Sum of roots and product of roots

Qno.1

a. (i) $\left(-\frac{a_{n-1}}{a_n} =\right) -\frac{1}{2}$ ***A1***

(ii) $\left((-1)^n \frac{a_0}{a_n} =\right) -\frac{36}{2} = (-18)$ ***A1A1***

Note: First ***A1*** is for the negative sign.

[3 marks]

b. **METHOD 1**

if λ satisfies $p(\lambda) = 0$ then $q(\lambda - 4) = 0$

so the roots of $q(x)$ are each 4 less than the roots of $p(x)$ ***(R1)***

so sum of roots is $-\frac{1}{2} - 4 \times 5 = -20.5$ ***A1***

METHOD 2

$p(x+4) = 2x^5 + 2 \times 5 \times 4x^4 \ldots + x^4 \ldots = 2x^5 + 41x^4 \ldots$ ***(M1)***

so sum of roots is $-\frac{41}{2} = -20.5$ ***A1***

[2 marks]

Tofal [5 marks]

Qno.2

Markscheme

$\alpha + \beta = 2k$ ***A1***

$\alpha\beta = k - 1$ ***A1***

$(\alpha + \beta)^2 = 4k^2 \Rightarrow \alpha^2 + \beta^2 + 2\underbrace{\alpha\beta}_{k-1} = 4k^2$ ***(M1)***

$\alpha^2 + \beta^2 = 4k^2 - 2k + 2$

$\alpha^2 + \beta^2 = 4 \Rightarrow 4k^2 - 2k - 2 = 0$ ***A1***

attempt to solve quadratic ***(M1)***

$k = 1, \ -\frac{1}{2}$ ***A1***

[6 marks]

Qno.3

a. for the equation to have real roots

$(y-1)^2 - 4y(y-1) \geqslant 0$ ***M1***

$\Rightarrow 3y^2 - 2y - 1 \leqslant 0$

(by sign diagram, or algebraic method) ***M1***

$-\frac{1}{3} \leqslant y \leqslant 1$ ***A1A1***

Note: Award first ***A1*** for $-\frac{1}{3}$ and 1, and second ***A1*** for inequalities. These are independer

[4 marks]

b. $f : x \to \frac{x+1}{x^2+x+1} \Rightarrow x + 1 = yx^2 + yx + y$ ***(M1)***

$\Rightarrow 0 = yx^2 + (y-1)x + (y-1)$ ***A1***

hence, from (a) range is $-\frac{1}{3} \leqslant y \leqslant 1$ ***A1***

[3 marks]

c. a value for y would lead to 2 values for x from (a) ***R1***

Note: Do not award ***R1*** if (b) has not been tackled.

Qno.4

a. $\text{sum} = 0$ ***A1***

$\text{product} = 6$ ***A1***

[2 marks]

b. $P(1) = 1 - 10 + 15 - 6 = 0$ ***M1A1***

$\Rightarrow (z - 1)$ is a factor of $P(z)$ ***AG***

Note: Accept use of division to show remainder is zero.

[2 marks]

c. **METHOD 1**

$(z-1)^3(z^2 + bz + c) = z^5 - 10z^2 + 15z - 6$ ***(M1)***

by inspection $c = 6$ ***A1***

$(z^3 - 3z^2 + 3z - 1)(z^2 + bz + 6) = z^5 - 10z^2 + 15z - 6$ ***(M1)(A1)***

$b = 3$ ***A1***

METHOD 2

α, β are two roots of the quadratic

$b = -(\alpha + \beta),\ c = \alpha\beta$ ***(A1)***

from part (a) $1 + 1 + 1 + \alpha + \beta = 0$ ***(M1)***

$\Rightarrow b = 3$ ***A1***

$1 \times 1 \times 1 \times \alpha\beta = 6$ ***(M1)***

$\Rightarrow c = 6$ ***A1***

Note: Award ***FT*** if $b = -7$ following through from their sum $= 10$.

METHOD 3

$(z^5 - 10z^2 + 15z - 6) \div (z - 1) = z^4 + z^3 + z^2 - 9z + 6$ ***(M1)A1***

Note: This may have been seen in part (b).

