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Best Coaching for UPSC MATHEMATICS

UPSC CSE 2021 Mathematics Optional Paper 2 Solution

S.No	UPSC Question	Topic	SuccessClap Question Bank Source
1	1b	Real Analysis	SC B29 Qn 49
2	1c	Real Analysis	SC B26 Qn 14
3	3b	Real Analysis	SC B07 Qn1
4	4b	Complex Analysis	SC H05 Qn55
5	5a	PDE	SC I01 Qn 10
6	5e	Fluid Dynamics	SC M06 Qn 1
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8	7a	PDE	SC I04 Qn 47
9	7c	Fluid Dynamics	SC M03 Qn 13
10	8b	Numerical Analysis	SC L02 Qn 2

Question Bank Solutions: Link

UPSC Mathematics Test Series (45 Full Length) 2022: Link

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① Let $m_1, m_2 \ldots n_k$ be positive integers &
$d > 0$ the GCD of $m_1, m_2 \ldots m_k$.
Show there exists integers $x_1, x_2 \ldots x_k$ s.t

$$d = x_1 m_1 + x_2 m_2 + \ldots x_k m_k \qquad [10]$$

GCD defn: If a, b integers
→ If $d|a$ & $d|b$
& → If $c|a$ & $c|b$ & $c|d$
then d is GCD of (a,b) $d = (a,b)$

→ Let $S = \{ a_1 m_1 + a_2 m_2 + a_3 m_3 \ldots a_k m_k \mid a_1, a_2, \ldots a_k$ are integers & $a_1 m_1 + a_2 m_2 + \ldots a_k m_k > 0 \}$

→ $m_1 > 0$ (given positive integers)
So $m_1 = m_1 . 1 + 0 . m_2 + 0 . m_3 + . 0 . m_k$
$> 0 \Rightarrow m_1 \in S$

→ Similarly $m_2 > 0$ so $m_2 \in S$

Conclusion S is Non-Empty

→ well ordered principle, there exists least element, so let it be d

So $d \in S \Rightarrow d = m_1 b_1 + m_2 b_2 + \dots m_k b_k$ for some integers $b_1, b_2, \dots b_k$

Also $d > 0$

Let $m_1 = dq + r \quad 0 \le r < d$ (Division Theorem)

↳ If $r \neq 0 \Rightarrow r = m_1 - dq$

$= m_1 - (m_1 b_1 + m_2 b_2 + \dots m_k b_k) q$

$r = m_1(1 - q b_1) + m_2(-b_2 q) + m_3(-b_3 q) + \dots m_k(-b_k q)$

$\Rightarrow r \in S$

But $r < d$ & d is least element So $r = 0$

$\Rightarrow m_1 = dq \Rightarrow d \mid m_1$

Similarly $d \mid m_2$

$d \mid m_3$

$\vdots$

$d \mid m_k$

Suppose $c|m_1$, $c|m_2$, $c|m_3$ & $c|m_k$

$\Rightarrow c \mid m_1b_1 + m_2b_2 + m_3b_3 + \dots m_kb_k$

$\Rightarrow c|d$

So d is GCD of $(m_1, m_2 \dots m_k)$

Brief Note:

→ In ALL TEXTBOOKS, this question is present for two numbers, in chapter 1 as preliminary reading part.

→ Most of the students SKIP as it seems Obvious.

→ This qn is extension of two numbers, and you will not get Solution in any book. You have to extend the theorem.

→ Abstract Algebra is Ocean.

↓ It is advisable to study from ANY GRADUATION TEXT BOOK of your State. UPSC asks 1 or 2 KNOWN qns. If such qn appears, you are Lucky.

Download Abstract Algebra FREE study material from SuccessClap website.

SuccessClap Paper 2

(1b) Test Uniform Convergence of

$$x^4 + \frac{x^4}{1+x^2} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$$ (10M)

[SuccessClap : SC-B29 Uniform Conver Qn-49
Question Bank Series]

S_n = nth partial sum $= f_1(x) + f_2(x) + \cdots f_n(x)$

$$= x^4 + \frac{x^4}{1+x^2} + \cdots \frac{x^4}{(1+x^4)^{n-1}} = \frac{x^4\left[1 - 1/(1+x^4)^n\right]}{1 - 1/(1+x^4)}$$

$$= 1 + x^4 - \frac{1}{(1+x^4)^{n-1}}$$

Sum function $S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^4 & x \neq 0 \\ 0 & x = 0 \end{cases}$

Sum function $S(x)$ is discontinuous at $x=0 \in [0,1]$

the series is not uniform convergent on $[0,1]$

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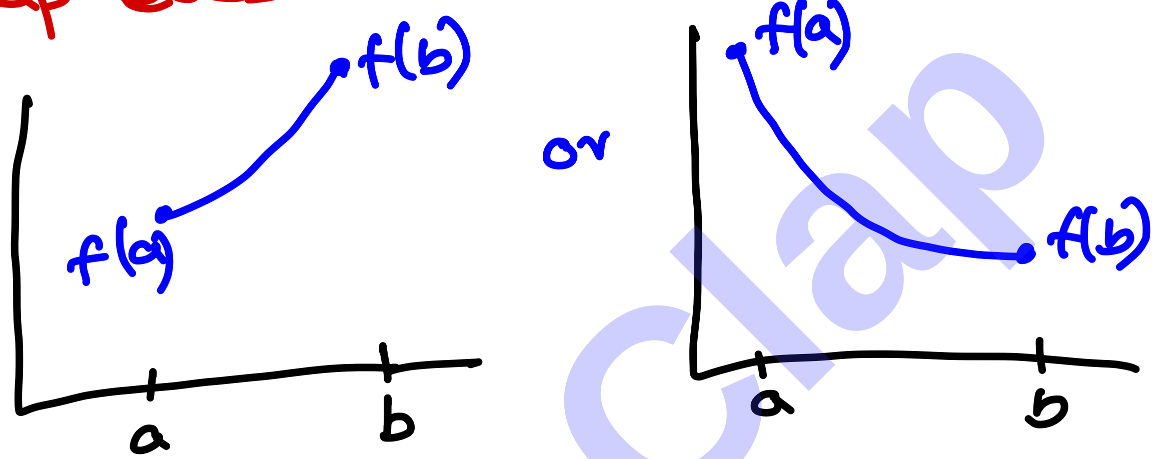
(1c) If a function f is monotonic in $[a,b]$, Prove f is Rieman integrable in $[a,b]$

SuccessClap Question bank SC-B26 Qn 14

Monotonic :



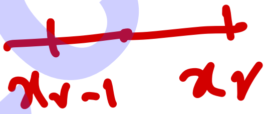
Given $\varepsilon > 0$

↳ If Increasing

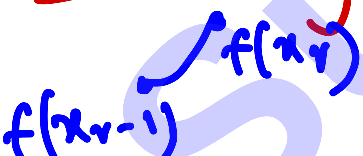
$P = [a = x_0, x_1, x_2 \ldots x_{r-1}, x_r \ldots b]$

↳ I will make partition such that each subinterval length is $< \dfrac{\varepsilon}{f(b)-f(a)-1}$

