Project #1

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Abstract

This project explored the dynamics of point vortices. A MATLAB model to simulate the motion of a group of vortices n, each of strength Γ_i. In fluid mechanics, a point vortex placed at an origin has an induced velocity field. The movement of each vortex is influenced by various parameters such as their own strength, position, as well as the strength and position of nearby vortices. This project examined symmetric co-rotating, counter-rotating, and symmetric co-rotating vortex pairs, a four-vortex system over ground at y = 0, simulating a plane during take-off or landing. It highlights the importance of modeling real-world applications using numerical methods.

1 Problem Statement

In this project, the goal was to create a model that simulated the movement of a group of vortices that simulated real world scenarios involving aircraft. In order to begin, each vortex is treated as a point vortex that induced its own velocity field and therefore influences other nearby vortices. Each vortex was characterized by their strength Γ_i and initial positions (x_i, y_i). To arrive at a numerical solution Euler's Method was used to create a timestep for each iteration of the simulation. The model is able to serve as a crucial understanding for vortex dynamics and possible ways to improve aircraft design.

2 Methodology

From fluid mechanics, a point vortex has the induced velocity field in polar coordinates

$$\mathbf{u_r} = (u_r, u_\theta) = \left(0, \frac{\Gamma}{2\pi r}\right) \tag{1}$$

We the convert this to Cartesian coordinates and now consider a group of n vortices of strength Γ_i.

$$\mathbf{u}_j(\mathbf{r}) = \frac{\Gamma_j}{2\pi r_j^2} k \times (r - r_j) = \frac{\Gamma_j}{2\pi[(x-x_j)^2 + (y-y_j)^2]}[-(y-y_j), (x-x_j)] \tag{2}$$

where $r = |\mathbf{r} - r_j|$ and $\mathbf{k}$ is the unit vector in the z direction. Now considering a group of n vortices of strength Γ_i, a vortex moves with the total induced velocity field due to the rest of the vortices as follows:

$$\mathbf{U_i} = \left[\sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi[(x_i-x_j)^2 + (y_i-y_j)^2]}[-(y_i-y_j)], \sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi[(x_i-x_j)^2 + (y_i-y_j)^2]}[(x_i-x_j)]\right] \tag{3}$$

According to Helmholtz' Law the vortex follows the fluid

$$\frac{dr_i}{dt} = \left(\frac{dx_i}{dt}, \frac{dy_i}{dt}\right) = U_i(r_i - r_j) \tag{4}$$

Equation (4) was approximated using Euler's Method and used to calculate the position of each vortex as such

$$\Delta r_i \approx \mathbf{U_i}\Delta t \Rightarrow r_i(t+\Delta t) = r_i(t) + \Delta r_i \tag{5}$$

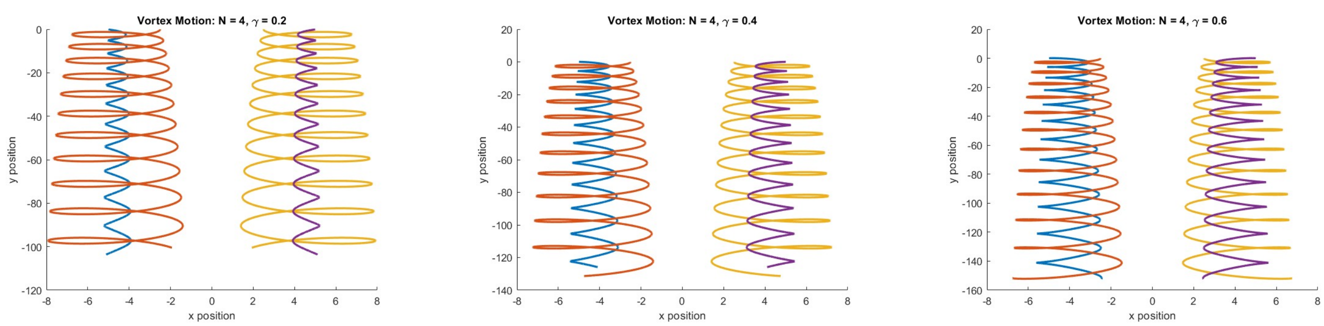
3 Model Simulation Calculations

In the following calculations, constants b and Γ are taken as positive values. The values of the parameter $\gamma = 0.2, 0.4, 0.6$

