$$= \lim_{h\to 0} \frac{0 - k^2 \tan^{-1}\left(\frac{h}{k}\right)}{h} \quad \frac{0}{0} \text{ form}$$

$$f_x(0,k) = \lim_{h\to 0} -k^2 \cdot \frac{1}{1+\frac{h^2}{k^2}} \cdot \frac{1}{k} = -k$$

$$f_x(0,0) = 0$$

$$\therefore f_{xy} = \lim_{k\to 0} \frac{-k - 0}{k} = -1 \rightarrow (1)$$

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$$f_{yx} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{h^2 \tan^{-1}(k/h) - k^2 \tan^{-1}\left(\frac{h}{k}\right) - 0}{k}$$

To be more careful at this step

$$= \lim_{k\to 0} \frac{h^2 \tan^{-1}(0)}{h - 0} - k \tan^{-1}\left(\frac{h}{k}\right) \quad (\because f(h,0) = 0)$$

$$h\left[\frac{\tan^{-1} k/h}{k/h}\right] \quad k \to 0$$

$$= \lim_{k\to 0} -k \tan^{-1}(\infty) = \cancel{0} = \cancel{0}\ h$$

$$\therefore f_{yx} = \lim_{h\to 0} \frac{h - 0}{h} = \cancel{0} \rightarrow (2)$$

$\therefore$ from (1) & (2) $\boxed{f_{xy} \neq f_{yx}}$

2(b) Find the nature and location of the singularities of the function

$f(z) = \frac{1}{z(e^z-1)}$. Prove that f (z) can be expanded in the form

$$\frac{1}{z^2} - \frac{1}{2z} + a_0 + a_2 z^2 + a_4 z^4 + \cdots,$$

Where $0 < |z| < 2\pi$ and find the values of a_0 and a_2. (20)

Given $f(z) = \frac{1}{z(e^z-1)}$

we know $e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$$f(z) = \frac{1}{z(\cancel{1}+z+\frac{z^2}{2!}+\cdots-\cancel{1})}$$

$$= \frac{1}{z^2\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)}$$

$$= \frac{1}{z^2}\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}\cdots\right)^{-1}$$

from $(1+x)^{-1} = 1-x+x^2-x^3 \cdots$

$$= \frac{1}{z^2}\left(1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\frac{z^4}{5!}\right)+\left(\frac{z}{2}+\frac{z^2}{3!}+\cdots\right)^2 - \cdots\right)$$

$$= \frac{1}{z^2}\left(1-\frac{z}{2!}-\frac{z^2}{6}+\frac{z^2}{4}-\frac{z^3}{4!}+\frac{z^4}{5!}+\frac{(z^4)}{(3!)^2}\cdots\right)$$

$$= \frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}-\frac{z}{24}+\frac{z^4}{z^2}\left(\frac{1}{36}-\frac{1}{120}\right)+\frac{O(z^5)}{z^2}\cdots$$

$$f(z) = \frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}-\frac{z}{24}+\left(\frac{7}{360}\right)z^2+O(z^3)+\cdots$$

$$\boxed{a_0 = \frac{1}{12} \quad a_2 = \frac{7}{360}}$$ $\rightarrow \frac{1}{720}$

Nature of Singularities.

at $z=0$ $\quad \lim_{z\to 0} z^2 f(z) = $ finite value

$\therefore$ f(z) has Simple pole of order 2. at $z=0$

Note: Such type of problems. Atleast 3 term should be correct

Square $\rightarrow \left(\frac{z^2}{4}+\frac{z^4}{36}+\frac{2z^3}{6}+\cdots\right)$

$z^2\left(-\frac{1}{6}+\frac{1}{4}\right)$

$+z^3\left(-\frac{1}{24}\right)$

Ist term $-\frac{1}{6}$

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2(c) Maximize $Z = 5x_1 - 2x_2 + 3x_3$,

Subject to $2x_1 + 2x_2 - x_3 \geq 2$,

$3x_1 - 4x_2 \leq 3$,

$x_2 + 3x_3 \leq 5$,

$x_1, x_2, x_3 \geq 0$. (20)

Constrains $2x_1+2x_2-x_3-S_1+A_1=2$

$3x_1-4x_2+S_2=3$

$x_2+3x_3+S_3=5$

$Z = 5x_1-2x_2+3x_3-MA_1+0S_2+0S_3$

By Big M-method :