$\delta_r = x_r - x_{r-1}$



Increasing so $M_r = f(x_r)$ $\quad m_r = f(x_{r-1})$ (obvious)



$$\omega(P,f) = \sum (M_r - m_r)\,\delta_r$$

$$= \sum [f(x_r) - f(x_{r-1})]\,\delta_r$$

$\delta_r < \dfrac{\varepsilon}{f(b)-f(a)-1}$

$$< \frac{\varepsilon}{f(b)-f(a)-1} \sum f(x_r) - f(x_{r-1})$$

$\sum f(x_r) - f(x_{r-1}) = f(b) - f(a)$

$$< \frac{\varepsilon}{f(b)-f(a)-1}\,[f(b)-f(a)]$$

$< \varepsilon$

$\omega(P,f) < \varepsilon$ So Riemann integrable

For Decreasing : $\delta \leq \dfrac{\varepsilon}{f(a)-f(b)-1}$ $\quad \omega(P,f) < \delta\,[f(a)-f(b)]$ $< \varepsilon$

(1d) $c:[0,1]\to\mathbb{C}$ be curve $c(t)=e^{4\pi i t}$ $0\le t<1$

Evaluate $\displaystyle\int_c \frac{dz}{2z^2-5z+2}$ [10]

$$\int_c \frac{1}{2z^2-5z+2}\,dz=\int_c f(z)\,dz \qquad c: |z|=1$$

$$f(z)=\frac{1}{2z^2-5z+2}=\frac{1}{2(z-1/2)(z-2)}$$

$2z^2-5z+2$
$2z^2-4z-z+2$
$2z(z-2)-1(z-2)$
$(2z-1)(z-2)$
$2(z-1/2)($

$z=1/2$ lie inside

$$f(z)=\frac{g(z)}{z-1/2} \qquad g(z)=\frac{1}{2(z-1)}$$

$$\int_c f(z)\,dz=\int_c \frac{g(z)}{z-1/2}\,dz$$

$$=2\pi i\, g(1/2)$$

$$=2\pi i\,\frac{1}{2(1/2-2)}$$

$\frac{1}{2}-2=$
$-3/2$
$2\times-3/2=-3$

$$=\frac{-2\pi i}{3}$$

Two times winding $\underline{4\pi i t}$, so multiply by 2

$$\int_c f(z)\,dz=\frac{-4\pi i}{3}$$

(1e) A dept of company has 5 employees with 5 jobs to be performed. The time (in hrs) that each man takes to perform each job is given in effective matrix. Assign all jobs to these five employees to minimise the total processing time:

Employee

jobs	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

[10]

Ans: A — II
B — V
C → III
D → I
E → IV

Value 22

Source: S.Chand OR : Qn. No 11 (Page 373)

(29) Find max/Min of $f(x) = x^3 - 9x^2 + 26x - 24$ in $0 \le x \le 1$ [15]

$$f(x) = x^3 - 9x^2 + 26x - 24$$

$$f'(x) = 3x^2 - 18x + 26$$

$$= 3[x^2 - 6x] + 26 = 3[x^2 - 6x + 9 - 9] + 26$$

$$= 3[(x-3)^2] - 1$$

$$= 3(x-3)^2 - 1$$

$9 \times 3 = 27$

$f'(x) = 0 \Rightarrow$ Critical pt $\quad 3(x-3)^2 - 1 = 0 \quad (x-3) = \pm\frac{1}{\sqrt{3}}$

$$x = 3 \pm \frac{1}{\sqrt{3}}$$

No Critical point in [0 1]

$f''(x) = 6(x-3)$

In [0 1] $f''(x) < 0$ So Increasing

$f(0) = -24$

$f(1) = 1 - 9 + 26 - 24 = -6$

Min at $x = 0$ & value -24

Max at $x = 1$ & value -6

2c) Find Laurent $f(z) = \dfrac{z^2 - z + 1}{z(z^2 - 3z + 2)}$ in power of $(z+1)$ in $|z+1| > 3$

[20]

Doubt: Is it compulsary to resolve in partial fraction & expand?

Ans: NO : expansion at denominator is important.

Ex $\dfrac{f(x)}{(x-a)(x^2+2x+3)(x-b)^2}$

$$f(x)\left[1-\frac{a}{x}\right]^{-1}\left(1+\frac{2x}{3}+\frac{x^2}{2}\right)^{-1}\left[1+\frac{b}{x}\right]^{-2} \equiv$$ Same as

$$A_1\left[1-\frac{a}{x}\right]^{-1} + B_1\left[1+\frac{2x}{3}+\frac{x^2}{3}\right]^{-1} + C_1\left[1-\frac{b}{x}\right] + D_1\left[1-\frac{b}{x}\right]^{-2}$$

General example

A_1, B_1, C_1, D_1 take care

$f(z) = \dfrac{z^2 - z + 1}{z(z^2 - 3z + 2)}$

we want in $z+1$

$u = z+1$

$z = u-1$

& $u > 3$

$$f(u) = \frac{u^2 - 2u + 1 - u + 1 + 1}{(u-1)(u^2 - 2u + 1 - 3u + 3 + 2)}$$

$$= \frac{u^2-3u+3}{(u-1)(u^2-5u+6)} = \frac{u^2-3u+3}{(u-1)(u-2)(u-3)}$$

$$= \frac{A}{u-1} + \frac{B}{u-2} + \frac{C}{u-3}$$

$$u^2-3u+3 = A(u-2)(u-3) + B(u-1)(u-3) + C(u-1)(u-2)$$

Put $u=1 \Rightarrow 1 = A(-1)(-2)$ $A = 1/2$

$u=2 \Rightarrow 1 = B(1)(-1)$ $B=-1$

$u=3 \Rightarrow 3 = C(2)(1)$ $C = 3/2$

4
-6
$+3$

9
-9
$+3$

$$f(u) = \frac{1/2}{u-1} - \frac{1}{u-2} + \frac{3}{2}\frac{1}{u-3}$$

For $u>3 \Rightarrow 1 > \frac{3}{u} > \frac{1}{u}$ $\frac{1}{u}<1$

$$f(u) = \frac{1}{2u(1-\frac{1}{u})} - \frac{1}{u(1-\frac{2}{u})} + \frac{3}{2u(1-\frac{3}{u})}$$

$1 > \frac{3}{u} > \frac{2}{u} > \frac{1}{u} \Rightarrow \frac{1}{u}<1,\ \frac{2}{u}<1\ \ \frac{3}{u}<1$

$$f(u) = \frac{1}{2u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1} + \frac{3}{2u}\left(1-\frac{3}{u}\right)^{-1}$$

$$\int \left(1-\frac{1}{u}\right)^{-1} = 1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots$$

$$\begin{cases} \left(1-\frac{2}{u}\right)^{-1} = 1+\frac{2}{u}+\frac{4}{u^2}+\frac{8}{u^3}+\cdots \\ \left(1-\frac{3}{u}\right)^{-1} = 1+\frac{3}{u}+\frac{9}{u^2}+\frac{27}{u^3}+\cdots \end{cases}$$

$$f(u) = \frac{1}{2u}+\frac{1}{2u^2}+\frac{1}{2u^3}+\frac{1}{2u^4}+\cdots$$

$$-\frac{1}{u}-\frac{2}{u^2}-\frac{4}{u^3}-\frac{8}{u^4}+\cdots$$

$$\frac{3}{2u}+\frac{9}{2u^2}+\frac{27}{2u^3}+\frac{81}{2u^3}+\cdots$$

27 3^2 81

$$= \frac{1}{u}\left(\frac{1}{2}-1+\frac{3}{2}\right)+\frac{1}{u^2}\left(\frac{1}{2}-2+\frac{9}{2}\right)+\frac{1}{u^3}\left(\frac{1}{2}-4+\frac{27}{2}\right)+\frac{1}{u^4}\left(\frac{1}{2}-8+\frac{81}{2}\right)$$

$$= \frac{1}{u}+\frac{3}{u^2}+\frac{10}{u^3}+\frac{33}{u^4}+\cdots$$

$\frac{28}{2}$ 14 $-\frac{4}{8}$

$$f(2) = \frac{1}{2+1}+\frac{3}{(2+1)^2}+\frac{10}{(2+1)^3}+\frac{33}{(2+1)^4}+\cdots$$

