On the stochastic flow of harmonic maps

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The Model

The domain $D \subseteq \mathbb{R}^3$ is **bounded**, $\dim D = 2$.

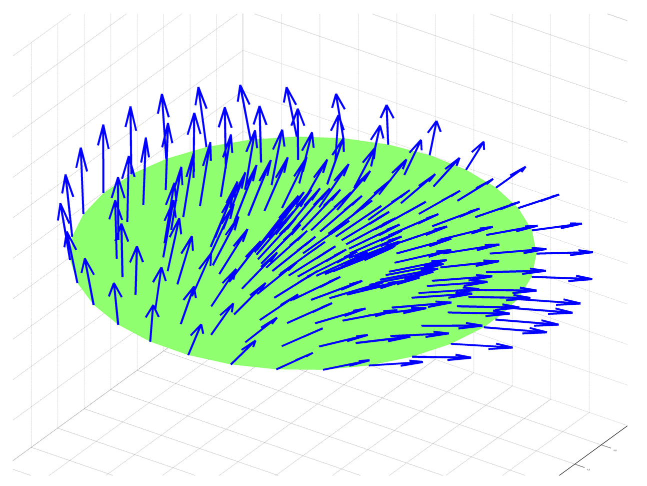


Figure 1: Example : $D =$ the unit disk

The field $u(t,x)$ has to respect the pointwise norm constraint

$$\boxed{|u(t,x)| = 1 \ , \ \text{for all } (t,x) \in \mathbb{R}^+ \times D \ .}$$

It tends to align to Δu to minimize the **Dirichlet energy**

$$E := \int_D |\nabla u|^2 dx \ .$$

The **flow of harmonic** maps writes :

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u|\nabla u|^2 \ , & (t,x) \in \mathbb{R}^+ \times D \ , \\ u(t=0,\cdot) = u_0 \ , \in H^1 & x \in D \ , \\ u(t,\cdot)_{|\partial D} = u_{0|\partial D} \ , & t \in \mathbb{R}^+ \ . \end{cases} \tag{1}$$

If u is a steady-state of (1) then u is a **harmonic map** between the two manifolds D and $\mathbb{S}^2$. It says that u_0 has an harmonic map in its homotopy class.

A new numerical scheme

Notation : Time step : $\Delta t := \frac{T}{N}$, $N \in \mathbb{N}^*$. Consider a subspace of finite elements $V_h \subseteq H^1$.

Algorithm [4] : *Fix $u^0 := u_0 \in V_h$ and for any $n \in \{0, \ldots, N-1\}$, and denote*

$$\mathbb{W}(u^n) := \{\psi \in V_h, \ \forall x \in D, \ \psi(x) \perp u^n(x)\} \ .$$

Let $\theta \in (\frac{1}{2}, 1]$, and $v^n \in \mathbb{W}(u^n)$ be the unique solution to the following variational problem: $\forall \varphi \in \mathbb{W}(u^n)$,

$$\langle v^n, \varphi \rangle_{L^2} + 2\theta\Delta t \langle \nabla v^n, \nabla \varphi \rangle_{L^2} = -2\Delta t \langle \nabla u^n, \nabla \varphi \rangle_{L^2} + \langle \sigma(u^n)\Delta W^n, \varphi \rangle_{L^2}$$

Renormalization step : we set almost surely, for all $x \in D$,

$$u^{n+1}(x) = \frac{u^n(x) + v^n(x)}{|u^n(x) + v^n(x)|} \ . \tag{3}$$

Theorem 1. *The algorithm converges up to a subsequence to a solution of (2) as $\Delta t, h \to 0$.*