C_{Bi}	C_j / Basic	5 (x_1)	-2 (x_2)	3 (x_3)	-M (A_1)	0 (S_2)	0 (S_3)	Sol	θ
-M	A_1	$\boxed{2}$	2	-1	1	0	0	2	1 →
0	S_2	3	-4	0	0	1	0	3	1
0	S_3	0	1	3	0	0	1	5	-
	Z_j	-2M	-2M	M	-M	0	0		
	C_j-Z_j	5+2M ↑	-2+2M	3-M	0	0	0		

x_1 incoming, A_1 out

C_{Bi}	Basic	x_1	x_2	x_3	A_1	S_2	S_3	Sol	θ
5	x_1	1	1	-1/2	[illegible]	0	0	1	–
0	S_2	0	-7	3/2	[illegible]	1	0	0	0
0	S_3	0	1	$\boxed{3}$	[illegible]	0	1	5	5/3
	Z_j	5	5	-5/2	-	0	0		
	C_j-Z_j	0	-7	11/2 ↑	-	0	0		

$x_3 \rightarrow$ incoming, $S_3 \rightarrow$ out

C_{Bi}	Basic	x_1	x_2	x_3	A_1	S_2	S_3	Sol
5	x_1	1	7/6	0	-	0	1/6	11/6
0	S_2	0	-15/2	0	-	1	-1/2	-5/2
3	x_3	0	1/3	1	-	0	1/3	5/3
	Z_j	5	$\frac{41}{6}$	3	-	0	11/6	
	C_j-Z_j	0	$-\frac{53}{6}$	0	-	0	-11/6	

Since all $C_j - Z_j \leq 0$ Optimality reached

∴ Solution is $x_1 = 11/6$ $x_2 = 0$

$x_3 = 5/3$

∴ Max of $z = 5\left(\frac{11}{6}\right)+3\left(\frac{5}{3}\right) = \frac{85}{6}$

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$\frac{85}{3}$

4(a) If G is a group of order 35, show that it cannot have two subgroups of order 7. (10)

By Lagranges theorem
Every subgroup order divides group order.

$\therefore$ 7/35 O(H) = 7 · is a subgrp.

~~And~~ also known Union of 2 proper subgroups of same order can not be a group.

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$\therefore$ let G has 2 subgrps of order 7
We ~~ab~~ have to prove contradiction

4(b) Test for convergence the following series:

(i) $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \cdots$

(ii) $\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots$ (20)

(i) $u_n = \frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2}$

$u_{n+1} = \frac{1^2 3^2 \cdots (2n-1)^2 (2n+1)^2}{2^2 4^2 \cdots (2n)(2n+2)^2}$

$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+1)^2} = 1$

D-Alemberts Ratio test fails

$\frac{u_n}{u_{n+1}} = (2n+2)^2 \frac{1}{(2n)^2}\left(1+\frac{1}{2n}\right)^{-2}$ $\quad (\because (1-x)^{-2} = 1+2x+3x^2+\cdots)$

$= \frac{(2n)^2}{(2n)^2}\left(1+\frac{1}{n}\right)^2 \left(1+2\left(\frac{1}{2n}\right)+3\left(\frac{1}{2n}\right)^2+4\left(\frac{1}{2n}\right)^3+\cdots\right)$

$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{1}{n}+\frac{3}{4n^2}+\frac{1}{2n^3}+\cdots\right)$

$= 1+\frac{1}{n}+\frac{2}{n}+\frac{3}{4n^2}+\frac{1}{n^2}+\cdots$

$= 1+\frac{3}{n}+O(n^2)\cdots$

From Binomial test series is convergent ?

By comparing $\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + O(n^2)$.

