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Ans 1(a)

$$\begin{vmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$
$R_5 \to R_5 - 2R_1$

$$\Rightarrow \begin{vmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{vmatrix}$$

$R_4 \to R_4 - 2R_3$
$R_5 \to R_5 - 3R_3$

$$\Rightarrow \begin{vmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{vmatrix}$$

$R_2 \rightleftarrows R_3$
$R_5 \rightleftarrows R_3$
$R_3 \to \frac{1}{-4} R_3$

$$\Rightarrow \begin{vmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 0 & 5/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

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which is echelon form, hence Dim W = 3

and Basis = $\{(1,2,-1,3,4), (0,1,3,-1,2), (0,0,1,0,5/4)\}$

Ans 1(b)(i) Given A is skew symmetric real matrix, hence

$$A^T = -A \quad -①$$

Now $A^2 + I = 0 \Rightarrow A \cdot A = -I \quad -②$

from ① $A^T = -A$

pre and post multiplying both sides by A respectively we get

$$A^T A = A A^T = -A \cdot A = I \quad (\text{from } ②)$$

✓

hence $\boxed{A^T A = A A^T = I \quad \therefore A \text{ is orthogonal}}$ ✓ 5

Ans 1(b)(ii) H is hermitian matrix so $H^\theta = H$

Given is matrix e^{iH} $\qquad i^\theta = -i$

then $(e^{iH})^\theta = e^{iH^\theta} = e^{iH}$ ✗ ✗

Hence e^{iH} is also Hermitian matrix.

Ans(1)(I) $f(x, y) = \begin{cases} \dfrac{\sin(x-y)}{|x|+|y|} & |x|+|y| \neq 0 \\ 0 & (x, y) = (0, 0) \end{cases}$

$$= 3ze^{z} + \frac{e^{3z}}{10}\left(1-\frac{D_1}{2}\right)\left(1-\frac{D_1}{2+i}\cdots\right)\left(1-\frac{D_1}{2-i}\cdots\right)z$$

$$= 3ze^{z} + \frac{e^{3z}}{10}\left(z-\frac{13}{10}\right) = 3ze^{z} + \frac{ze^{3z}}{10} - \frac{13e^{3z}}{100}$$

hence complete solution is $\boxed{c_1x + x(c_2\cos\log x + c_3\sin\log x) + 3x\log x + \frac{x^3\log x}{10} - \frac{13x^3}{100}}$

Ans 1(d) $w=(x,y)$ $\quad x=u+v,\ y=u-v$

Now $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$

$$\frac{\partial^2 w}{\partial u\partial v} = \frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial x\partial y}\frac{\partial y}{\partial u} + \frac{\partial^2 w}{\partial y\partial x}\frac{\partial x}{\partial v} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial v}$$

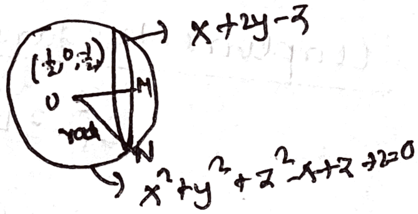
$$= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 w}{\partial y\partial x} - \frac{\partial^2 w}{\partial y^2}$$

✓ 8

Hence $\boxed{\frac{\partial^2 w}{\partial u\partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}}$ Ans

Ans 1(e) Equ of sphere: $x^2+y^2+z^2-x+z+2=0$ —(1)

Equ. of plane: $x+2y-z=4$ —(2)



Now centre of sphere: $\left(\frac12, 0, -\frac12\right)$

radius of sphere $= \sqrt{\frac14+\frac14+2} = \sqrt{\frac52}$

Distance $OM = \left|\frac{\frac12+\frac12-4}{\sqrt{1+4+1}}\right| = \sqrt{\frac32}$