3.1 Symmetric Co-Rotating Vortex Pairs

In this simulation, the vortex wake of a wing with inboard flaps deployed is modeled. To calculate the trajectories, the initial configuration $(x, y, \Gamma)_i$ and constants b and $\Gamma = 10$ with $\Delta t = 0.1$ and the total time T = 500. For this calculation, the configuration is as follows

$$\left[(-\frac{b}{2}, 0, -\Gamma), (-\frac{b}{4}, 0, -\gamma\Gamma), (\frac{b}{4}, 0, +\gamma\Gamma), (\frac{b}{2}, 0, +\Gamma)\right] \tag{6}$$



(a) $\gamma = 0.2$ (b) $\gamma = 0.4$ (c) $\gamma = 0.6$

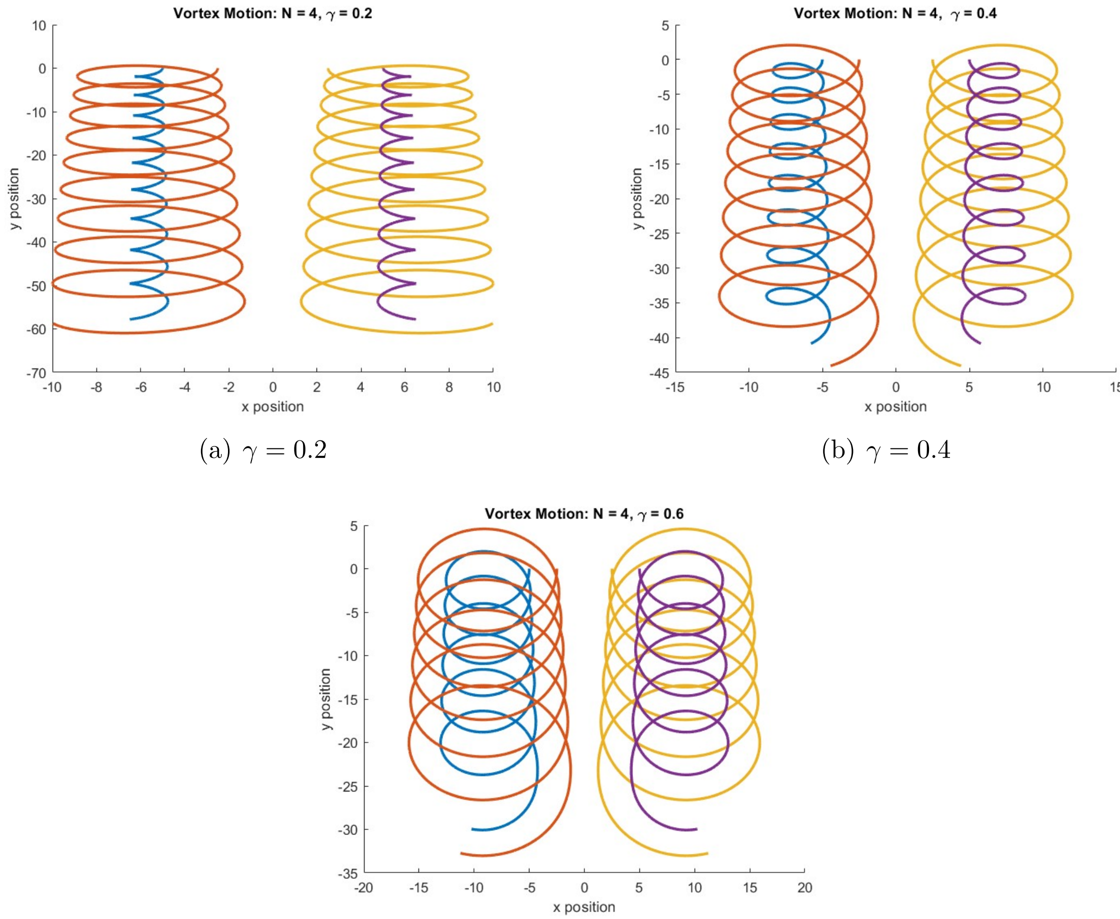
Figure 1: Calculation 1 with increasing γ values

From the above figures, there is a noticeable difference between the vortices represented by the blue and purple lines. As γ increases, the diameter of the vortices also increases. This represents the stronger interactions the other nearby vortices have on them.