41 -8 33 $\quad \frac{82}{2}$ 41

3a) Let f be entire function with Taylor series expansion with centre $z=0$ has infinitly many terms. Show $z=0$ is essential singularity of $f(\frac{1}{z})$ [15]

Taylor series expansion of $f(z)$ at $z=a$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad a_n = \frac{f^n(a)}{n!}$$

at $z=0 \rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

$$f(\tfrac{1}{z}) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3}$$

Clearly $z=0$ is essential singular for $f(\frac{1}{z})$ as it contains infinite number of terms with negative power of z

As per qn, this is my explanation.

If you have any other explanation, let me know.

(36) Find stationary values of $x^2+y^2+z^2$ s.t $ax^2+by^2+cz^2=1$ & $lx+my+nz=0$. Interpret geometrically [20]

SuccessClap Question Bank SC-B07 Qn 1

$u=x^2+y^2+z^2$ s.t $ax^2+by^2+cz^2=1$ & $lx+my+nz=0$

$du=2x\,dx+2y\,dy+2z\,dz$ $\quad 2ax\,dx+2by\,dy+2cz\,dz=0$

$l\,dx+m\,dy+n\,dz=0$

Multiply $1, \lambda_1, \lambda_2$

$(x\,dx+y\,dy+z\,dz)+\lambda_1(ax\,dx+by\,dy+cz\,dz)+\lambda_2(l\,dx+m\,dy+n\,dz)=0$

$$\left.\begin{array}{l} x+a\lambda_1 x+l\lambda_2=0 \\ y+b\lambda_1 y+m\lambda_2=0 \\ z+c\lambda_1 z+n\lambda_2=0 \end{array}\right\}$$ Multiply by x, y, z & adding

$(x^2+y^2+z^2)+\lambda_1(ax^2+by^2+cz^2)+\lambda_2(lx+my+nz)=0$

$u+\lambda_1(1)+\lambda_2(0)=0 \Rightarrow u+\lambda_1=0 \quad \lambda_1=-u$

$$\left.\begin{array}{ll} x-axu+l\lambda_2=0 & \Rightarrow x=\dfrac{l\lambda_2}{au-1} \\ y-byu+m\lambda_2=0 & \Rightarrow y=\dfrac{m\lambda_2}{bu-1} \\ z-czu+n\lambda_2=0 & \Rightarrow z=\dfrac{n\lambda_2}{cu-1} \end{array}\right\}$$ Put values

$\dfrac{l^2}{au-1}+\dfrac{m^2}{bu-1}+\dfrac{n^2}{cu-1}=0$ gives max/Min of u

(39) Convert LPP to dual LPP

Minimize $Z = x_1 - 3x_2 - 2x_3$

s.t $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 - 4x_2 \ge 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \ge 0$

x_3 is unrestricted

(15)

Multiply Eq ① by $-1 \Rightarrow -3x_1 + x_2 - 2x_3 \ge -7$

Eqn 3 $\rightarrow = \rightarrow \ge$ & $\le$

$-4x_1 + 3x_2 + 8x_3 \ge 10$

$-4x_1 + 3x_2 + 8x_3 \le 10 \Rightarrow 4x_1 - 3x_2 - 8x_3 \ge -10$

$x_3 = x_3' - x_3''$ s.t $x_3', x_3'' \ge 0$

Min $Z = x_1 - 3x_2 - 2x_3' + 2x_3''$

s.t $-3x_1 + x_2 - 2x_3' + 2x_3'' \ge -7$

$2x_1 - 4x_2 \ge 12$

$-4x_1 + 3x_2 + 8x_3' - 8x_3'' \ge 10$

$4x_1 - 3x_2 - 8x_3' + 8x_3'' \ge -10$

$x_1, x_2, x_3', x_3'' \ge 0$

Its Dual

Max $W = -7y_1 + 12y_2 - 10y_3 + 10y_4$

s.t $-3y_1 + 2y_2 + 4y_3 - 4y_4 \le 1$

$y_1 - 4y_2 - 3y_3 + 3y_4 \le -3$

$-2y_1 + 0y_2 - 8y_3 + 8y_4 \le -2$

$2y_1 + 0y_2 + 8y_3 - 8y_4 \le 2$

$y_1, y_2, y_3, y_4 \ge 0$

$y_3 - y_4 = y_1$

$$\text{Max } W = -7y_1 + 12y_2 + 10y_1$$

$$\text{s.t} \quad -3y_1 + 2y_2 + 4y_1 \leq 1$$

$$y_1 - 4y_2 - 3y_1 \leq -3$$

$$-2y_1 + 8y_1 = -2$$

$y_1, y_2 \geq 0$

y_1 is unrestricted

(4a) Show there are infinitely many subgroups of additive group $\mathbb{Q}$ of rational numbers [15]

Consider set of rational numbers of form $1/p$ where p is prime so $1/2, 1/3, 1/5, 1/7$

Now consider any subset of these. Using these as generators we can form a corresponding additive subgroups. So if for example we choose $1/2$ & $1/3$, we will have a subgroup that include $\{1/2, 1/3, 1/2 + 1/3, \frac{1}{2} + \frac{1}{3} + \frac{1}{3}\}$ etc as members.

So every subset of $\{1/p\}$ generates a corresponding subgroups. As $\{1/p\}$ is countably infinite, it has an uncountable number of subsets.

Now we have to do is prove that the subgroups they generate are all different.

If some subgroup generated by a subset of $\{1/2, 1/3, 1/5, 1/7, 1/11 \ldots\}$ has the property that some element e has the property that $p * e = 1$, then $1/p$ must be an element of subgroup. So if we compare two subgroups, they can only be isomorphic if they are generated by the same subset of $\{1/2, 1/3, 1/5 \ldots\}$ and different choices of subset will generate different additive subgroups.

As there is an uncountable number of additive subgroups formed in this manner, all unique, there must be uncountable number of them.