$\therefore \boxed{MN^2 = ON^2 - OM^2 = \frac52 - \frac32 = 1}$, hence proved

Equation of any sphere through ① and ② is

$$x^2+y^2+z^2-x+z-2+\lambda(x+2y-z-4)=0$$

Its centre $= \left(\frac{1}{2}-\frac{\lambda}{2}, -\lambda, \frac{\lambda}{2}-\frac{1}{2}\right)$

and radius $= \sqrt{\left(\frac{1-\lambda}{2}\right)^2+\left(\frac{2\lambda}{2}\right)^2+\left(\frac{\lambda-1}{2}\right)^2+2+4\lambda} = 1$

$$1+\lambda^2-2\lambda+4\lambda^2+\lambda^2+1-2\lambda+8+16\lambda=4$$

$$6\lambda^2+12\lambda+6=0 \Rightarrow \lambda^2+2\lambda+1=0$$

$$\Rightarrow (\lambda+1)^2=0 \Rightarrow \lambda=-1$$

$$\boxed{\text{Hence sphere is } x^2+y^2+z^2-2x-2y+2z+2=0}$$

Ans 5(a) $(x^3D^3+2xD-2)y = x^2\ln x+3x$

Let $x=e^z \Rightarrow \log x = z \Rightarrow \frac{1}{x}\frac{\partial x}{\partial y} = \frac{\partial z}{\partial y} \Rightarrow D_1 \equiv xD$

where $D_1 \equiv \frac{\partial}{\partial z}$ and $D \equiv \frac{\partial}{\partial x}$, hence equation becomes

$$(D_1(D_1-1)(D_1-2)+2D_1-2)y = z.e^{2z}+3e^z$$

Complementary function:

$$D_1^3-3D_1^2+2D_1+2D_1-2=0 \Rightarrow D_1 = 1, 1\pm i$$

Hence Cf is $y = c_1e^z + e^z(c_2\cos z+c_3\sin z)$ or

$$y = c_1x+x(c_2\cos\log x+c_3\sin\log x) \quad \text{— ①}$$

Particular Integral:

$$= \frac{z.e^{3z}+3e^z}{(D_1-1)(D_1-(1+i))(D_1-(1-i))} = \frac{3e^z}{(D_1-1)(1)} + e^{3z}\frac{z}{(D_1+2)(D_1+(2-i))(D_1+(2+i))}$$

$$= 3ze^z + \frac{e^{3z}}{2(2-i)(2+i)}\left(1+\frac{D_1}{2}\right)^{-1}\left(1+\frac{D_1}{2-i}\right)^{-1}\left(1+\frac{D_1}{2+i}\right)^{-1}z$$

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$$= 3xe^{z} + \frac{e^{3z}}{10}\left(1-\frac{D_1}{2}\ldots\right)\left(1-\frac{D_1}{2+1}\ldots\right)\left(1-\frac{D_1}{2-1}\right)\cdot(z)$$ (5)

$$= 3ze^{z} + \frac{ze^{3z}}{10} - \frac{13e^{3z}}{100}$$

$$\boxed{\begin{array}{l}\text{hence complete solution is} \\ y = c_1x + x(c_2\sin\log x + c_3\cos\log x) + 3x\log x + \frac{x^3\log x}{10} - \frac{13x^3}{100}\end{array}}$$

$$\frac{1}{D_1^3+3D_1^2+4D_1+2} = \frac{1}{2}\left\{\left(1+\frac{D_1^3+3D_1^2+4D_1}{2}\right)^{-1}\right\} \quad 2$$

Best approach

donot know which element is missed

Ans 5(b) $r = a(1-\cos\theta)$

Taking log both sides and then diff wrt θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get