$z^4 + z^3 + z^2 - 9z + 6 \div (z - 1) = z^3 + 2z^2 + 3z - 6$ ***(M1)***

$z^3 + 2z^2 + 3z - 6 \div (z - 1) = z^2 + 3z + 6$ ***A1A1***

[5 marks]

d. $z^2 + 3z + 6 = 0$ ***M1***

$z = \frac{-3 \pm \sqrt{9 - 4 \bullet 6}}{2}$ ***M1***

$= \frac{-3 \pm \sqrt{-15}}{2}$

$z = -\frac{3}{2} \pm \frac{\mathrm{i}\sqrt{15}}{2}$ ***A1***

(or $z = 1$)

$z^4 + z^3 + z^2 - 9z + 6 \div (z-1) = z^3 + 2z^2 + 3z - 6$ ***(M1)***

$z^3 + 2z^2 + 3z - 6 \div (z-1) = z^2 + 3z + 6$ ***A1A1***

[5 marks]

d. $z^2 + 3z + 6 = 0$ ***M1***

$z = \frac{-3 \pm \sqrt{9 - 4 \bullet 6}}{2}$ ***M1***

$= \frac{-3 \pm \sqrt{-15}}{2}$

$z = -\frac{3}{2} \pm \frac{\mathrm{i}\sqrt{15}}{2}$ ***A1***

(or $z = 1$)

[redacted]

Notes: Award the second ***M1*** for an attempt to use the quadratic formula or to complete the square.

Do not award ***FT*** from (c).

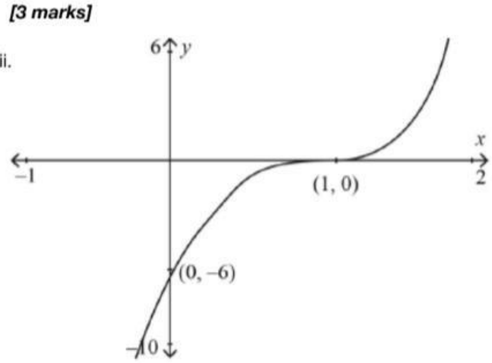
[3 marks]

e.i. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 20x^3 - 20$ ***M1A1***

for $x > 1,\ 20x^3 - 20 > 0 \Rightarrow$ concave up ***R1AG***

[3 marks]

e.ii.



x-intercept at $(1,\ 0)$ ***A1***

y-intercept at $(0,\ -6)$ ***A1***

stationary point of inflexion at $(1,\ 0)$ with correct curvature either side ***A1***

[3 marks]

Qno.5

a. (i)-(iii) given the three roots $\alpha,\ \beta,\ \gamma$, we have

$x^3 + px^2 + qx + c = (x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$= \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)(x-\gamma)$ ***A1***

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha+\beta+\gamma)$ ***AG***

$q = (\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$c = -\alpha\beta\gamma$ ***AG***

[3 marks]

b. **METHOD 1**

(i) Given $-\alpha-\beta-\gamma = -6$

And $\alpha\beta+\beta\gamma+\gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$

So $\beta-\alpha = \gamma-\beta$ ***M1***

or $2\beta = \alpha+\gamma$

Attempt to solve simultaneous equations: ***M1***

$\beta + 2\beta = 6$ ***A1***

$\beta = 2$ ***AG***

(ii) $\alpha+\gamma = 4$

$2\alpha + 2\gamma + \alpha\gamma = 18$

$\Rightarrow \gamma^2 - 4\gamma + 10 = 0$

$\Rightarrow \gamma = \frac{4\pm i\sqrt{24}}{2}$ ***(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\frac{4+i\sqrt{24}}{2}\right)\left(\frac{4-i\sqrt{24}}{2}\right)2 = -20$ ***A1***

METHOD 2

(i) let the three roots be $\alpha,\ \alpha-d,\ \alpha+d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) α is a root, so $2^3 - 6\times2^2 + 18\times2 + c = 0$ ***M1***

$8 - 24 + 36 + c = 0$

$c = -20$ ***A1***

METHOD 3

(i) let the three roots be $\alpha,\ \alpha-d,\ \alpha+d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2-d) + (2-d)(2+d) + 2(2+d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}i$

$\Rightarrow c = -20$ ***A1***

[5 marks]

METHOD 3

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2 - d) + (2 - d)(2 + d) + 2(2 + d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}\mathrm{i}$

$\Rightarrow c = -20$ ***A1***

[5 marks]

c. **METHOD 1**

Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$.

So $\frac{\beta}{\alpha} = \frac{\gamma}{\beta}$ ***M1***

or $\beta^2 = \alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta + \gamma\beta + \beta^2 = 18$

$\beta(\alpha + \beta + \gamma) = 18$

$6\beta = 18$

$\beta = 3$ ***A1***

$\alpha + \gamma = 3,\ \alpha = \frac{9}{\gamma}$

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