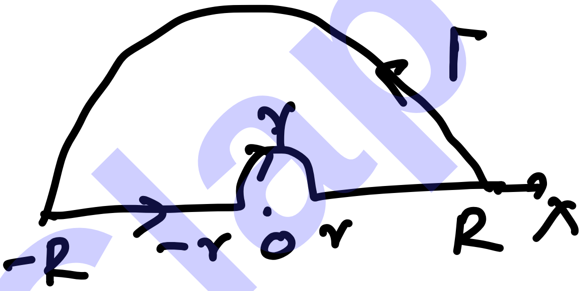
Source: Quora

(4b) Use Contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)} \quad a>0 \qquad [20]$$

SuccessClap : Question bank SC-HOS QnSS

Consider $\int_C f(z) = \int_C \frac{e^{iz}}{z(z^2+a^2)}dz$



Simple poles
$z=0$ lie outside
$z=ai$ lie inside
$z=-ai$ lie outside

$$\text{Residue}(z=ai) = \underset{z\to ai}{Lt}(z-ai)f(z) = \underset{z\to ai}{Lt}\frac{e^{iz}}{z(z+ai)}$$
$$= \frac{e^{-a}}{-2a^2}$$

Residue Thm

$$\int_C f(z)\,dz = 2\pi i \sum R^+ = \int_r^R + \int_\Gamma + \int_{-R}^{-r} + \int_\gamma$$

Jordan lemma : $\underset{R\to\infty}{Lt}\int_\Gamma f(z)\,dz = 0$

$$\underset{z\to 0}{Lt}\, z f(z) = \underset{z\to a}{Lt}\frac{e^{iz}}{z^2+a^2} = \frac{1}{a^2}$$

$$\Rightarrow \lim_{r\to 0} \int_{\gamma} f(z)\,dz = i \cdot \frac{1}{a^2}(0-\pi) = \frac{-\pi i}{a^2}$$

$r \to 0,\ R \to \infty \Rightarrow$

$$\int_0^{\infty} f(x)\,dx + \int_{-\infty}^{0} f(x)\,dx - \frac{\pi i}{a^2} = 2\pi i \left(\frac{-e^{-a}}{2a^2}\right)$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{i\pi}{a^2}\left(1-e^{-a}\right)$$

Imaging part equate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}\left(1-e^{-a}\right)$$

(4) Solve by Big M

Maximize $Z = 4x_1 + 5x_2 + 2x_3$

s.t $2x_1 + x_2 + x_3 \geq 10$

$x_1 + 3x_2 + x_3 \leq 12$

$x_1 + x_2 + x_3 = 6$

[15]

Maximize $Z = 4x_1 + 5x_2 + 2x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

s.t $2x_1 + x_2 + x_3 - S_1 + 0S_2 + A_1 + 0A_2 = 10$

$x_1 + 3x_2 + x_3 + 0S_1 + S_2 + 0A_1 + 0A_2 = 12$

$x_1 + x_2 + x_3 + 0S_1 + 0S_2 + 0A_1 + A_2 = 6$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

S_1 is surplus variable S_2 is slack variable

IBFS $x_1 = x_2 = x_3 = S_1 = 0$ $A_1 = 10$ $A_2 = 6$ $S_2 = 12$

C_j		4	5	2	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	A_2	b	θ
$-M$	A_1	$\boxed{2}$	1	1	-1	0	1	0	10	5 →
0	S_2	1	3	1	0	1	0	0	12	12
$-M$	A_2	1	1	1	0	0	0	1	6	6
$Z_j = \Sigma C_B A_j$		$-3M$	$-2M$	$-2M$	M	0	$-M$	$-M$	$-16M$	
$C_j = C_j - Z_j$		$3M+4$ ↑	$2M+5$	$2M+2$	$-M$	0	0	0		

	C_j	4	5	2	0	0	$-M$		
CB	Basis	x_1	x_2	x_3	S_1	S_2	A_2	b	θ
4	x_1	1	$1/2$	$1/2$	$-1/2$	0	0	5	10
0	S_2	0	$5/2$	$1/2$	$1/2$	1	0	7	2.8
$-M$	A_2	0	$1/2$	$1/2$	$1/2$	0	1	1	2 →
	$Z_j = \Sigma C_B a_{ij}$	4	$2-\frac{M}{2}$	$2-\frac{M}{2}$	$-2-\frac{M}{2}$	0	$-M$	$20-M$	
	$C_j = c_j - Z_j$	0	$3+\frac{M}{2}$ ↑	$\frac{M}{2}$	$2+\frac{M}{2}$	0	0		

	C_j	4	5	2	0	0	
CB	Basis	x_1	x_2	x_3	S_1	S_2	b
4	x_1	1	0	0	-1	0	4
0	S_2	0	0	-2	-2	1	2
5	x_2	0	1	1	1	0	2
	$Z_j = \Sigma C_B a_{ij}$	4	5	5	1	0	26
	$C_j = c_j - Z_j$	0	0	-3	-1		

Optimal BFS $x_1 = 4$ $x_2 = 2$ $x_3 = 0$

$\max Z = 26$

Section B

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(5a) Obtain PDE by eliminating f, from [10]

$f(x+y+z,\ x^2+y^2+z^2)=0$

SuccessClap Question Bank SC-I-01 Qn 10

→ Several methods to solve

→ use direct formula to solve such problems

FORMULA

For $f(u,v)=0$ → Its PDE is $Pp+Qq=R$

$$P=\frac{\partial(u,v)}{\partial(y,z)} \qquad Q=\frac{\partial(u,v)}{\partial(z,x)} \qquad R=\frac{\partial(u,v)}{\partial(x,y)}$$

$u=x+y+z \qquad v=x^2+y^2+z^2$

$$P=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2y & 2z\end{vmatrix}=2z-2y$$

$$Q=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial u}{\partial x}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2z & 2x\end{vmatrix}=2x-2z$$

$$R=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2x & 2y\end{vmatrix}=2y-2x$$

Soln is $Pp+Qq=R$

$$(z-y)p+(x-z)q=y-x$$

(5b) Find positive root of $3x = 1 + \cos x$ with initial values $0, \pi/2$. & improve result by Newton-Raphson correct to 8 significant figures [10]

$f(x) = 3x - 1 - \cos x$

↳ Given two initial values $(0, \pi/2)$ → use Bisection method

x_0	x_1	$x_2 = \frac{x_0 + x_1}{2}$	$f(x_2)$
0	$\pi/2$	$\pi/4$	> 0
0	$\pi/4$	$\pi/8$	< 0
$\pi/8$	$\pi/4$	$3\pi/16$	< 0
$\frac{3\pi}{16}$	$\pi/4$	0.68722	> 0
$\frac{3\pi}{16}$	0.687	0.638136	> 0

Soln is 0.6381

Newton Raphson

$f'(x) = 3 + \sin x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$x_0 = 0.60$

$x_1 = 0.60701$

$x_2 = 0.60710$

(5c) (i) Convert $(3798.3875)_{10}$ to octal & hexadecimal equivalents

(ii) Obtain principal conjunctive normal form of $(\neg P \to R) \wedge (Q \rightleftarrows R)$

(i)

$$\begin{array}{r|l l}
8 & 3798 & \\
8 & 474 & 6 \\
8 & 59 & 2 \\
 & 7 & 3
\end{array}$$

$0.3875 \times 8 = 3.1$ 3

$0.1 \times 8 = 0.8$ 0

$0.8 \times 8 = 6.4$ 6

$0.4 \times 8 = 3.2$ 3

7326   3063

$(3798.3875)_{10} = (7326.3063)_8$

Hexa

$$\begin{array}{r|l}
16 & 3798 \\
16 & 237 - 6 \\
 & 14 - D
\end{array}$$

$0.3875 \times 16 = 6.2$ 6

$0.2 \times 16 = 3.2$ 3

$0.2 \times 16 = 3.2$ 3

633

$14 \equiv E$

(ED6)