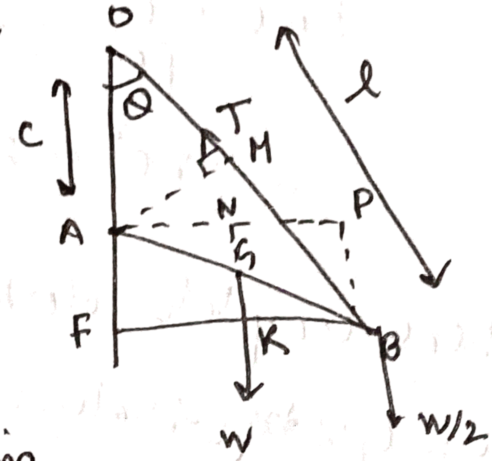
$$-r\frac{d\theta}{dr} = \cot\theta/2 \Rightarrow \frac{dr}{r} + \frac{d\theta\sin\theta/2}{\cos\theta/2} = 0$$

(8) $\log r + 2\log\sec\theta/2 = \log c$ (where c is some constant.

$$\Rightarrow r\cdot\sec^2\theta/2 = c \Rightarrow \boxed{r = c\cos^2\theta/2}$$

which is required orthogonal trajectory

Ans 5(c) Let AB be given rod with centre of gravity at G. Let OB be string of length l with Tension T and making angle θ with vertical.



Now To avoid reaction at A, taking moment around A, we get

$$T.AM = W.AN + \frac{W}{2}.AP$$

$$T.c\sin\theta = W.FK + \frac{W}{2}.BF$$ BF = $l\sin\theta$

$$T.c\sin\theta = W.\frac{l\sin\theta}{2} + \frac{W}{2}.l\sin\theta$$

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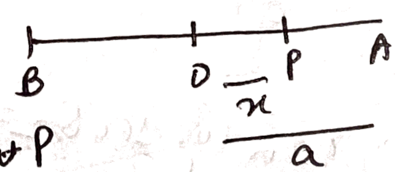
How $FK = \frac{l\sin\theta}{2}$?

Since $\sin\theta \neq 0$ hence

$$T.c = Wl \Rightarrow \boxed{T = \frac{Wl}{c}}$$

Ans 5(d) Let the origin be at O and particle be at rest at A.



Now let particle be at time t at P

$$\therefore m\frac{d^2x}{dt^2} = -m\mu\left[x + \frac{a^4}{x^3}\right]$$

Integrating both sides

$$\left(\frac{dx}{dt}\right)^2 = -\mu\left[x^2 - \frac{a^4}{x^2}\right] + C$$

At time $t=0$, $\frac{dx}{dt}=0$, $x=a$, hence $C=0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = +\mu\left[\frac{a^4}{x^2} - x^2\right]$$

$$\frac{x\,dx}{a^2\sqrt{1-\frac{x^4}{a^4}}} = \sqrt{u}\,dt$$

Let $x^2 \to a^2\cos p \Rightarrow 2x\,dx = -a^2\sin p\,dp$, we get

$$\frac{-a^2\sin p\,dp}{2a^2\sqrt{1-\cos^2 p}} = \sqrt{u}\,dt \Rightarrow -dp = \sqrt{u}\,dt$$

$$-p = 2\sqrt{u}\,t + c$$

$$-\cos^{-1}\frac{x^2}{a^2} = 2\sqrt{u}\,t + c \quad \text{(where } c \text{ is some constant)}$$

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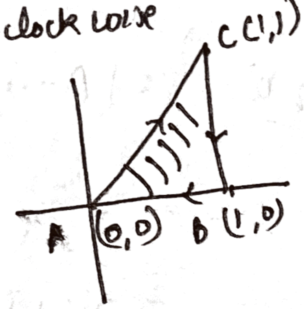
At $t=0$ $x=a$, hence $c=0$

$\therefore$ $\boxed{\text{we have at } x=0,\ t=\frac{\pi}{4\sqrt{u}}}$ Ans.

Ans 5(e) Let ABC be triangle with counter-clock wise orientation



To evaluate: $\int_C (x^2y\,dx + x^2dy)$

Now using Green's theorem we have

$$\int_C (x^2y\,dx + x^2dy) = \iint_R (2x - x^2)\,dx\,dy$$ where C is triangle's circumference and R is shaded region

$$= \int_{x=0}^{1}\int_{y=0}^{y=x} (2x-x^2)\,dx\,dy = \int_0^1 (2x-x^2)\,x\,dx$$

$$= \left.\frac{2x^3}{3} - \frac{x^4}{4}\right|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \boxed{\frac{5}{12}}$$ Ans

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Ans 2(a) $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$ Ans.