3.2 Symmetric Counter-Rotating Vortex Pairs

The following calculation was to model the vortex wake of a wing with tip flaps deployed. The same values for b and Γ as the previous statement were used. The configuration is now

$$\left[(-\frac{b}{2}, 0, -\Gamma), (-\frac{b}{4}, 0, +\gamma\Gamma), (\frac{b}{4}, 0, -\gamma\Gamma), (\frac{b}{2}, 0, +\Gamma)\right] \tag{7}$$



(a) $\gamma = 0.2$

(b) $\gamma = 0.4$

(c) $\gamma = 0.6$

Figure 2: Calculation 2 with increasing γ values

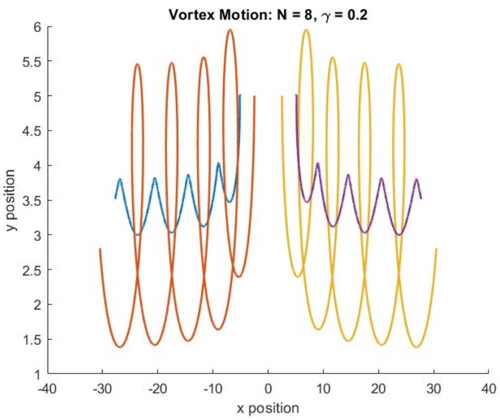
From these calculations it is clear that as γ increases, the blue and purple vortices follow much greater interactions and more tightly spiraled compared to when $\gamma = 0.2$. Although not shown in the diagrams, likely due to the total timestep in the model, long term behavior of the vortex pairs will likely converge to a single line due to the opposite circulation.

3.3 Symmetric Co-Rotating Vortex Pair: 4 Vortex System

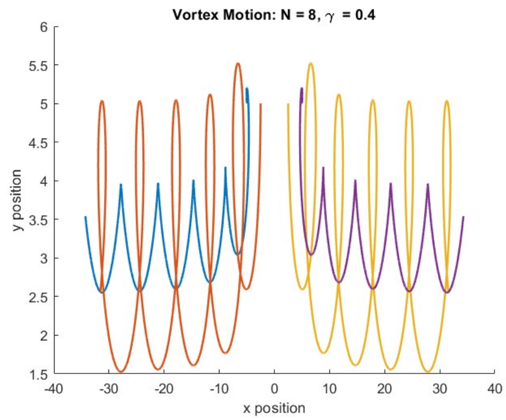
The following calculation simulated a four vortex system over ground at $y = 0$ representing an aircraft during take off and landing. The timestep was altered such that $\Delta t = 0.01$ to allow for a smoother plot. The same values for b and Γ were used as the prior calculations. The configuration used was

$$\left[(-\frac{b}{2}, \frac{b}{2}, -\Gamma), (-\frac{b}{4}, \frac{b}{2}, -\gamma\Gamma), (\frac{b}{4}, \frac{b}{2}, +\gamma\Gamma), (\frac{b}{2}, \frac{b}{2}, +\Gamma)\right] \tag{8}$$

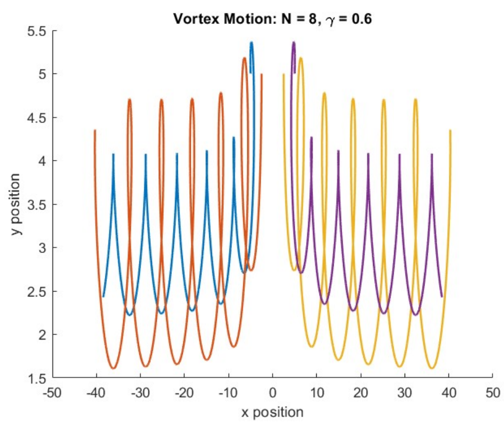
In this simulation, we must also consider the image vortices located at $y = -\frac{b}{2}$ to satisfy the impermeability condition at the ground plane and so it is really an eight vortex system. The impermeability condition requires the flow velocity normal to a boundary to be zero. In this case, no fluid can penetrate the ground and therefore maintaining the zero-slip condition. The 4 vortices located at $y = -\frac{b}{2}$ are fictitious and placed at mirror locations with respect to the ground plane $y = 0$. These fictitious vortices ensure that the boundary condition is met at the ground plane. Since these 4 vortices are fictitious they are not represented by the plots below.