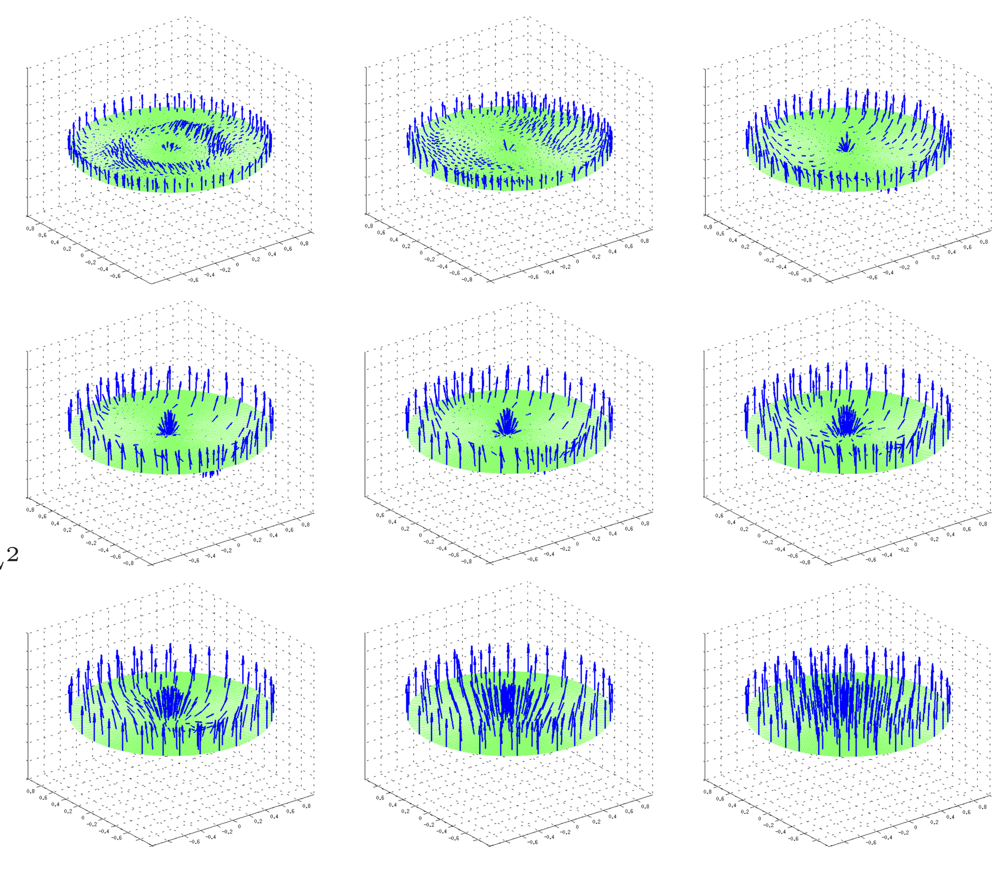


Figure 2: Example of simulation of (2)

Blow-up Phenomena

Deterministic case. In this box we consider equivariant fields

$$u(t,x) = \left(\frac{x_1}{|x|}\sin h(t,|x|), \frac{x_2}{|x|}\sin h(t,|x|), \cos h(t,|x|)\right) \ .$$

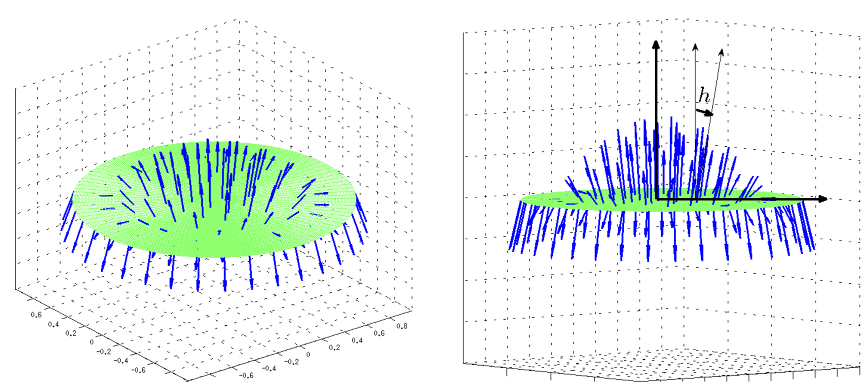


Figure 4: Equivariant field

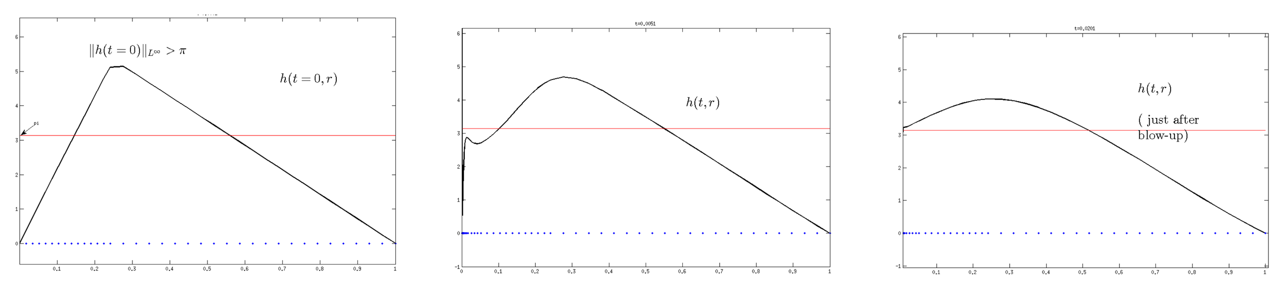


Figure 5: Blow-up of h in the deterministic case

Theorem 3. *Every deterministic equivariant solution u such that $\|h_0\|_{L^\infty} \le \pi$ is a **global regular solution**. Conversely, there exist h_0 with $\|h_0\|_{L^\infty} > \pi$ that blow up in finite time t^* in the following sense :*