$I + aA + bA^2 = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 2a \\ -2a & a \end{bmatrix} + \begin{bmatrix} -3b & 4b \\ -4b & -3b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $1 + a - 3b = 0$

$4b + 2a = 0$ $\Rightarrow$ $a = -2b$

$-2a - 4b = 0$ $\quad 1 - 5b = 0$

$1 + a - 3b = 0$ $\Rightarrow$ $\boxed{b = \frac{1}{5},\ a = \frac{-2}{5}}$

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Ans 2(b) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$

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$(2-\lambda)[(\lambda-4)(\lambda-1)+2] - 1[0] = 0$

$(\lambda-2)(\lambda^2 - 5\lambda + 4 + 2) = 0 \Rightarrow (\lambda-2)(\lambda-3)(\lambda-2) = 0$

$\boxed{\text{hence minimal polynomial is } (x-2)^2(x-3) = 0}$ $(x-2)(x-3)$

Minimal polynomial means Minimum

Ans 2(c) Let $I = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$ (1^∞ form)

Take log both sides $\log I = \lim_{x \to 0} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$ $\left(\frac{0}{0} \text{ form}\right)$

Applying L hopital rule, $\log I = \lim_{x \to 0} \frac{x}{\tan x} \left(\frac{\sec^2 x \cdot x - \tan x}{x^2}\right)$

$= \cancel{\lim_{x \to 0} \frac{1}{\sin x \cos x} - \frac{1}{x}}$

$$\log I = \lim_{x\to 0} \frac{2\sec^2 x \tan x \cdot x + \sec^2 x - \sec^2 x}{\sec^2 x \cdot x + \tan x}$$ 9

Again applying L Hopital rule

$$\log I = \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x \, x + 2\sec^2 x(\tan x + \sec^2 x \cdot x)}{\sec^2 x + \sec^2 x + 2\sec^2 x \tan x \cdot x}$$

$$= 0$$

$$\boxed{\text{hence } I = e^0 = 1}$$

Ans 2(c) (ii) $\quad V = A t^{-1/2} e^{-\frac{x^2}{4a^2 t}}$

Take log both sides we get

$$\log V = \log A - \frac{1}{2}\log t - \frac{x^2}{4a^2 t} \quad -(1)$$

Now diff (1) wrt t

$$\frac{1}{V}\frac{dV}{dt} = -\frac{1}{2t} + \frac{x^2}{4a^2t^2} \quad -(2)$$

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Diff (1) wrt x we get

$$\frac{1}{V}\frac{dV}{dx} = -\frac{x}{2a^2 t} \quad -(3)$$

Again diff (3) wrt x

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial x}\left(-\frac{x}{2a^2t}\right) + V\left(-\frac{1}{2a^2t}\right)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{V x^2}{4a^4t^2} - \frac{V}{2a^2t} \Rightarrow a^2\frac{\partial^2 V}{\partial x^2} = \frac{Vx^2}{4a^2t^2} - \frac{V}{2t} \quad -(4)$$

Hence from (4) (2) and (3) we have

$$\boxed{\frac{dV}{dt} = a^2\frac{\partial^2 V}{\partial x^2}}$$

Hence proved

Ans 2 (d) Let P be (α, β, γ), hence equation of generator is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \quad -(1)$$

$\therefore$ Any point on it with $z=0$ is $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

If it lies on guiding curve we have

$$\frac{1}{a^2}\left(\alpha - \frac{l\gamma}{n}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