$\beta > 1$ cgt.

here $\beta = 3$ $\therefore$ convergent Series

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(ii) $u_n = \dfrac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)}$

$u_{n+1} = \dfrac{a(a+1)\cdots(a+n)(a+n+1)}{b(b+1)\qquad(b+n)(b+n+1)}$

$\dfrac{u_n}{u_{n+1}} = \dfrac{b+n+1}{(a+n+1)}$ $\qquad \lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = 1$

D Alembert's test fails to conclude

Considering Raabe's test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{b+n+1}{a+n+1} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{b+\cancel{n}+\cancel{1}-a-\cancel{n}-\cancel{1}}{(a+n+1)}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{\left(\frac{a+1}{n}+1\right)}$$

$$= (b-a)$$

$\therefore$ given series is cgt if $(b-a) > 1$
dgt if $(b-a) < 1$
at $(b-a) = 1$ fails.

$$\frac{u_n}{u_{n+1}} = \frac{(b+n+1)}{n}\left(\frac{1+a+n}{n}\right)^{-1}$$

$$= \left(1+\frac{1+b}{n}\right)\left(1+\frac{(1+a)}{n}\right)^{-1} = \left(1+\frac{(1+b)}{n}\right)\left(1-\frac{(1+a)}{n}+\frac{(1+a)}{n^2}-\frac{(1-a)}{n^3}\cdots\right)$$

$$= 1 - \frac{(1+a)}{n} + \frac{(1+b)}{n} + O(n^2)$$

By Binomial test $\quad = 1 + \dfrac{(b-a)}{n} + O(n^2)$

cgt if $(b-a) > 1$
dgt $(b-a) \le 1$

4(c) If a function f is continuous in [0, 1],

show that $\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} dx = \frac{\pi}{2} f(0)$ (20)

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} dx$$

By fundamental principle $\int_a^b f(x)\,dx = F(b) - F(a)$

$f(x)$ is primitive of $F(x)$

from modified fund. theorem ~~of~~

$$\int_a^b f(x)g(x) = \mu \int_a^b g(x)\,dx \quad \text{where } f(a) \le \mu \le f(b)$$

& $f(x)$ is a cont. function

Let $g(x) = \frac{n}{1+n^2x^2}$

$$\therefore \int_0^1 \frac{n}{1+n^2x^2} = n \cdot \frac{1}{n} \tan^{-1}(nx) \Big]_0^1$$

$$= \tan^{-1}(n)$$

(16)

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} = \lim_{n\to\infty} \mu \tan^{-1}(n) = \lim_{n\to\infty} f(0) \tan^{-1}(n)$$

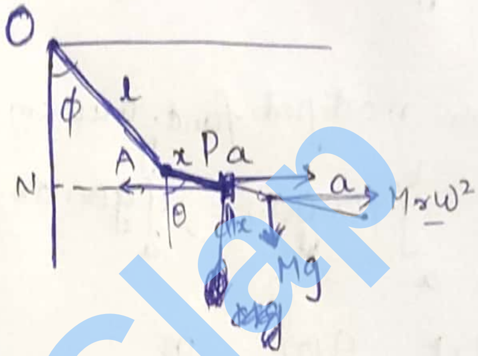
~~Since upper & lower limit~~

$$= \lim_{n\to\infty} f(0) \tan^{-1}(n) = \frac{\pi}{2} f(0)$$

Section B

5(a) A rod of length 2a, is suspended by a string of length l, attached to one end, if the string and rod revolve about the vertical with uniform angular velocity, and their inclination to the vertical θ and φ respectively, show that

$$\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\varphi)\sin\varphi}{(\tan\varphi - \tan\theta)\sin\theta}.$$ (10)



Consider small element dx

$\Rightarrow dm = \rho\, dx = \frac{M}{2a} dx$

$\therefore (dm)(PN)\omega^2 = \int \frac{M}{2a}(l\sin\phi + x\sin\theta)\,\omega^2 dx$

$= \int_0^{2a} \frac{M\omega^2}{2a}(l\sin\phi + x\sin\theta)\, dx$

$= \frac{M}{2a}\omega^2\left(l\, 2a\sin\phi) + \frac{(2a)^2}{2}\sin\theta\right)$

$= M\omega^2(l\sin\phi + a\sin\theta)$

By D'Alembert's $= Mg \Rightarrow \frac{g}{\omega^2} = (l\sin\phi + a\sin\theta)$ $\rightarrow (1)$

Taking Moment about "O"

$Mg(l\cos\phi + a\cos\theta) = M\omega^2(l\sin\phi + a\sin\theta) \rightarrow (2)$

from (1) & (2)

$$(l\sin\phi + a\sin\theta) = \frac{(l\sin\phi + a\sin\theta)}{(l\cos\phi + a\cos\theta)}$$

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5(b) A two – dimensional flow field is given by $\psi = xy$.