(a) $\gamma = 0.2$



(b) $\gamma = 0.4$



(c) $\gamma = 0.6$

Figure 3: Calculation 3 with increasing γ values

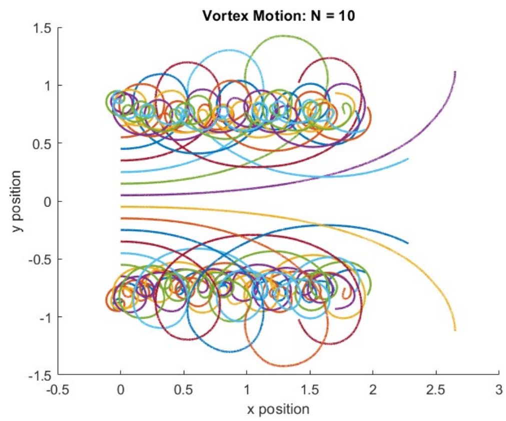
It is clear from the plots that as γ increases, the inner vortices increase in diameter that contribute to a more turbulent wake pattern. When $\gamma = 0.2$ the oscillations are more gentle which indicate weaker interactions and less turbulent wake patterns and they increase as γ gets larger.

3.4 Vortex Sheet Roll-Up

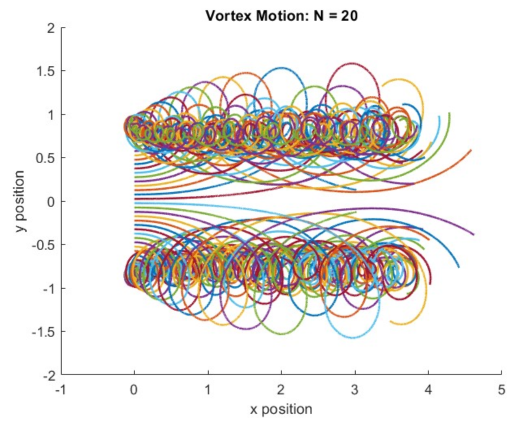
The vortex wake of an airplane can be modeled as a distribution of vortices on a line segment. We consider $2N$ equally spaced vortices of varying strengths along $y \in [-1, +1]$ with the configuration:

$$(y_i, \Gamma_i) = \left(\frac{i+1/2}{N}, \frac{(2i+1)/N}{\sqrt{1-(i+1/2)^2/N^2}} \right) \quad \text{for} \quad i = -N, ..., 0, ..., N-1 \tag{9}$$

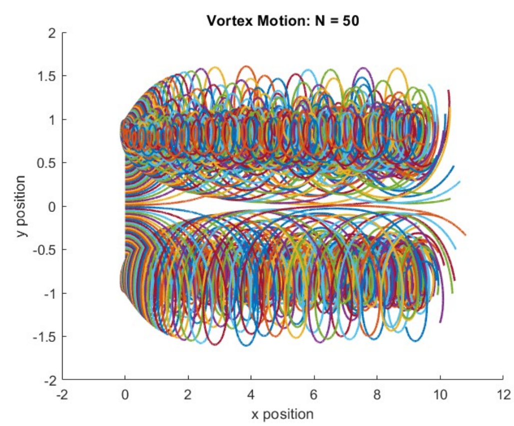
This distribution of vortices mimics the elliptical loading condition. The calculation is conducted for $N = 10, 20, 50$. In order to acheive smooth plots to visualize the vortex motions, Δt is set to 0.0001 while T is set to 1.



(a) $N = 10$ Vortices



(b) $N = 20$ Vortices



(c) $N = 50$ Vortices

Figure 4: Wake Turbulence with N = 10, 20, 50

From the plots above it demonstrates the wake turbulence of an aircraft. As N increases, the density of the vortices increases significantly. For each plot there is symmetry that result in a consistent wake turbulence pattern to mimic the elliptical loading condition.

4 Conclusion

As a result of this project, it is evident that numerical solutions to model simple vortex dynamics is possible. Understanding vortex motion provides valuable insight into aircraft design to minimize drag, wake turbulence, and vortex shedding to make aircraft safer and efficient.

References

[1] Savas. O *ME 163/Z Engineering Aerodynamics*. Department of Mechanical Engineering at University of California, Berkeley, Spring 2025.

[2] Bermudez. L *Matlab Code.* University of California, Berkeley