Using (1) we have $\frac{1}{a^2}\left(\alpha - \frac{(x-\alpha)\gamma}{(z-\gamma)}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{(y-\beta)\gamma}{(z-\gamma)}\right)^2 = 1$

$$\frac{1}{a^2}\left(\frac{\alpha z - x\gamma}{z-\gamma}\right)^2 + \frac{1}{b^2}\left(\frac{\beta z - y\gamma}{z-\gamma}\right)^2 = 1 \quad -(2)$$

If it passes by $x=0$, we have from (2)

$$\Rightarrow \frac{1}{a^2}\frac{\alpha^2 z^2}{(z-\gamma)^2} + \frac{1}{b^2}\left(\frac{\beta z - \gamma y}{z-\gamma}\right)^2 = 1$$

If it is rectangular hyperbola, then coeff y^2 + coeff $z^2 = 0$

ie $\frac{\gamma^2}{b^2} + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 = 0$

$$\boxed{\text{Hence Locus of P is } \frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1}$$

Hence proved.

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Ans 7(a) $\quad xy_1 - y = (x-1)(y_2 - x + 1)$

$$\Rightarrow \quad y_2 - y_1\left(\frac{x}{x-1}\right) + \frac{y}{x-1} = (x-1)$$

Comparing equation with $(D^2 + PD + Q)y = R$

$$P = -\frac{x}{x-1},\ Q = \frac{1}{x-1},\ R = x-1$$

Now $1 + P + Q = 0$, hence e^x will be one solution

Let the complete solution be $y = uv$ where $u = e^x$ and

v is given by $\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(P + \frac{2}{u}\frac{du}{dx}\right) = R/u$

hence $\frac{dq}{dx} + q\left(-\frac{x}{x-1} + \frac{2}{e^x}\cdot e^x\right) = \frac{x-1}{e^x}$ where $q = \frac{dv}{dx}$

$$\frac{dq}{dx} + q\left(\frac{x-2}{x-1}\right) = \frac{x-1}{e^x} \quad -(1)$$

Integrating factor is $e^{\int\frac{x-2}{x-1}dx} = e^{\int 1 - \frac{1}{x-1}dx} = \frac{e^x}{x-1} \quad -(2)$

hence we have from (1) and (2) we have

$$q \cdot \frac{e^x}{x-1} = \int \frac{x-1}{e^x}\cdot\frac{e^x}{x-1}\,dx = x + c \text{ where } c \text{ is constant}$$

$$q = \int\frac{dv}{dx} = \int\frac{x(x-1)}{e^x} + \int c\frac{(x-1)}{e^x}$$

$$v = -x(x-1)e^{-x} + \int(2x-1)e^{-x} + c\left[(x-1)-e^{-x} + \int e^{-x}\right]$$

$$v = x(1-x)e^{-x} + (2x-1)-e^{-x} + \int 2e^{-x}dx + c(x-1)e^{-x} - e^{-x} + c_2$$

$$v = x(1-x)e^{-x} + (1-2x)e^{-x} - 2e^{-x} + (1-x)e^{-x}c - e^{-x}c + c_2$$

$$v = x(1-x)e^{-x} + (1-2x)e^{-x} - 2e^{-x} + c(1-x)e^{-x} - e^{-x}c + c_2$$

$$v = e^{-x}\left[x(1-x) + 1 - 2x - 2 + c(-x)\right] + c_2$$

$$\boxed{\text{hence complete solution} = (-x^2 - x - 1 - cx) + c_2e^x}$$

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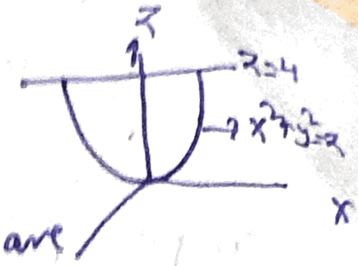
Ans 7(b) To evaluate: $\iint_S \vec{F}\cdot n\, ds$

Using Gauss divergence theorem we have



$$\iint_S \vec{F}\cdot n\, ds = \iiint_V (\nabla\cdot F)\, dV$$

where S is surface of region above xy plane and bounded by plane and cone and

V is volume enclosed.