(i) Show that the flow is irrotational.

(ii) Find the velocity potential.

(iii) Verify that ψ and ϕ satisfy the Laplace equation.

(iv) Find the streamlines and potential lines. (10)

(i) $\psi = xy$

Velocity components $q = -\psi_y i + \psi_x j = ui + vj$

$= -xi + yj$

$$\nabla \times q = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -x & y & 0 \end{vmatrix} = i(0) - j(0) + k(0) = \vec{0}$$

$\therefore \nabla \times q = \vec{0}$ irrotational

(ii) from Cauchy Riemann Eqns

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\therefore \frac{\partial \phi}{\partial x} = x \Rightarrow \phi = \frac{x^2}{2} + f(y)$$

$$\frac{\partial \phi}{\partial y} = f'(y) = -y \Rightarrow f(y) = -\frac{y^2}{2}$$

$$\therefore \phi = \frac{(x^2 - y^2)}{2} + c$$

(iii) $$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 + 0 = 0$$

$\phi_{xx} = 1 \quad \phi_{yy} = -1 \Rightarrow \phi_{xx} + \phi_{yy} = 0$

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$\therefore \phi$ & ψ satisfies laplace

(iv) from (i) streamlines $\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{-x} = \frac{dy}{y}$

$\ln x + \ln y = \ln c$

$\boxed{c = xy}$

5(c) (i) Suggest a value c so that the iteration formula $x = x + c(x^2 - 3)$ may converge at a good rate. Given that $x = \sqrt{3}$ is a root. (10)

from iteration formula

$\phi(x) = x$ converges with good rate

if $|\phi'(x)| < 1$

$\Rightarrow |1 + c(2x)| < 1$

$-1 < 1 + c(2x) < 1$

$-2 < 2cx < 0$

$-1 < cx < 0$

$\Rightarrow$ given $x = \sqrt{3}$ root

$\therefore -\frac{1}{\sqrt{3}} < c < 0$ $\therefore c \in (0, -\frac{1}{\sqrt{3}})$

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5(d) The moment of inertia about its axis of a solid rubber type of mass M and circular cross section of radius a is $(M/4)(4b^2 + 3a^2)$ where b is the radius of the curve. If the tyre be hollow and of small uniform thickness, Show that the corresponding result is $(M/2)(2b^2 + 3a^2)$. (10)

5(e) Find the surface which is orthogonal to the one parameter system $z = cxy\,(x^2 + y^2)$ which passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$. (10)

$\Rightarrow$ let $\dfrac{z}{xy(x^2+y^2)} = c = f(x,y,z) \Rightarrow \dfrac{\partial f}{\partial x} = \dfrac{z - (x^2+y^2) - 2x^2)}{y(x(x^2+y^2))^2}$

$$\frac{\partial f}{\partial x} = \frac{z\,(x^2-y^2)}{y x^2 (x^2+y^2)^2}$$

$$\therefore \frac{\partial f}{\partial z} = \frac{1}{xy(x^2+y^2)} \qquad \frac{\partial f}{\partial y} = \frac{z\,(y^2-x^2)}{x y^2 (x^2+y^2)}$$

By Lagranges ~~auxiliary~~ method

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z}$$

Since $f_x P + f_y q = f_z$ is ortugonal to f.

$$\frac{dx}{\frac{z(x^2-y^2)}{yx^2(x^2+y^2)^2}} = \frac{dy}{\frac{z(y^2-x^2)}{xy^2(x^2+y^2)^2}} = \frac{dz}{\frac{1}{xy(x^2+y^2)}}$$

$$\frac{dx}{1/x} = \frac{-dy}{1/y} \Rightarrow x\,dx + y\,dy = 0$$

$$x^2 + y^2 = c_1$$

Let multipliers (x, y, z)

$$\frac{x\,dx + y\,dy + z\,dz}{\frac{z}{xy(x^2+y^2)}} = \frac{dx}{\frac{z(x^2-y^2)}{x^2y(x^2+y^2)}}$$

$=$

3(a) Solve by Charpit's method the partial differential equation
$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0.$ (10)