$$\|\partial_r h\|_{L^\infty} \to \infty \text{ as } t \to t^* \ .$$

Case with noise. Noise can be added to the equivariant case considering a real noise $\dot{W}(t) \in L^2(D;\mathbb{R})$ and

$$\sigma(x,u) = u^\perp \dot{W} \ , \text{ where}$$

$$u^\perp := \left(\frac{x_1}{|x|}\cos h(t,|x|), \frac{x_2}{|x|}\cos h(t,|x|), -\sin h(t,|x|)\right) \ ,$$

leading to

$$\begin{cases} dh = (\partial_{rr}h + \frac{1}{r}\partial_r h - \dfrac{\sin 2h/2}{r^2})dt + dW \ , \\ \qquad (r,t) \in (0,1) \times \mathbb{R}^+ \ , \\ h(r \in \{0,1\}, t) = 0 \ , \ t \in \mathbb{R}^+ \ , \\ h(\cdot, t=0) = h_0 \ . \end{cases} \tag{4}$$

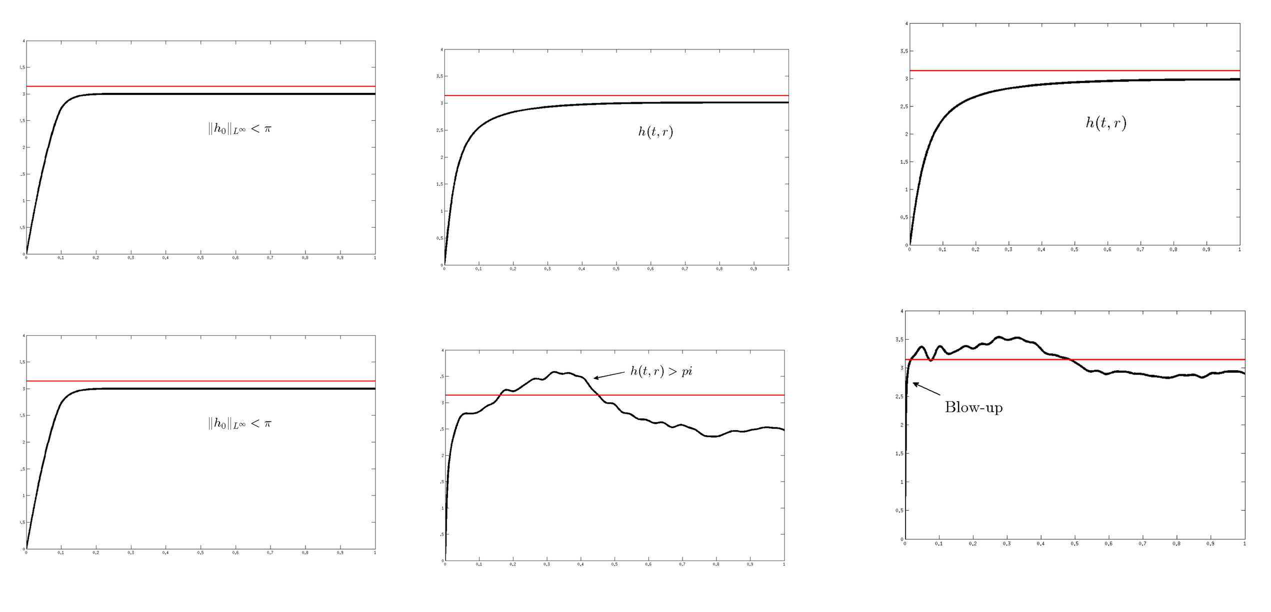


Figure 6: Example of blow-up that can not occur without noise

The Noise

The **stochastic flow of harmonic maps** writes :

$$du = \left(\Delta u + u|\nabla u|^2\right) dt + \sigma(u) \circ dW \ , \tag{2}$$

with boundary conditions as in (1).

Conditions on the noise.

1. The term $\sigma(u)\dot{W}(t,x)$ has to be orthogonal to the vector $u(t,x)$.
2. The product $\sigma(u) \circ dW$ is understood in the Stratonovitch sense.