Hence $\iiint_V (\nabla\cdot F)\, dv = \iiint (4z + xz^2 + 3)\, dx\, dy\, dz$

Limits of limit are $\sqrt{x^2+y^2} \to 4$

$$\therefore \iiint (\nabla\cdot F)\, dv = \iint \left(2z^2 + \frac{xz^3}{3} + 3z\right)\Big|_{\sqrt{x^2+y^2}}^{4} dx\, dy$$

$$= \iint \left[2(16) + \frac{2\cdot 64}{3} + 3\cdot 4 - 2(x^2+y^2) + \frac{x(x^2+y^2)^{3/2}}{3} + 3\sqrt{x^2+y^2}\right] dx\, dy$$

$$= \iint \left(\frac{260}{3} - 2(x^2+y^2) + \frac{x(x^2+y^2)^{3/2}}{3} + 3\sqrt{x^2+y^2}\right) dx\, dy \quad (1)$$

Now projection of x and y plane will be $x^2+y^2=16$

Using polar Let $x \to r\cos\theta$ and $y \to r\sin\theta$ we have

from (1)

$$\iiint (\nabla\cdot F)\, dv = \int_0^{2\pi}\int_0^4 \left[\frac{260}{3} - 2r^2 + \frac{r\cos\theta\cdot r^3}{3} + 3r\right] r\, d\theta\, dr$$

$$= \int_0^{2\pi} \left[\frac{130}{3}r^2 - \frac{2r^4}{4} + \frac{\cos\theta\cdot r^6}{3\cdot 6} + r^3\right]_0^4 d\theta$$

$$= 2\pi\left[\frac{130}{3}(16) - 2\cdot 64 + 0 + 64\right]$$

$$= 2\pi\cdot\left[\frac{2080 - 64}{3}\right] = \boxed{1888\left[\frac{2\pi}{3}\right]}$$

320π

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Ans 7(b) Let P be central acceleration, hence

$$\frac{P}{\mu^2} = \mu\left(\frac{1}{\mu^7} - \frac{c^4}{\mu^3}\right) = h^2\left(u + \frac{\partial^2 u}{\partial \theta^2}\right) \quad \text{where } \mu = \frac{1}{r}$$

Integrating both sides wrt. θ

$$\mu\left(\frac{-1}{3\mu^6} + \frac{c^4}{\mu^2}\right) + c_1 = h^2\left[u^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right] = (\text{velocity at apse})^2$$

At apse $u = \frac{1}{c}$ and $v^2 = c^6\left(\frac{2\mu}{3}\right)$ and $\frac{\partial u}{\partial \theta} = 0$

∴ we have $\mu\left(-\frac{c^6}{3} + c^6\right) + c_1 = h^2\left[\frac{1}{c^2}\right] = c^6\left(\frac{2\mu}{3}\right)$

∴ $c_1 = 0$ and $h^2 = c^8\left(\frac{2\mu}{3}\right)$

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Hence we have

$$\mu\left(\frac{-1}{3\mu^6} + \frac{c^4}{\mu^2}\right) = \mu \cdot \frac{2c^8}{3}\left(\mu^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right)$$

$$\frac{-1 + 3\mu^4 c^4 - 2c^8\mu^8}{2c^8\mu^6} = \left(\frac{du}{d\theta}\right)^2$$

$$\Rightarrow \frac{du \cdot \sqrt{2}\, c^4 u^3}{\sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 - \left(\sqrt{2}\, c^4 u^4 - \frac{3}{2\sqrt{2}}\right)^2}} = d\theta$$

Taking $\sqrt{2}\, c^4 u^4 - \frac{3}{2\sqrt{2}} \to t \Rightarrow 4\sqrt{2}\, c^4 u^3\, du = dt$

$$\therefore \frac{dt}{4\sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 - t^2}} = d\theta$$

Integrating we have $\frac{1}{4}\sin^{-1} t \cdot 2\sqrt{2} = \theta + B$.