Charpit's method

$$\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-qf_q}=\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}$$

$f_p = 2px(x-1)+2qxy-2xz$ $\quad f_x = p^2(2x-1)+2pqy-2pz$

$f_q = 2qy(y-1)+2pxy-2yz$ $\quad f_y = q^2(2y-1)+2pqx-2qz$

$f_z = -2px-2qy+2z$

$$\frac{dz}{-2(p^2x(x-1)+q^2y(y-1)+2pqxy-2pxz-2qyz)}=\frac{dz}{-2(-z^2)}$$

$f_x+pf_z = \cancel{2xp^2}-p^2+\cancel{2pqy}-\cancel{2pz}-\cancel{2p^2x}-\cancel{2pqy}+\cancel{2pz}$

$$\frac{dx}{-p^2}=\frac{dy}{-q^2}=\frac{dz}{2z^2}$$

$dz = pdx+qdy$

$$dz = p\frac{-p^2}{2z^2}dz+q\frac{-q^2}{2z^2}dz$$

$p^3+q^3=-2z^2$

$\underline{p^3+q^3+2z^2=0}$

8(b) A uniform bar of length 2l and mass M is suspended from one end by a spring. The spring is constrained to move in vertical direction and the bar is constrained to rotate in a horizontal plane. If k is the force constant of the spring, construct the Hamiltonian of the system and obtain the equation of motion.

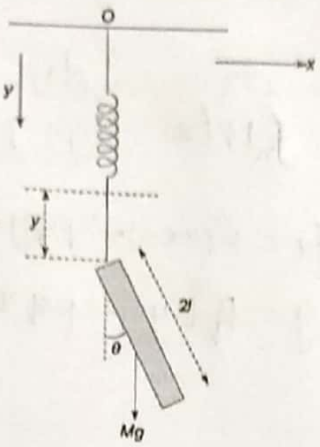

Fig. 3.6

(20)

Consider for rod.

$x_G = l\sin\theta$

$y_G = y + l\cos\theta$

$V_G^2 = \dot{x}_G^2 + \dot{y}_G^2$

$\dot{x}_G = l\cos\theta\cdot\dot{\theta}$

$\dot{y}_G = \dot{y} + l(-\cos\theta)\dot{\theta}$

$= (l\cos\theta\dot{\theta})^2 + (\dot{y} - l\sin\theta\dot{\theta})^2$

$= l^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + (\dot{y})^2 - 2l\sin\theta\cdot\dot{\theta}$

θ small $\sin\theta \simeq \theta$

$V_G^2 = l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{\theta}$

$T = \frac{1}{2}MV_G^2 + \frac{1}{2}I\omega^2$

$= \frac{1}{2}M(l^2\dot{\theta}^2 + \dot{y}^2 - 2l\dot{\theta}) + \frac{1}{2}(\frac{4}{3}Ml^2\dot{\theta}^2)$

For bar $I = \frac{4M}{3}l^2$

$T = Ml^2(\frac{1}{2} + \frac{2}{3})\dot{\theta}^2 - 2l\dot{\theta} + \dot{y}^2$

For $V = \underbrace{-ky^2}_{\text{Sprint}} - \underbrace{(y + l\cos\theta)Mg}_{\text{Potential}}$

$I = \frac{ML^2}{12}$

Note

$L = 2l$

$\frac{1}{2}I\omega^2$

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$$T = \frac{7}{6}Ml^2\dot\theta^2 - 2l\dot\theta + \dot y^2 \qquad V = -ky^2 - Mg(y + l\cos\theta)$$

$\Rightarrow L = T - V$

$$P_k = \frac{\partial L}{\partial \dot q_k} \Rightarrow P_\theta = \frac{\partial L}{\partial \dot\theta} \qquad P_y = \frac{\partial L}{\partial \dot y}$$

$$\therefore P_\theta = \frac{7}{6} 2Ml^2\dot\theta \qquad \frac{3P_\theta}{7Ml^2} = \dot\theta \qquad P_y = \frac{7}{3}\dot y$$

$$\dot y = \frac{3}{7}P_y$$

$$\Rightarrow T = \frac{7}{6}\left(\frac{3}{7}\right)^2 \frac{P_\theta^2}{(Ml^2)^2} Ml^2 - 2l\frac{3}{7}\frac{P_\theta}{Ml^2} + \left(\frac{3}{7}P_y\right)^2$$

$$T = \frac{3}{14}\frac{P_\theta^2}{Ml^2} - \frac{6}{7}l\frac{P_\theta}{Ml^2} + \left(\frac{3}{7}\right)^2 P_y^2$$