$(3798.3875)_{10} = (ED6.633)_{16}$

(ii) Obtain principal conjunctive normal form of $(\sim P \to R) \wedge (Q \rightleftarrows P)$ [10]

$A \to B \equiv \sim A \vee B$

$A \rightleftarrows B \equiv (\sim A \vee B) \wedge (A \vee \sim B)$

$(\sim(\sim P) \vee R) \wedge (\sim Q \vee P) \wedge (Q \vee \sim P)$

$(P \vee R) \wedge (P \vee \sim Q) \wedge (\sim P \vee Q)$

$[P \vee R \vee (Q \wedge \sim Q)] \wedge [P \vee \sim Q \vee (R \wedge \sim R)]$

$\wedge [\sim P \vee Q \vee [R \wedge \sim R]]$

add $Q \wedge \sim Q$ will not chge expression

$(P \vee R \vee Q) \wedge (P \vee R \vee \sim Q) \wedge (P \vee \sim R \vee R)$

$\wedge (P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge$

$(\sim P \vee Q \vee \sim R)$

$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ De-Morgan

$(P \vee R \vee Q) \wedge (P \vee \sim Q \vee \sim R) \wedge (P \vee \sim Q \vee R) \wedge$

$(\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$ $\quad A \wedge A = A$

$(P \vee Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee \sim R) \wedge$

$(\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$

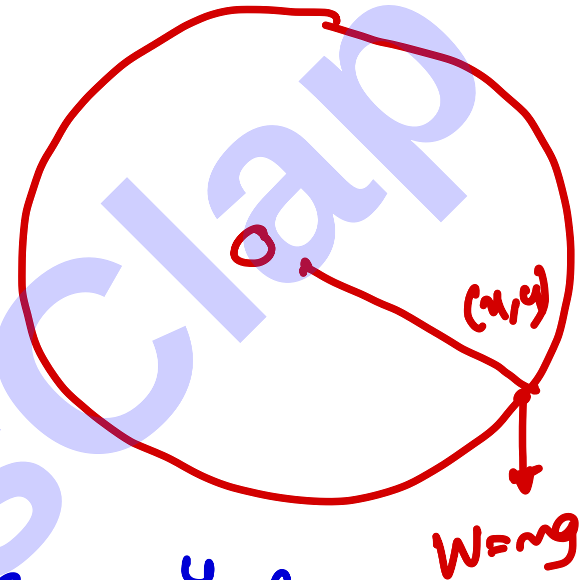
This is Principal Conjuctive normal form

5d) A particle is constrained to move along a circle lying in vertical xy-plane. Using D-Alembert show equation of motion is [10]

$\ddot{x}y - \ddot{y}x - gx = 0$ g is gravity

Particle mass m move in circle radius r in xy plane

O is centre



Constraints $x^2+y^2=r^2$

$$2x\,\delta x + 2y\,\delta y = 0 \Rightarrow \delta x = \frac{-y}{x}\delta y$$

D-Alembert $(F - m\ddot{r})\,\delta r = 0$

Component form

$$(F_x - m\ddot{x})\,\delta x + (F_y - m\ddot{y})\,\delta y = 0$$

$F_x = 0 \quad F_y = -mg$

$$-m\ddot{x}\,\delta x - (mg + m\ddot{y})\,\delta y = 0$$

$$m(-\ddot{x}y + \ddot{y}x + gx)\,\delta x = 0$$

$$\ddot{x}y - \ddot{y}x - gx = 0$$

(5e) What arrangements of source & sinks can have velocity potential $w = \log_e\left(z - \frac{a^2}{z}\right)$? Draw sketch of streamlines & prove that two of them subdivide into circle $r = a$ & axis of y. [10]

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① $w = \log\left(z - \frac{a^2}{z}\right) = \log\left[\frac{(z-a)(z+a)}{z}\right]$

$= \log(z-a) + \log(z+a) - \log z$ $\qquad z = x+iy$

$= \log((x-a)+iy) + \log((x+a)+iy) - \log(x+iy)$

$= \phi + i\psi$

$\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$

Imaginary part

$\psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$

$= \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}$

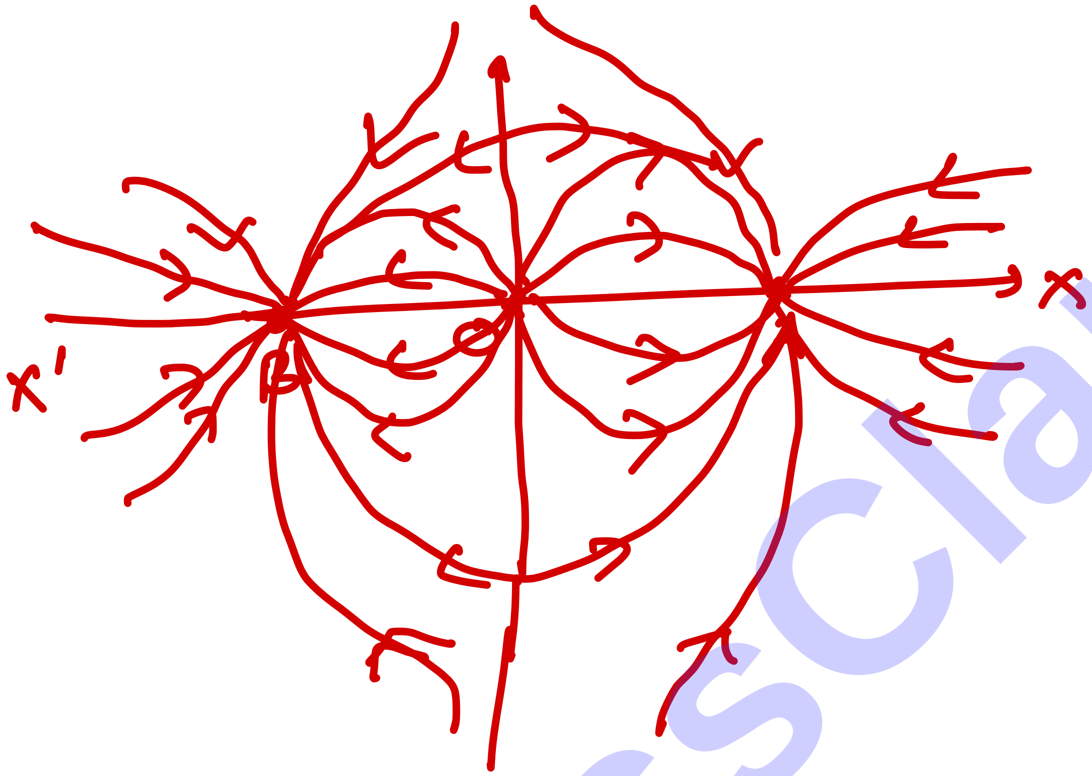
Streamline $\psi = \text{constant} \Rightarrow \frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = C$