$$\frac{1}{4}\sin^{-1}\left(\sqrt{2}\, c^4 u^4 - \frac{3}{2\sqrt{2}}\right) 2\sqrt{2} = \theta + B \Rightarrow \frac{1}{4}\sin^{-1}\left(4c^4u^4 - 3\right) = \theta + B$$

At $\theta = 0$, $\mu = \frac{1}{c}$ hence we have $\frac{1}{4}\cos^{-1}\left(4c^4u^4 - 3\right) = \theta$

Ans 8(a) Given $(D^2+2D+5)y = e^{-t}\sin t$

Taking Laplace both sides we have

$s^2F(s) - sf(0) - f'(0) + 2[sF(s) - f(0)] + 5F(s) = \frac{1}{(s+1)^2+1}$

Give $f(0)=0$ and $f'(0)=1$, $\therefore$ we have

$(s^2+2s+5)F(s) - 1 = \frac{1}{s^2+2+2s}$

$F(s) = \frac{1}{(s+1)^2+2^2} + \frac{1}{[(s+1)^2+1][(s+1)^2+2^2]}$

$= \frac{1}{(s+1)^2+2^2} + \frac{1}{3}\left[\frac{1}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+2^2}\right]$

Taking Inverse Laplace both sids we have

$y = \frac{e^{-t}\sin 2t}{2} + \frac{e^{-t}}{3}\sin t - \frac{1}{3\cdot 2}e^{-t}\sin 2t$

hence $\boxed{y = \frac{e^{-t}}{3}[\sin t + \sin 2t]}$

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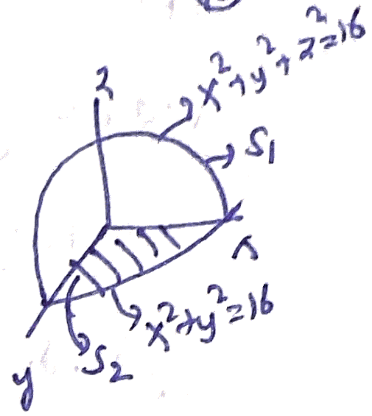
Ans 8(ii) Stokes Theorem says



$$\iint_{S_1} (\nabla \times F)\cdot n\,ds = \iint_C \vec{F}\cdot dr$$

where S_1 is upper half of sphere and C is curve $x^2+y^2=16$ and let S_2 be region bounded by C

Taking LHS

$\iint_{S_1} (\nabla\times F)\cdot n\,ds$, Now we know by gauss divergence Theorem, we have $\iiint_V (\nabla\cdot\nabla\times F)\,dv = \iint_{S_1} (\nabla\times F)\cdot n\,ds + \iint_{S_2} (\nabla\times F)\cdot n\,ds = 0$

Hence $\iint_{S_1} (\nabla\times F)\cdot n\,ds = -\iint_{S_2} (\nabla\times F)\cdot n\,ds$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(2z) + \hat{k}(3y-1)$$

and $\hat{n} = -\hat{k}$

$\therefore \iint_{S_1} (\nabla\times F)\cdot n\,ds = \iint (3y-1)\,ds$ — ①

In polar $x \to r\cos\theta$, $y \to r\sin\theta$, $r: 0\to 4$ and $\theta: 0 \to 2\pi$

hence from ① we have $\iint_{S_1} (\nabla\times F)\cdot n\,ds = \int_0^{2\pi}\int_0^4 (3r^2\cos\theta - r)\,d\theta\,dr$

$$= \int_0^{2\pi} \left[r^3\cos\theta - \frac{r^2}{2} \right]_0^4 d\theta = \int_0^{2\pi} (64\cos\theta - 8)\,d\theta$$

$$\therefore \boxed{\iint_{S_1} (\nabla\times F)\cdot n\,ds = -16\pi}$$ — ②

✓

Taking RHS

$$\int_C \vec{F}\cdot dr = \int (x^2+y-4)\,dx + 3xy\,dy + 0\,dz \qquad (\because z=0 \text{ on } C)$$