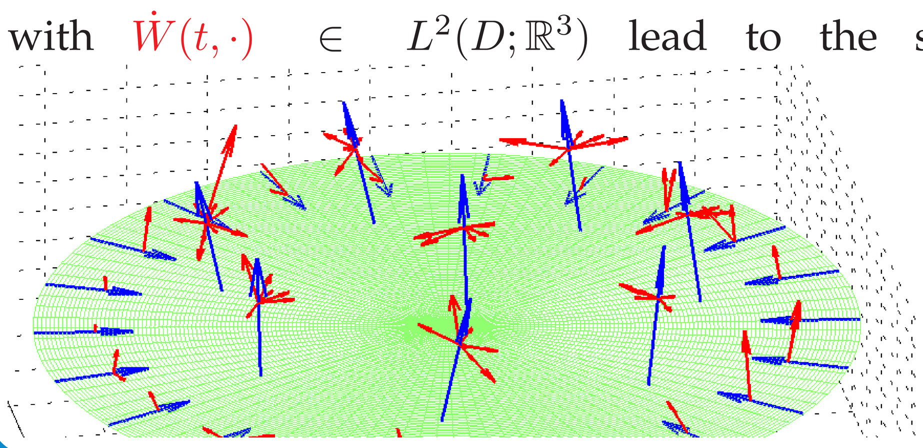
Equivalent **Itô** formulation :

$$du = \left(\Delta u + u|\nabla u|^2 - g(x)u\right) dt + \sigma(u)dW \ .$$

The following two choices

$$\sigma(u(t,x))\dot{W}(t,x) = \begin{cases} u(t,x) \times \dot{W}(t,x) \ , \\ P_{u(t,x)^\perp}\dot{W}(t,x) \ , \ \text{(orth. proj.)} \end{cases}$$

with $\dot{W}(t,\cdot) \in L^2(D;\mathbb{R}^3)$ lead to the same laws.



Global solutions

Theorem 2. *For $u_0 \in H^1$, there exist a global solution to (2) in the weak sense and a sequence of stopping times $T^1 < T^2 < \cdots < T^n$ such that $\mathbb{P}\left(T^n \xrightarrow{n\to\infty} \infty\right) = 1$, and one has almost surely*

$$u \in \bigcup_{n\in\mathbb{N}} \mathcal{C}\left([T^n, T^{n+1}); H^1\right) \cap L^2\left([T^n, T^{n+1}); H^2\right) \ ,$$

*and this u is **unique** in this class of solutions. Moreover, the times T^k at which u may blow-up are caracterized by **bubbles**:*

$$\inf_{R>0} \sup_{\substack{x\in D \\ t\in[T^{k-1},T^k)}} \int_{B(x,R)} |\nabla u(t,y)|^2 dy > \varepsilon_1 \ .$$

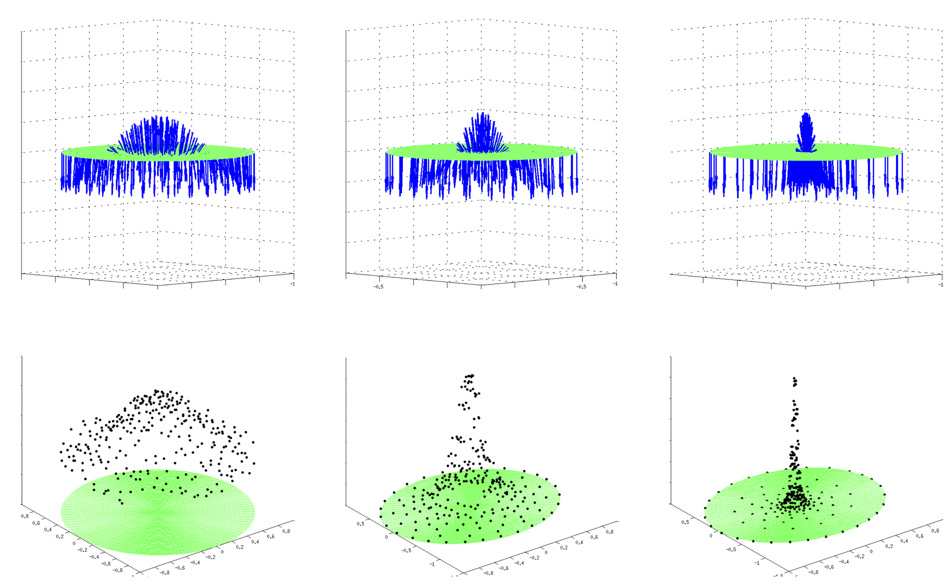


Figure 3: A bubble at the centre

The solution $u(t,x)$ is extended after T^k with a loss of energy expressed as a fixed quantum $\varepsilon_1 > 0$.

Future developments

•Blow-up for any initial data
Numerical studies show that initial data that are close to equivariant explosive initial data may also blow up, even if the noise term is multidimensional. Thus, if the noise approaches a control that brings a solution close to an initial explosive data, any solution of (2) may blow up in finite time with a positive probability.

•Blow-up for stochastic LLG
Such blow-up phenomena are related to the blow-up of the stochastic Landau-Lifshitz-Gilbert equation. In this case we have to understand how the so called gyromagnetic term that must be added in (2) interacts with bubbles.

References

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