$C = 0 \Rightarrow y = 0 \Rightarrow$ x-axis is streamline

$C \to \infty \Rightarrow (x^2+y^2-a^2)x = 0$

$\Rightarrow x=0$

& $x^2+y^2=a^2$ | $r=a$ streamline



[6a] Solve wave eqn $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$ $\quad 0<x<L$, $t>0$

s.t $\quad u(0,t)=0 \quad u(L,t)=0$

$u(x,0)=\frac{1}{4}x(L-x) \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0$

[20]

wave eqn $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$

↳ Soln is $\quad u(x,t)=\sum B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$

$t=0 \quad \frac{1}{4}x(L-x)=\sum B_n \sin\frac{n\pi x}{L}$

$$B_n=\frac{2}{L}\int_0^L \frac{1}{4}x(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$=\left(\frac{1}{2L}\right)\int_0^L (Lx-x^2)\sin\frac{n\pi x}{L}\,dx$$

$$=\left(\frac{1}{2L}\right)\left[(Lx-x^2)\left\{\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right\}-(L-2x)\left\{\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right\}\right.$$
$$\left.(-2)\left\{\frac{\cos\frac{n\pi x}{L}}{n^3\pi^3/L^3}\right\}\right|_0^L$$

$$=\left(\frac{1}{2L}\right)\left[-\frac{L}{n\pi}(Lx-x^2)\cos\frac{n\pi x}{L}+\frac{L^2}{n^2\pi^2}(L-2x)\sin\frac{n\pi x}{L}\right.$$
$$\left.-\frac{2L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$= \left(\frac{1}{2L}\right)\left[\frac{-L}{n\pi}\left(L^2 - L^2\right)\cos n\pi + \frac{L^2}{n^2\pi^2}(L-2L)\sin n\pi - \frac{2L^3}{n^3\pi^3}\cos n\pi - \left(0 - \frac{2L^3}{n^3\pi^3}\cos 0\right)\right]$$

$$= \left(\frac{1}{2L}\right)\left[\frac{2L^3}{n^3\pi^3} - \frac{2L^3}{n^3\pi^3}\cos n\pi\right]$$

$\cos n\pi = (-1)^n$

$$= \frac{L^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

n is odd $(-1)^n = -1$ $\qquad B_n = \dfrac{2L^2}{n^3\pi^3}$

n is even $(-1)^n = 0$ $\qquad B_n = 0$

$\Big\}$ only for odd value exist

$$u(x,t) = \frac{2L^2}{\pi^3}\sum_{n=1,3,5,7} \frac{1}{n^3}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

(6b) Obtain Boolean Fn $F(x,y,z)$. Simplify & draw GATE network [15]

x	y	z	F(x,y,z)
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

$$F(x,y,z) = xyz + xyz' + xy'z + x'yz$$

$$= xy(z+z') + xy'z + x'yz$$

$$= xy + xy'z + x'yz$$

$$= x(y+y'z) + x'yz$$

$(A + \bar{A}B = A+B)$

$$= x(y+z) + x'yz$$

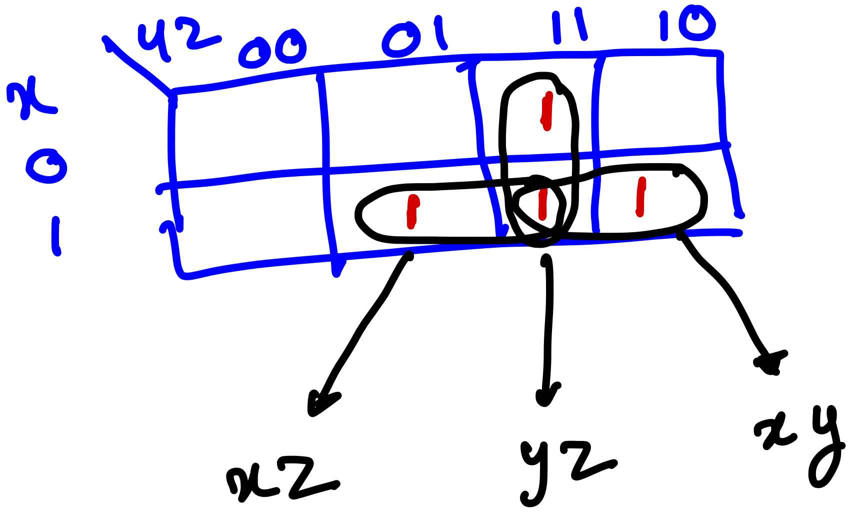
$$= xy + xz + x'yz$$

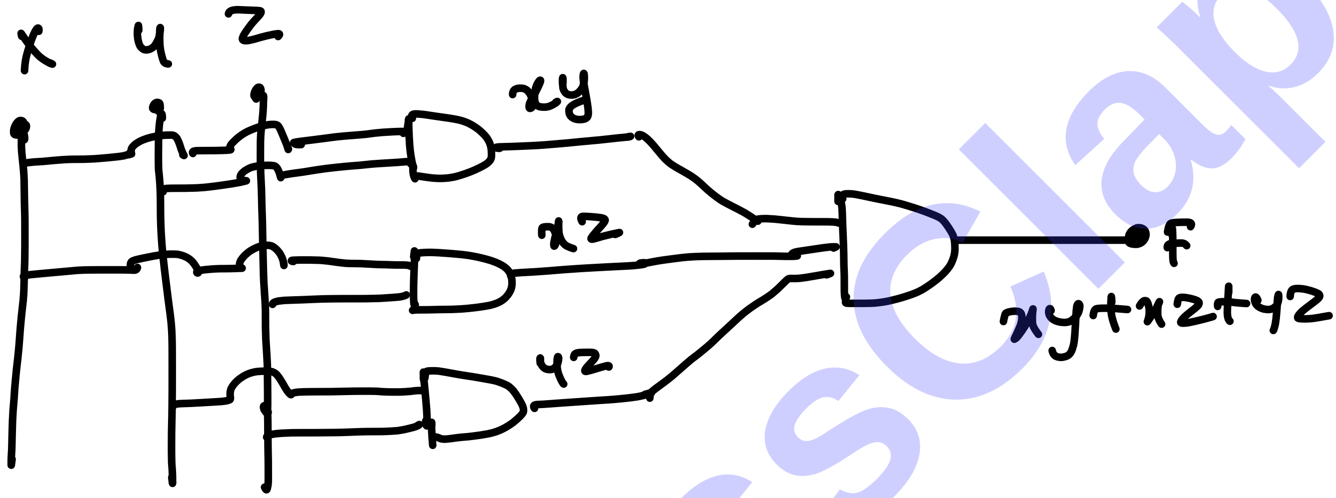
$$= xy + z(x+x'y)$$

$$= xy + z(x+y)$$

$$= xy + xz + yz$$

K-maps :

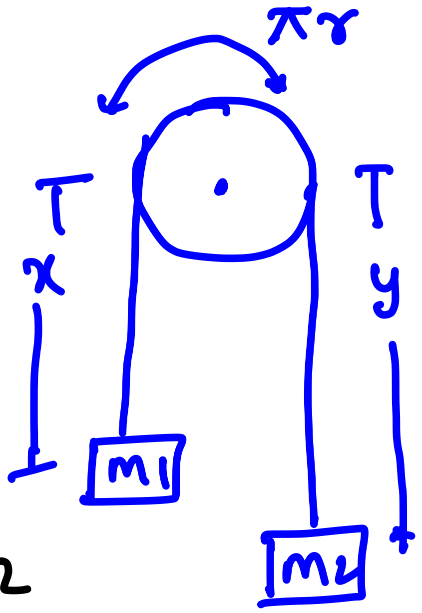




6(c) Obtain lagrange for two particles of unequal masses connected by inextensible string passing through small smooth pulley [15]