Taking $x \to 4\cos\theta$, $y \to 4\sin\theta$, $\theta: 0 \to 2\pi$ we have

$$\int_C \vec{F}\cdot dr = \int \left((16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta) + 192\sin\theta\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} \left(64\cos^2\theta\sin\theta - 16\sin^2\theta + 16\sin\theta + 192\sin\theta\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} -16\sin^2\theta\, d\theta \qquad \left[\because \int_0^{x} f(x)\,dx = 0 \text{ if } f(-x) = -f(x)\right]$$

$$= -\frac{64}{2}\cdot \beta\left(\frac{3}{2},\frac{1}{2}\right) = \boxed{-16\pi} \quad \text{(b)}$$

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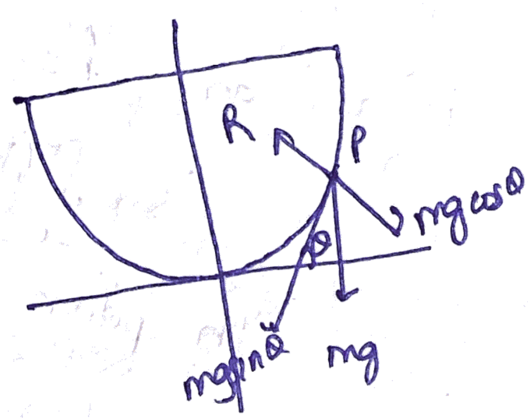
Since (a) = (b) = -16π

hence Stokes Theorem is verified

Ans 8(b) Given cycloid with particle P at

ant time t at P



hence we have motion equations!

$$-mg\sin\psi = m\frac{d^2 s}{dt^2} \quad (1)$$

$$R - mg\cos\psi = \frac{mv^2}{\rho} \quad (2)$$

for cycloid we have $s = 4a\sin\psi$, $\therefore$ we have

$$-\frac{gs}{4a} = \frac{d^2 s}{dt^2}$$, Integrating both sides we have

$$-\frac{gs^2}{4a} + c = \left(\frac{ds}{dt}\right)^2.$$

Now at time $t=0$, $s=4a$ and $v=0$

$\therefore$ we have $c = \frac{16a^2 g}{4a}$

$$\therefore \quad \frac{g}{4a}\left((4a)^2 - s^2\right) = \left(\frac{\partial s}{\partial t}\right)^2$$

$$\Rightarrow \sqrt{\frac{g}{4a}} \frac{ds}{\sqrt{(4a)^2 - s^2}} = \partial t$$

Integrating both sides

$$\sqrt{\frac{g}{4a}} \sin^{-1}\frac{s}{4a} = t + c_2$$

At time $t=0$ $s=4a$ hence $c_2 = 0$

$\therefore$ time taken at any time from top $= \sqrt{\frac{g}{4a}} \sin^{-1}\frac{s}{4a}$

$\therefore$ t_1 (time taken in falling down first hay of vertical height ie $y=a$ and $s = 2\sqrt{2}a$) $= \left[\sqrt{\frac{g}{4a}} \sin^{-1}\frac{1}{\sqrt{2}}\right] = \frac{g\pi}{16a}$

t_2 time taken in falling

Integration both sides for $s: 4a \to 2\sqrt{2}a$ we have

$$t_1 = \sqrt{\frac{g}{4a}}\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}1\right] = \sqrt{\frac{g}{4a}} \cdot \frac{\pi}{4}$$

Similarly integrating both sides for $s: 2\sqrt{2}a \to 0$ we have

$$t_2 = \sqrt{\frac{g}{4a}}\left[\sin^{-1}0 - \sin^{-1}\frac{1}{\sqrt{2}}\right] = \sqrt{\frac{g}{4a}} \frac{\pi}{4}$$

$\boxed{\text{Since } t_1 = t_2 = \sqrt{\frac{g}{4a}} \cdot \frac{\pi}{4}}$ hence proved

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