$H = T + V$ Since V is free from 't'

$$H = \frac{3}{14}\frac{P_\theta^2}{Ml^2} - \frac{6P_\theta}{7Ml} + \frac{9}{49}P_y^2 - ky^2 - Mg(y + l\cos\theta)$$

$$\dot P_\theta = -\frac{\partial H}{\partial\theta} \Rightarrow \dot P_\theta = -(+Mgl\sin\theta) = -Mgl\sin\theta$$

$$\dot\theta = \frac{\partial H}{\partial P_\theta} \qquad \dot\theta = \frac{3}{7}\frac{P_\theta}{Ml^2} - \frac{6}{7Ml}$$

$$\dot P_y = -\frac{\partial H}{\partial y} \Rightarrow \dot P_y = -(-2ky - Mg) = 2ky + Mg$$

$$\dot y = \frac{\partial H}{\partial P_y} \qquad \dot y = \frac{18}{49}P_y$$

$$\ddot y = \frac{18}{49}\dot P_y$$

$$\ddot y = \frac{18}{49}(2ky + Mg)$$

8(c) (i) Express F = A + B'C as sum of minterms.

(ii) Express the Boolean function $F = XY + \bar{X}Z$ in product of maxterm.

(iii) Convert the given expression in standard POS form.

f(A, B, C) = (A + B) (B + C) (A + C)

(iv) Convert the given expression in standard SOP form.

f(A, B, C) = AC + AB + BC

(v) If f(A, B, C) = (A' + B) (B' + C), find standard SOP and Standard POS. (20)

(i) $F = A + B'C$

minterms $F = (A)(B+B')(C+C') + B'C(A+A')$

$= A(BC + BC' + B'C + B'C') + B'CA + A'B'C$

$F = ABC + ABC' + AB'C + AB'C' + A'B'C$

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(ii) $F = XY + \bar{X}Z$

$= XY(Z+\bar{Z}) + \bar{X}Z(Y+\bar{Y})$

$= XYZ + XY\bar{Z} + \bar{X}YZ + \bar{X}\bar{Y}Z$

Complement of these terms will be ~~POS~~ max terms

$= (\bar{X}+\bar{Y}+\bar{Z})(\bar{X}+\bar{Y}+Z)(X+\bar{Y}+\bar{Z})(X+Y+\bar{Z})$

(iii) $f(A,B,C) = (A+B)(B+C)(A+C)$

$= (AB + AC + B + BC)(A+C)$

$= (AB + AC + AB + ABC)C$

$= ABC + AC + ABC + ABC$

$= ABC + AC$

$= AC(1+B)$

$= AC(B+B') = ABC + AB'C$

(iv) SOP form is Sum of products

$$f(A,B,C) = AC + AB + BC$$
$$= AC(B+B') + AB(C+C') + BC(A+A')$$
$$= ABC + AB'C + ABC + ABC' + ABC + A'BC$$
$$= ABC + A'BC + AB'C + ABC'$$

(iv)

$$f(A,B,C) = (A'+B')(B'+C)$$
$$= A'B' + A'C + B' + B'C$$
$$= \cancel{A'B'C + A'B'C'} + \cancel{A'CB + A'CB'} + \cancel{B'(1+C)}$$
$$= B'(1+C+A') + A'C$$
$$= B' + A'C = B'(A+A')(C+C') + A'C$$
$$= B'(AC + AC' + A'C + A'C') + A'C(B+B')$$
$$= AB'C + AB'C' + A'B'C + A'B'C' + A'BC + A'B'C$$
$$f(A,B,C) = A'BC + AB'C + AB'C' + A'B'C + A'B'C'$$

SOP form

(v) Complement of (iv) gives POS form

$$f(A,B,C) = ((\cancel{A'B}$$
$$= (A+B'+C')(A'+B+C')(A'+B'+C)(A+B+C)$$
$$(A'+B+C')$$