Video Soln is present in Mechanics Course of SuccessClap. Checkout in SuccessClap Youtube videos

→ Note given small smooth pulley, so pulley related mass, Moment of Inertia can be ignored (Not given)



→ Length of string

$L = \pi r + x + y$

$\dot{L} = \dot{x} + \dot{y} = 0 \Rightarrow \dot{x} = -\dot{y}$

$T_{m_1} = \frac{1}{2} m_1 \dot{x}^2 \qquad T_{m_2} = \frac{1}{2} m_2 \dot{y}^2$

$V_{m_1} = -mgx \qquad V_{m_2} = -mgy$

$T = T_{m_1} + T_{m_2} = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 \qquad \dot{y} = -\dot{x}$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2$$

$$L = \pi r + x + y$$

$$V = V_{m_1} + V_{m_2} = -m_1 g x - m_2 g y \qquad y = L - \pi r - x$$

$$= -m_1 g x - m_2 g L + m_2 g \pi r + m_2 g x$$

$$= (m_2 - m_1) g x - m_2 g L + m_2 g \pi r$$

$$L = T - V$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 - (m_2-m_1) g x + m_2 g L - m_2 g \pi r$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} \qquad \frac{\partial L}{\partial x} = -(m_2-m_1) g$$

L-eqn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$(m_1+m_2)\ddot{x} = (m_2-m_1) g$$

$$\ddot{x} = \frac{(m_2-m_1) g}{m_1+m_2}$$

(7a) Find General Soln of

$$(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$$ [15]

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$$D^2 - D'^2 - 3D + D' = (D+D')(D-D') - 3(D-D')$$

$$= (D+D'-3)(D-D')$$

$$(D-D')(D+D'-3)z = xy + e^{x+2y}$$

$$\hookrightarrow CF = \phi_1(y+x) + e^{3x}\phi_2(y-x)$$

$$PI_1 = \frac{1}{(D-D')(D+D'-3)} xy = \frac{-1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)^{-1} xy$$

$$= \frac{-1}{3D}\left(1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right)\left(1+\frac{D+D'}{3}+\frac{2DD'}{9}+\cdots\right)xy$$

$$= \frac{-1}{3D}\left(1+\frac{D}{3}+\frac{D'}{3}+\frac{D'}{D}+\frac{D'}{3}+\frac{2DD'}{9}+\cdots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{y}{3}+\frac{2x}{3}+\frac{1}{D}x+\frac{2}{9}\right)$$

$$= -\frac{1}{9}\left(\frac{x^2y}{2}+\frac{xy}{2}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right)$$

$$PI_2 = \frac{1}{(D+D'-3)(D-D')} e^{x+2y} = \frac{1}{(D+D'-3)(1-2)} e^{x+2y}$$

$$= -\frac{1}{D+D'-3} e^{x+2y} = -e^{1.x+2.y} \frac{1}{(D+1)+(D'+2)-3}$$

$$= -e^{x+2y} \frac{1}{D+D'} . 1$$

$$= -e^{x+2y} \frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1} 1$$

$$= -e^{x+2y} \frac{1}{D}(1+ \cdots) 1 = -xe^{x+2y}$$

$$z = CF + PI$$

7b) Use Gauss Seidel Solve

$$3x_1+9x_2-2x_3=11$$
$$4x_1+2x_2+13x_3=24$$
$$4x_1-2x_2+x_3=-8$$

(15)

Rearrange

$$4x_1-2x_2+x_3=-8$$
$$3x_1+9x_2-2x_3=11$$
$$4x_1+2x_2+13x_3=24$$

diagonal term high

$$x_1^{k+1}=\frac{1}{4}\left(-8+2x_2^{k}-x_3^{k}\right)$$
$$x_2^{k+1}=\frac{1}{9}\left(11-3x_1^{k+1}+2x_3^{k}\right)$$
$$x_3^{k+1}=\frac{1}{13}\left(24-4x_1^{k+1}-2x_2^{k+1}\right)$$

Iterate	x_1	x_2	x_3
Initial	0	0	0
1	-2	1.8888	2.17094
2	-1.59829	2.2374	1.993717
3	-1.37972	2.12517	1.94373
4	-1.4233	2.1286	1.9566
5	-1.4248	2.13197	1.95657
6	-1.4231	2.1314	1.9561

(7c) Show $q = \frac{\lambda(-y\hat{i} + x\hat{j})}{x^2+y^2}$ (λ = const) is possible incompressible fluid motion.
Determine streamlines.
Is kind of motion potential? If yes, find velocity potential.

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$q = u\hat{i} + v\hat{j} + w\hat{k}$ $\quad u = \frac{-\lambda y}{x^2+y^2}$ $\quad v = \frac{\lambda x}{x^2+y^2}$ $\quad w=0$

Eqn of Continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{2\lambda xy}{(x^2+y^2)^2}$ $\quad \frac{\partial v}{\partial y} = \frac{-2\lambda xy}{(x^2+y^2)^2}$ $\quad \frac{\partial w}{\partial z} = 0$

$\Rightarrow$ Continuity Eqn is satisfied $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$\Rightarrow$ Possible motion

Streamline $\frac{dx}{-\lambda y/x^2+y^2} = \frac{dy}{\lambda x/x^2+y^2} = \frac{dz}{0}$

$\Rightarrow \frac{dz}{0} \Rightarrow z = c_1$ $\quad dx/-y = \frac{dy}{x}$ $\Rightarrow x^2+y^2 = c_2$

Streamline eqns $z = c_1$
$x^2+y^2 = c_2$

$$\nabla \times q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-\lambda y}{x^2+y^2} & \frac{\lambda x}{x^2+y^2} & 0 \end{vmatrix} = \lambda \left| \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \right| = 0$$

Flow is potential kind

$$q = -\nabla \varphi$$

$$\frac{\partial \varphi}{\partial x} = -u = \frac{\lambda y}{x^2+y^2}$$

$$\frac{\partial \varphi}{\partial y} = -v = \frac{-\lambda x}{x^2+y^2}$$

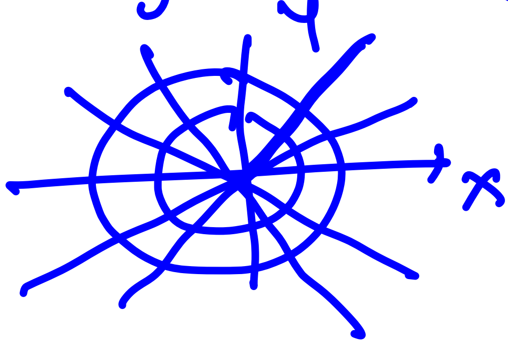
$$\frac{\partial \varphi}{\partial z} = 0$$

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} = 0$$

$$= \lambda \left(\frac{y\,dx - x\,dy}{x^2+y^2} \right) = \lambda\, d \tan^{-1} \frac{x}{y}$$

$$\varphi = \lambda \tan^{-1} \frac{x}{y}$$

Equipotential $\varphi = \text{constant} \Rightarrow \frac{x}{y} = c \Rightarrow x = cy$



(8b) Derive Newton backward difference interpolation formula & also do error analysis [15]

Note:

① Plz see the checklist of all Derivations and Error Analysis given in SuccessClap website for Numerical Analysis

② SuccessClap Question Bank SC-L02 Qn 2

x	a	$a+h$	$a+nh$
$f(x)$	$f(a)$	$f(a+h)$	$f(a+nh)$

Let $f(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)(x-a-(n-1)h) + A_3(a-x-nh)(x-a-(n-1)h)(x-a(n-2)h) + \dots A_n(x-a-nh)(x-a-(n-1)h)\dots(x-a-h)$

$A_0, A_1, A_2 \dots A_n$ cost

$\rightarrow x = a+nh \Rightarrow f(a+nh) = A_0$

$\rightarrow x = a+(n-1)h \Rightarrow f(a+(n-1)h) = A_0 + A_1[a+(n+1)h - a - nh]$

$= A_0 - A_1 h$

$A_1 h = A_0 - f(a+(n+1)h)$

$= f(a+nh) - f(a+(n-1)h)$

$$= \nabla f(a+nh)$$

$$A_1 = \frac{\nabla f(a+nh)}{h}$$

$\Rightarrow x = a + (n-2)h$

$$f(a+(n-2)h) = A_0 + A_1[a+(n-2)h - a - nh] + A_2[a+(n-2)h - a - nh][a+(n-2)h - a - (n-1)h]$$

$$= A_0 + A_1(-2h) + A_2(-2h)(-h)$$

$$A_2 2h^2 = f(a+(n-2)h) - A_0 + 2A_1 h$$

$$= f(a+(n-2)h) - f(a+nh) + 2\nabla f(a+nh)$$

$$= f(a+(n-2)h) - f(a+nh) + 2\{f(a+nh) - f(a+(n-1)h)\}$$

$$= \{f(a+(n-2)h - f(a+(n-1)h)\} - \{f(a+(n-1)h) - f(a+nh)\}$$

$$= \nabla f[a+(n-1)h] - \nabla f(a+nh)$$

$$= \nabla^2 f(a+nh)$$

$$A_2 = \frac{\nabla^2 f(a+nh)}{h^2 \cdot 2!}$$

Similarly $\quad A_3 = \dfrac{\nabla^3 f(a+nh)}{h^3 3!} \qquad A_n = \dfrac{\nabla^n f(a+nh)}{h^n n!}$

Also → Let $u = \dfrac{x-(a-nh)}{h}$

$\Rightarrow x-a-(n-1)h = uh+h = (u+1)h$

$x-a-h = (a+nh+uh) - a - h = (u+n-1)h$

$$f(x) = f(a+nh) + (x-a-nh)\frac{\nabla f(a+nh)}{h \cdot 1!} + (x-a-nh)(x-a(n-1)h\frac{\nabla^2 f(a+nh)}{2!\,h^2} + (x-a-nh)(x-a(n-1)h)\ldots(x-a-h)\frac{\nabla^n f(a+nh)}{h^n n!}$$

↗ Put in u → format

$$f(a+nh+uh) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \cdots \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(a+nh)$$

Reminder term R_n

$$R_n = u(u+1)\ldots(u+n)\frac{h^{n+1}}{(n+1)!}f^{n+1}(\theta)$$

Since $f(x)$ cannot be expressed as $f^{n+1}(\theta)$ $\quad u = \dfrac{x-x_n}{h}$

$$R_n = \frac{\nabla^{n+1} f(x_n)}{(n+1)!}u(u+1)(u+2)\ldots(u+n)$$

(8c) Show that complex potential $\tan^{-1} z$, the streamlines and equipotential curves are circles. Find velocity at any pt & check singularities at $z = \pm i$ [20]

$$w = \tan^{-1} z = \phi + i\psi$$
$$\bar{w} = \tan^{-1} \bar{z} = \phi - i\psi$$

Subtract $\quad 2i\psi = \tan^{-1} z - \tan^{-1} \bar{z}$

$$= \tan^{-1} \frac{z - \bar{z}}{1 + z\bar{z}}$$

$z = x + iy$

$\bar{z} = x - iy$

$$= \tan^{-1} \frac{2iy}{1 + x^2 + y^2}$$

$$\tan 2i\psi = \frac{2iy}{1 + x^2 + y^2}$$

$$x^2 + y^2 + 1 = 2y \coth 2\psi$$

Streamlines $\quad \psi = \text{constant} \quad$ Let $\psi = c_1$

Eqn $\quad x^2 + y^2 + 1 = 2y \coth 2c_1$

$x^2 + y^2 = 2y \coth 2c - 1 \longrightarrow$ Circle

Similarly $w+\bar{w} = 2\phi = \tan^{-1} z + \tan^{-1} \bar{z}$

$$= \tan^{-1} \frac{z+\bar{z}}{1-z\bar{z}}$$

$$= \tan^{-1} \frac{2x}{1-(x^2+y^2)}$$

$z = x+iy$
$\bar{z} = x-iy$

$$\tan 2\phi = \frac{2x}{1-(x^2+y^2)}$$

$$1-(x^2+y^2) = 2x \cot 2\phi$$

Equipotential $\phi = \text{Constant}$ Let $\phi = C_2$

Eqn $1-(x^2+y^2) = 2x \cot 2C_2$

$$x^2+y^2 = 1-2x\cot 2C_2 \longrightarrow \text{Circle}$$

velocity $q = \frac{dw}{dz} = \frac{1}{1+z^2}$ $\quad w = \tan^{-1} z$

Denominator $1+z^2$ vanish at $z = \pm i$

↳ Singularity at $z = \pm i$

(8b) Find complete Integral of $p=(z+qy)^2$ by Charpit (15 Marks)

→ M.D. Raisinghania unsolved question

→ SuccessClap question Bank - similar qn $q=(z+px)^2$
SC-IO2-Orthogonal- Qn. No 16

$$f=(z+qy)^2-p$$

$f_x=0$ $\quad f_y=2q(z+qy)$ $\quad f_z=2(z+qy)$ $\quad f_p=-1$

$f_q=2(z+qy)y$ $\quad f_z=2(z+qy)$

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$$

$$\frac{dx}{-1}=\frac{dy}{2y(z+qy)}=\frac{dz}{2qy(z+qy)-p}=\frac{dp}{-2p(z+qy)}=\frac{dq}{-4q(z+qy)}$$

$$\frac{dy}{y}=\frac{dp}{-p} \Rightarrow py=a \quad p=\frac{a}{y}$$

Put $p=\frac{a}{y}$ in $p=(z+qy)^2 \Rightarrow \frac{a}{y}=(z+qy)^2$

$$q=\frac{\sqrt{a}}{y^{3/2}}-\frac{z}{y}$$

$$dz=pdx+qdy$$

$$=\frac{a}{y}dx+\frac{\sqrt{a}}{y^{3/2}}dy-\frac{z}{y}dy$$

$$ydz+zdy=adx+\sqrt{a}\,y^{-1/2}dy$$

$\downarrow$

$d(yz)$ $\quad$ ∫ Integr

$$\boxed{yz=ax+2\sqrt{ay}+b}$$